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H^2 spaces of generalized half-planes

by

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Abstract. New proofs are given of the following assertions about the Hardy space H^2 on Siegel domains of type II: H^2 is a Hilbert space and has a reproducing kernel. Elements of H^2 have "L^2-boundary values", and admit a Paley–Wiener type representation formula.

1. Introduction. The basic elementary facts in the theory of H^2 spaces are: (a) that these spaces are Hilbert spaces, (b) that H^2 functions have "L^2 boundary values", (c) that a Paley–Wiener type representation formula holds, and (d) that H^2 spaces have reproducing kernels. For tube domains over regular cones these results were proved by S. Bochner [2]. For Siegel domains of type II they were obtained by S. G. Gindikin [4]. Gindikin's arguments, however, were not conclusive, and the first complete derivation of his results was given—using methods different from his—by A. Korányi and E. M. Stein [7]. The purpose of this paper is to present, for Siegel domains of type II, a new and, maybe, simpler approach to the proof of the four facts listed above.

2. Definitions, notation, and statement of results. Let W and V be finite dimensional complex vector spaces of positive dimensions with $\dim W = m$ and $\dim V = n$. Let U be a real form of W chosen once and for all. Elements of U and W will be, usually, denoted by l.c. *latin characters*, elements of V always by ζ and ω. The conjugate of $z \in W$ relative to U will be written as $\bar{z}$. The value of an element λ of the dual space U' of U at the vector a of U or W will be denoted by $\langle \lambda, a\rangle$. We select once and for all Haar measures dx and $d\zeta$ on the vector groups U and V. The Fourier transform on $L^1(U)$ is defined by

$$\hat{f}(\lambda) = \int_U \exp(-2\pi i\langle \lambda, x\rangle) f(x)\,dx.$$

The Haar measure $d\lambda$ on U' is normalized so that the Fourier inversion

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formula reads

$$f(x) = \int_{U'} \exp(2\pi i \langle \lambda, x\rangle)\hat{f}(\lambda)\,d\lambda.$$

A *regular cone* Ω in U is an open convex cone with vertex at the origin which contains no affine line. Let $\bar{\Omega}$ denote the closure of Ω. If Ω is regular, then so is its dual cone $\Omega' = \{\lambda \in U' : \langle \lambda, y\rangle > 0,\, y \in \bar{\Omega} - \{0\}\}$. A Hermitean bilinear map $\Phi: V \times V \to W$ is said to be *Ω-positive* if for all $\zeta \in V$, $\Phi(\zeta, \zeta) \in \bar{\Omega}$, and if $\Phi(\zeta, \zeta) = 0$ implies that $\zeta = 0$. For $\lambda \in \Omega'$ define a positive definite Hermitean form on $V \times V$ by $H_\lambda(\zeta, \omega) = 4\langle \lambda, \Phi(\zeta, \omega)\rangle$, and set $\varrho(\lambda) = \det H_\lambda$.

The *tube domain* over Ω in W is $T_\Omega = \{z \in W : \operatorname{Im} z \in \Omega\}$. The *Siegel domain of type II* determined in $W \times V$ by Ω and Φ is the set $D = \{(z, \zeta) \in W \times V : \operatorname{Im} z - \Phi(\zeta, \zeta) \in \Omega\}$. The *distinguished boundary* of D is the subset $B = \{(z, \zeta) : \operatorname{Im} z - \Phi(\zeta, \zeta) = 0\}$ of the topological boundary of D. The map $(x, \zeta) \mapsto (x + i\Phi(\zeta, \zeta), \zeta)$ is a homeomorphism of $U \times V$ onto B. The topological and measure theoretical structures of B are those of $U \times V$, transferred to B by the above map. Now L^p spaces for $1 \leqslant p < \infty$ can be defined on B. The $L^p(B)$ norm of a measurable function on B is explicitly

$$\|f\|_{L^p(B)} = \Big(\int_{U \times V} |f(x + i\Phi(\zeta, \zeta), \zeta)|^p\,dx\,d\zeta\Big)^{1/p}.$$

If $F: D \to C$ and $t \in \Omega$, then the function $F_t: D \cup B \to C$ is defined by $F_t(z, \zeta) = F(z + it, \zeta)$. Finally, for $1 \leqslant p < \infty$ the space $H^p = H^p(D)$ is defined as the set of all holomorphic functions $f: D \to C$ such that

$$\|F\|_{H^p} = \sup_{t \in \Omega} \|F_t|_B\|_{L^p(B)} < \infty.$$

The function $F \mapsto \|F\|_{H^p}$ is a norm on H^p. By abuse of notation we shall write $\|F_t|_B\|_{L^p(B)}$ as $\|F_t\|_{L^p(B)}$. General references about the facts reviewed here are [5], [6], and [8].

We now introduce a function space which will play an important part in our proofs. Consider the set of functions $\hat{F}: \Omega' \times V \to C$ subject to the following two conditions:

(A) For every $\zeta \in V$ $\hat{F}(\cdot, \zeta)$ is a measurable on Ω'.

(B) For every $\lambda \in \Omega'$ $\hat{F}(\lambda, \cdot)$ is a holomorphic entire function on V.

This set clearly forms a linear space. By a result of H. D. Ursell ([12], Theorem 8), the following statement is true; we record it for future reference as

Remark 1. A function $f: \Omega' \times V \to C$ satisfying conditions (A) and (B) is measurable on $\Omega' \times V$.

In view of the remark it is meaningful to impose the following, third condition on our functions:

(C) $$\|\hat{F}\|^2_{\hat{H}^2} = \int_{\Omega' \times V} e^{-\pi H_\lambda(\zeta, \zeta)} |\hat{F}(\lambda, \zeta)|^2\,d\lambda\,d\zeta < \infty.$$

We define the space $\hat{H}^2_0$ to be the set $\{\hat{F}: \Omega' \times V \to C : \hat{F}$ satisfies (A), (B), (C)$\}$. If $\hat{F}$ and $\hat{G}$ belong to $\hat{H}^2_0$, we say that $\hat{F}$ and $\hat{G}$ are *equivalent* ($\hat{F} \sim \hat{G}$) if $\hat{F}(\lambda, \zeta) = \hat{G}(\zeta, \lambda)$ for almost all $(\lambda, \zeta) \in \Omega' \times V$. In view of condition (C) we have

Remark 2. If $\hat{F}, \hat{G} \in \hat{H}^2_0$, then

$$\hat{F} \sim \hat{G} \Leftrightarrow \int_{\Omega'} \exp(-\pi H_\lambda(\zeta, \zeta))\,|\hat{F}(\lambda, \zeta) - \hat{G}(\lambda, \zeta)|^2\,d\lambda = 0$$

for almost every $\zeta \in V$

$$\Leftrightarrow \int_V \exp(-\pi H_\lambda(\zeta, \zeta))\,|\hat{F}(\lambda, \zeta) - \hat{G}(\lambda, \zeta)|^2\,d\zeta = 0$$

for almost every $\lambda \in \Omega'$.

Now $\hat{H}^2$ is defined as the set of equivalence classes (relative to $\sim$) of elements of $\hat{H}^2_0$. Clearly, $\hat{H}^2$ is an inner product space with the norm defined by (C). We can now state our results.

Lemma 1. *The space $\hat{H}^2$ is a Hilbert space.*

Theorem. (i) *Let $\hat{F} \in \hat{H}^2_0$, and let $(z, \zeta) \in D$. Define $U\hat{F}(z, \zeta)$ by*

(1) $$U\hat{F}(z, \zeta) = \int_{\Omega'} e^{2\pi i \langle \lambda, z\rangle} \hat{F}(\lambda, \zeta)\,d\lambda.$$

The integral in (1) is absolutely convergent, $U\hat{F}$ belongs to H^2, and if $G \in \hat{H}^2_0$ is equivalent to $\hat{F}$, then $U\hat{F} = U\hat{G}$.

(ii) *The integral (1) defines a map, also denoted by U, from $\hat{H}^2$ into H^2. It is a unitary map from $\hat{H}^2$ onto H^2, and H^2 is a Hilbert space.*

(iii) *If $F \in H^2$, then for $t \in \Omega$ tending to 0, $F_t|_B$ converges in the norm of $L^2(B)$ to an element $\tilde{F}$ of $L^2(B)$, and $\|F\|_{H^2} = \|\tilde{F}\|_{L^2(B)}$.*

(iv) *If (w, ω), $(z, \zeta) \in D$, then the function $(z, \zeta) \mapsto S_{(w,\omega)}(z, \zeta)$ defined by*

(2) $$S_{(w,\omega)}(z, \zeta) = \int_{\Omega'} e^{2\pi i \langle \lambda, z - \bar{w} - 2\Phi(\zeta, \omega)\rangle} \varrho(\lambda)\,d\lambda$$

belongs to H^2, and for every $F \in H^2$

(3) $$F(w, \omega) = \langle F \mid S_{(w,\omega)}\rangle_{H^2},$$

where $\langle \cdot \mid \cdot \rangle_{H^2}$ denotes the inner product in H^2.

Equation (3) states that $S_{(w,\omega)}$ is (a, and hence by general principles) the reproducing kernel of H^2, the so called *Szegő kernel of D*.

3. Proof of Lemma 1. Fix $\lambda \in \Omega'$. Define $\|f\|_\lambda$ for measurable functions on V by

$$\|f\|_\lambda^2 = \int_V e^{-\pi H_\lambda(\zeta,\zeta)}|f(\zeta)|^2 d\zeta.$$

and define $\mathscr{H}^\lambda$ to be the set of entire holomorphic functions on V for which $\|f\|_\lambda$ is finite. The space $\mathscr{H}^\lambda$ is an inner product space which obviously contains all the constants, and it is easily checked that it contains all polynomials. The proof of Lemma 1 consists in showing that $\mathscr{H}^\lambda$ is complete, and that $\hat{H}^2$ can be identified with the direct integral $\int_{\Omega'}^{\oplus} \mathscr{H}^\lambda d\lambda$. The basic facts about $\mathscr{H}^\lambda$, *viz.* the existence of a reproducing kernel and completeness are due to V. Bargmann [1]. For the sake of completeness, and also because some of the technical details of the proofs will be needed, we reprove these facts (with simplified proofs).

For $\zeta \in V$, and $f \in \mathscr{H}^\lambda$ define $(A_\zeta f)(\omega)$ to be $\exp\left(\pi H_\lambda(\omega, \zeta) - \frac{1}{2}\pi H_\lambda(\zeta, \zeta)\right) f(\omega - \zeta)$. Clearly, $\omega \mapsto (A_\zeta f)(\omega)$ is an entire function on V. A simple calculation shows that for $f, g \in H^\lambda$ one has (writing the inner product in $\mathscr{H}^\lambda$ as $\langle \cdot | \cdot \rangle_\lambda$)

$$\langle A_\zeta f | A_\zeta g\rangle_\lambda = \langle f | g\rangle_\lambda, \tag{4}$$

and that in particular for $f \in \mathscr{H}^\lambda$, $\|A_\zeta f\|_\lambda = \|f\|_\lambda$. Another easy calculation checks that $A_{-\zeta}$ is the inverse of A_ζ. Therefore A_ζ is a unitary transformation of $\mathscr{H}^\lambda$ onto itself. Let now $\theta \in \boldsymbol{R}$, for $f \in \mathscr{H}^\lambda$ define f_θ by $f_\theta(\zeta) = f(e^{i\theta}\zeta)$. Clearly, $f_\theta \in \mathscr{H}^\lambda$. The change of variable $\zeta \mapsto e^{i\theta}\zeta$ and the fact that $H_\lambda(e^{-i\theta}\zeta^{-i\theta}\zeta) = H_\lambda(\zeta, \zeta)$ show that $\langle f_\theta | 1\rangle_\lambda = \langle f | 1\rangle_\lambda$. Therefore, using first Fubini's and then Cauchy's theorem we have

$$\langle f | 1\rangle_\lambda = \frac{1}{2\pi}\int_0^{2\pi} \langle f_\theta | 1\rangle_\lambda d\theta = \left\langle \frac{1}{2\pi}\int_0^{2\pi} f_\theta d\theta \,\Big|\, 1\right\rangle_\lambda = f(0)\langle 1 | 1\rangle_\lambda. \tag{5}$$

By evaluating $\langle 1|1\rangle_\lambda$ in a coordinate system in which H_λ is diagonal, we find that $\langle 1|1\rangle_\lambda = \varrho(\lambda)^{-1}$. Using this in (5) we have

$$f(0) = \varrho(\lambda)\langle f | 1\rangle_\lambda. \tag{6}$$

Since $f(\zeta) = \exp\left(\frac{1}{2}\pi H_\lambda(\zeta, \zeta)\right)(A_{-\zeta}f)(0)$, we obtain from equality (6) $f(\zeta) = \varrho(\lambda)\exp\left(\frac{1}{2}\pi H_\lambda(\zeta, \zeta)\right)\langle A_{-\zeta}f | 1\rangle_\lambda$. Applying (4) to the right-hand side of the last equality we have

$$f(\zeta) = \varrho(\lambda)e^{\frac{1}{2}\pi H_\lambda(\zeta,\zeta)}\langle f | A_\zeta 1\rangle_\lambda. \tag{7}$$

Since $(A_\zeta 1)(\omega) = \exp\left(\pi H_\lambda(\omega, \zeta) - \frac{1}{2}\pi H(\zeta, \zeta)\right)$, we can rewrite (7) by setting $\varrho(\lambda)\exp\left(\pi H_\lambda(\omega, \zeta)\right) = K_\zeta^\lambda(\omega)$ as

$$f(\zeta) = \langle f | K_\zeta^\lambda\rangle_\lambda. \tag{8}$$

(Note that since K_ζ^λ is a numerical multiple of $A_\zeta 1$, it is an element of $\mathscr{H}^\lambda$.) We have proved that $\mathscr{H}^\lambda$ has a reproducing kernel given by K_ζ^λ. An easy calculation shows that $\|K_\zeta^\lambda\|_\lambda = \varrho(\lambda)^{1/2}\exp\left(\frac{1}{2}\pi H_\lambda(\zeta, \zeta)\right)$. Using this value of $\|K_\zeta^\lambda\|_\lambda$ and applying Schwarz's inequality to (8), we get

$$|f(\zeta)| \leqslant \varrho(\lambda)^{1/2}e^{\frac{1}{2}\pi H_\lambda(\zeta,\zeta)}\|f\|_\lambda. \tag{9}$$

If $K \subset V$ is compact and $C_K = \sup\{\exp\left(\frac{1}{2}\pi H_\lambda(\zeta, \zeta)\right): \zeta \in K\}$, then for $\zeta \in K$ (9) yields $|f(\zeta)| \leqslant \varrho(\lambda)^{1/2} C_K \|f\|_\lambda$. This inequality immediately implies the completeness of $\mathscr{H}^\lambda$.

We shall now derive another consequence of (9) which will be needed in the proof of the theorem. Let $\hat{F} \in \hat{H}_0^2$, then by condition (C) $\hat{F}(\lambda; \cdot)$ belongs to $\mathscr{H}^\lambda$ for almost every $\lambda \in \Omega'$. In view of (9) we then have

Remark 3. If $\hat{F} \in \hat{H}_0^2$, then for every $\zeta \in V$, $\lambda \mapsto \varrho(\lambda)^{-1/2}e^{-\frac{1}{2}\pi H_\lambda(\zeta,\zeta)} \times \times \hat{F}(\lambda, \zeta)$ belongs to $L^2(\Omega')$.

Let us also observe the following fact: if ζ_j, $j = 1, 2, 3, \ldots$ is a dense sequence in V and $f \in \mathscr{H}^\lambda$ is such that $\langle f | K_{\zeta_j}^\lambda\rangle = 0$ for $j = 1, 2, 3, \ldots$, then, by (8), $f = 0$. Consequently, we have the following

Remark 4. If ζ_j, $j = 1, 2, 3, \ldots$ is a dense sequence in V, then $K_{\zeta_j}^\lambda$, $j = 1, 2, 3, \ldots$ is a total sequence in $\mathscr{H}^\lambda$.

We now prove that $\hat{H}^2$ is complete. Let $\mathfrak{F} = \prod\{H^\lambda: \lambda \in \Omega'\}$, and let $\mathfrak{G} = \{f: \Omega' \times V \to C: f$ satisfies (A), and for every $\lambda \in \Omega', f(\lambda, \cdot) \in \mathscr{H}^\lambda\}$.

Note first that $\mathfrak{G}$ can be identified in an obvious way with a linear subspace of $\mathfrak{F}$. Also note that for fixed $\zeta \in V$, $(\lambda, \omega) \mapsto K_\zeta^\lambda(\omega)$ belongs to $\mathfrak{G}$. We shall now verify that the Hilbert spaces $\mathscr{H}^\lambda$ form a measurable field of Hilbert spaces ([3], page 142). To this end we must check three conditions.

(1) *If $f \in \mathfrak{G}$, then $\lambda \mapsto \|f(\lambda, \cdot)\|_\lambda$ is a measurable function on Ω'.*

To prove this note that, by Remark 1, f is a measurable function on $\Omega' \times V$. Then approximate the integral giving $\|f(\lambda, \cdot)\|_\lambda^2$ to within $\varepsilon/2$ by an integral over a large cube in V. Now approximate the integral over the cube to within $\varepsilon/2$ by a Riemann sum. This Riemann sum is a measurable function of λ. Therefore $\|f(\lambda, \cdot)\|_\lambda$ is the pointwise limit of measurable functions, and hence measurable.

(2) *If $g \in \mathfrak{F}$ is such that $\lambda \mapsto \langle g | f\rangle_\lambda$ is measurable for every $f \in \mathfrak{G}$, then $g \in \mathfrak{G}$.*

Proof. $g(\lambda)(\zeta) = \langle g | K_\zeta^\lambda\rangle_\lambda$ is measurable for every $\zeta \in V$ because $K_\zeta^\lambda \in \mathfrak{G}$. Now use Remark 1.

(3) *There is a sequence f_j of elements of $\mathfrak{G}$ such that for every $\lambda \in \Omega'$ the sequence $f_j(\lambda, \cdot)$ is total in $\mathscr{H}^\lambda$.*

Proof. Remark 4.

The elements of $\mathfrak{G}$ are called *measurable vector fields*. A measurable vector field f is said to be *square integrable* if $\int_{\Omega'} \|f(\lambda,\cdot)\|_\lambda^2 d\lambda$ is finite. Two square integrable measurable vector fields f and g are *equivalent* if $\int_{\Omega'} \|f(\lambda,\cdot)-g(\lambda,\cdot)\|_\lambda^2 d\lambda = 0$. The *direct integral* $\int_{\Omega'}^{\oplus} \mathscr{H}^\lambda d\lambda$ is defined as the set of equivalence classes of measurable, square integrable vector fields. The norm of $f\in\int_{\Omega'}^{\oplus} \mathscr{H}^\lambda d\lambda$ is $(\int_{\Omega'} \|f(\lambda,\cdot)\|_\lambda^2 d\lambda)^{1/2}$.

If $f\in\mathfrak{G}$ is square integrable, then, clearly, f belongs to $\hat{H}_0^2$. If g is another square integrable element of $\mathfrak{G}$, and g is equivalent to f, then (Remark 2) f and g are also equivalent in $\hat{H}_0^2$. The norm of a square integrable $f\in\mathfrak{G}$ equals its $\hat{H}^2$-norm. So far we have shown that $\int^{\oplus}\mathscr{H}^\lambda d\lambda$ can be identified with a subspace of $\hat{H}^2$. To prove that the subspace is actually all of $\hat{H}^2$ let $\hat{F}\in\hat{H}^2$ and select a representative $\hat{F}_1$ of $\hat{F}$ in H_0^2. The set of λ's in Ω' for which $\hat{F}_1(\lambda,\cdot)$ does not belong to $\mathscr{H}^\lambda$ is of measure zero. Now define $\hat{F}_2$ as follows:

$$\hat{F}_2(\lambda,\zeta) = \hat{F}_1(\lambda,\zeta) \text{ if } \hat{F}_1(\lambda,\cdot)\in\mathscr{H}^\lambda, \quad \text{and} \quad \hat{F}_2(\lambda,\zeta) = 0 \text{ otherwise.}$$

By Remark 1, $\hat{F}_2$ belongs to $\mathfrak{G}$, and hence to $\hat{F}_0^2$, and by Remark 2, it is equivalent to $\hat{F}_1$. This proves that $\hat{H}^2$ can be identified with the direct integral of the $\mathscr{H}^\lambda$'s. Since the direct integral of Hilbert spaces is a Hilbert space, Lemma 1 is proved.

4. Proof of the Theorem. In addition to Lemma 1 and Remark 3 two technical results will be needed which we now list.

LEMMA 2. *Let $F\in H^p$, $1\leqslant p<\infty$. Let $\zeta\in V$, and $\delta\in\Omega$ such that $\delta-\Phi(\zeta,\zeta)\in\Omega$. Then $z\mapsto F_\delta(z,\zeta) = F(z+i\delta,\zeta)$ belongs to $H^p(T_\Omega)$.*

LEMMA 3. *Let $\varepsilon>0$, $0<a<\frac{1}{2}$, and let $\lambda_1,\lambda_2,\ldots,\lambda_m$ be a basis of U' contained in Ω' which is compatible with the Haar measure $d\lambda$ on U'. Then $G^\varepsilon(z,\zeta) = \exp\{-\varepsilon\sum_{j=1}^m\langle\lambda_j,z\rangle^a\}$ belongs to $H^2(D)$, and is bounded and continuous on $\bar{D}$.*

Lemma 2 is actually true for all positive p, but we only need it for $p=1,2$. It is due to E. M. Stein [10]. Lemma 3 is a special case of Lemma 8.1 in [9].

Let now $\hat{F}\in\hat{H}_0^2$ and $(z,\zeta)\in D$, with $z=x+it+i\Phi(\zeta,\zeta)$ where $t\in\Omega$. In any coordinate system $\varrho(\lambda)$ is a homogeneous polynomial of degree n, and one can show readily that $\varrho(\lambda)^{1/2}\exp(-2\pi\langle\lambda,t\rangle)$ is square integrable on Ω'. Therefore by Remark 3

$$(10)\qquad \int_{\Omega'} e^{2\pi i\langle\lambda,z\rangle}\hat{F}(\lambda,\zeta)\,d\lambda = \int_{\Omega'} e^{2\pi i\langle\lambda,x\rangle}e^{-2\pi\langle\lambda,t\rangle}e^{-\frac{1}{2}\pi H_\lambda(\zeta,\zeta)}\hat{F}(\lambda,\zeta)\,d\lambda$$

is absolutely convergent for every $\zeta\in V$. Choosing coordinates in $W\times V$, and applying Morera's theorem in combination with Fubini's theorem one shows that $U\hat{F}$ is holomorphic in each coordinate of (z,ζ), and hence by Hartogs's theorem holomorphic in D. Since $\varrho(\lambda)^{1/2}\exp(-2\pi\langle\lambda,t\rangle)$ is a bounded function of λ on Ω', by Remark 3 one concludes that the quantity multiplying $\exp(2\pi i\langle\lambda,x\rangle)$ in (10) is square integrable on Ω'. Therefore, by Plancherel's theorem for every $\zeta\in V$,

$$\int_U |U\hat{F}(x+it+i\Phi(\zeta,\zeta),\zeta)|^2 dx = \int_{\Omega'} e^{-4\pi\langle\lambda,t\rangle}e^{-\pi H_\lambda(\zeta,\zeta)}|\hat{F}(\lambda,\zeta)|^2 d\lambda.$$

Integrating this equality on V, we get

$$(11)\qquad \|(U\hat{F})_t\|_{L^2(B)}^2 = \int_{\Omega'\times V} e^{-4\pi\langle\lambda,t\rangle}e^{-\pi H_\lambda(\zeta,\zeta)}|\hat{F}(\lambda,\zeta)|^2 d\lambda d\zeta \leqslant \|\hat{F}\|_{\hat{H}^2}^2.$$

From (11) we conclude that $U\hat{F}\in H^2$. If $t_k\in\Omega$ is a sequence tending to 0, then, by the dominated convergence theorem, we have that $\|(U\hat{F})_t\|_{L^2(B)}^2$ converges to $\|\hat{F}\|_{\hat{H}^2}^2$ and that therefore

$$(12)\qquad \|U\hat{F}\|_{\hat{H}^2}^2 = \|\hat{F}\|_{\hat{H}^2}^2.$$

If $\hat{G}\in H_0^2$ and $\hat{F}\sim\hat{G}$, then (12) implies that $\|U\hat{F}-U\hat{G}\|_{H^2} = \|\hat{F}-\hat{G}\|_{\hat{H}^2} = 0$, i.e., that equivalent $\hat{F}$'s give rise to the same $U\hat{F}$. Therefore, U defines a linear map from $\hat{H}^2$ to H^2 which we continue to write U. The equation (12) shows that U maps $\hat{H}^2$ isometrically into H^2. Now let $t_k\in\Omega$ be a sequence converging to 0, then (11) (with $(U\hat{F})_{t_k}-(U\hat{F})_{t_l}$ instead of $(U\hat{F})_t$) and the dominated convergence theorem show that $(U\hat{F})_{t_k}|_B$ is a Cauchy sequence in $L^2(B)$. Therefore, $(U\hat{F})_{t_k}|_B$ converges in $L^2(B)$ norm to an element of $L^2(B)$. We omit the proof that the sequential limit can be replaced by $t\in\Omega'$ tending to 0. We therefore have

Remark 5. Assertion (iii) of the theorem holds for every $F\in H^2$ which admits the representation (1).

By Lemma 3 and Schwarz's inequality $H^2\cap H^1\neq\{0\}$. Let $F\in H^2\cap H^1$, and let $\zeta\in V$ be arbitrary but fixed. Set $\Omega_\zeta = \{\delta\in\Omega\colon \delta-\Phi(\zeta,\zeta)\in\Omega\}$. For $\delta\in\Omega_\zeta$ $z\mapsto F_\delta(z,\zeta)$ belongs to $(H^2\cap H^1)(T_\Omega)$ by Lemma 2. By the theory of H^p spaces on tube domains ([11], Chapter 3) the boundary function of F_δ, *viz.* $x\mapsto F_\delta(x,\zeta)$ belongs to $(L^2\cap L^1)(U)$. We can therefore define a function $\hat{F}_\delta(\lambda,\zeta)$ by

$$(13)\qquad \hat{F}_\delta(\lambda,\zeta) = \int_U e^{-2\pi i\langle\lambda,x\rangle}F_\delta(x,\zeta)\,dx.$$

Remark 6. By the H^2 theory for tube domains, $\hat{F}_\delta(\cdot,\zeta)$ is supported in Ω'. Since $F_\delta(\cdot,\zeta)\in L^1(U)\cap C^\infty(U)$, $\hat{F}_\delta(\cdot,\zeta)$ is continuous and integrable. Therefore Fourier inversion holds for every $x\in U$.

If δ' is another element of Ω_ζ, then, for $\lambda \in \Omega'$ by the H^2 theory for tube domains,

$$\hat{F}_{\delta+\delta'}(\lambda, \zeta) = \exp(-2\pi\langle\lambda, \delta\rangle)\hat{F}_{\delta'}(\lambda, \zeta) = \exp(-2\pi\langle\lambda, \delta'\rangle)\hat{F}_\delta(\lambda, \zeta).$$

Therefore for $\delta \in \Omega_\zeta$, $\exp(2\pi\langle\lambda, \delta\rangle)\hat{F}_\delta(\lambda, \zeta)$ is independet of δ. Denote this function by $\hat{F}(\lambda, \zeta)$.

Now let $z = x+iy \in W$ be such that $(z, \zeta) \in D$, i.e., $y \in \Omega_\zeta$. Note that $F(z, \zeta) = F_y(x, \zeta)$. By Remark 6, we can apply Fourier inversion to (13). If we now express $\hat{F}_y$ in terms of $\hat{F}$ in the Fourier inversion formula, we get

$$F(z, \zeta) = \int_{\Omega'} e^{2\pi i\langle\lambda, z\rangle}\hat{F}(\lambda, \zeta)\,d\lambda. \tag{14}$$

Since ζ was arbitrary, (14) holds for every $(z, \zeta) \in D$.

We now prove that $\hat{F} \in \hat{H}^2$. Again fix $\zeta_0 \in V$, and also $\lambda \in \Omega'$. If $\delta \in \Omega_{\zeta_0}$, then there is a polydisc $\Delta \subset V$ centered at ζ_0 such that $\delta \in \Omega_\zeta$ for $\zeta \in \Delta$. Now by (13) and by the definition of $\hat{F}$ we have for $\zeta \in \Delta$ that

$$\hat{F}(\lambda, \zeta) = e^{2\pi\langle\lambda, \delta\rangle}\int_U e^{-2\pi i\langle\lambda, x\rangle}F_\delta(x, \zeta)\,dx.$$

Exactly as before by combining the theorems of Fubini, Morera and Hartogs, we can show that $\zeta \mapsto \hat{F}(\lambda, \zeta)$ is holomorphic in Δ. Since ζ_0 was arbitrary in V, it follows that $\hat{F}(\lambda, \cdot)$ is an entire function. By Remark 6 we know that $F(\cdot, \zeta)$ is continuous for every $\zeta \in V$, therefore, by Remark 1, $\hat{F}$ is measurable on $\Omega' \times V$. Now let $t \in \Omega$, then Plancherel's theorem applied to (13) gives for every $\zeta \in V$

$$\int_U |F(x+it+\Phi(\zeta, \zeta), \zeta)|^2 dx = \int_{\Omega'} e^{-4\pi\langle\lambda, t+\Phi(\zeta,\zeta)\rangle}|\hat{F}(\lambda, \zeta)|^2 d\lambda.$$

Integrating this equality on V we have

$$\|F_t\|^2_{L^2(B)} = \int_{\Omega'\times V} e^{-4\pi\langle\lambda, t\rangle}e^{-\pi H_\lambda(\zeta,\zeta)}|\hat{F}(\lambda, \zeta)|^2\,d\lambda d\zeta \leqslant \|F\|^2_{H^2}.$$

By Fatou's lemma it follows that $\hat{F} \in \hat{H}^2$. Taking the supremum over Ω we see that $\|\hat{F}\|_{H^2} = \|\hat{F}\|_{\hat{H}^2}$. We conclude that the map $F \mapsto \hat{F}$ maps the subspace $H^2 \cap H^1$ of H^2 isometrically into $\hat{H}^2$. By Lemma 1, the range of this map is contained in a complete space, and therefore, if we denote by M the closure in H^2 of $H^2 \cap H^1$, it extends uniquely to an isometry $\boldsymbol{V}$ of M into $\hat{H}^2$. Now for $F \in H^2 \cap H^1$ (14) holds, and therefore for such F, $\boldsymbol{UV}F = F$, i.e., $\boldsymbol{UV}$ is the identity of $H^2 \cap H^1$. By continuity, it follows that $\boldsymbol{UV}$ is the identity on all of M, and hence if $F \in M$, then $F = \boldsymbol{U}(\boldsymbol{V}F)$,

i.e., $\boldsymbol{U}$ maps $\hat{H}^2$ isometrically *onto* M: M is a Hilbert space, and the unitary maps $\boldsymbol{U}$ and $\boldsymbol{V}$ are inverses of each other. By Remark 5 it follows that assertion (iii) holds for every $F \in M$.

We now prove that $M = H^2$. Let $F \in H^2$, and let G^ε be the function introduced in Lemma 3. By that lemma and Schwarz's inequality $G^\varepsilon F \in H^2 \cap H^1$. Since assertion (iii) of the theorem holds in M, there exists an element $(G^\varepsilon F)^\sim$ of $L^2(B)$ such that $(G^\varepsilon F)_t|_B$ tends to $(G^\varepsilon F)^\sim$ in $L^2(B)$ as $t \in \Omega$ tends to zero. Consider first the case $\varepsilon = 1$. For some sequence $t_k \in \Omega$, $t_k \to 0$ (fixed once and for all in this proof), $(G^1 F)_{t_k}|_B \to (G^1 F)^\sim$ almost everywhere on B. Since $G^1_{t_k}|_B \to G^1|_B = \tilde{G}^1$ everywhere on B, and $\tilde{G}^1$ does not vanish anywhere, we can conclude that $F_{t_k}|_B$ converges almost everywhere on B to a limit $\tilde{F}$. Since $\|F_{t_k}\|_{L^2} \leqslant \|F\|_{H^2}$, it follows form Fatou's lemma that $\tilde{F} \in L^2(B)$. Now let ε be arbitrary positive. Since $(G^\varepsilon F)_{t_k}|_B = G^\varepsilon_{t_k}|_B\, F_{t_k}|_B \to \tilde{G}^\varepsilon\tilde{F}$ almost everywhere, and $(G^\varepsilon F)_{t_k}|_B \to (G^\varepsilon F)^\sim$ in $L^2(B)$, it follows that $(G^\varepsilon F)^\sim = \tilde{G}^\varepsilon\tilde{F}$ almost everywhere on B.

Now let $\varepsilon_\nu \to 0$, then

$$\|G^{\varepsilon_\nu}F - G^{\varepsilon_\mu}F\|^2_{H^2} = \int_{U\times V} |G^{\varepsilon_\nu} - G^{\varepsilon_\mu}|^2|\tilde{F}|^2\,dx d\zeta$$

because (iii) holds in M. Since $G^{\varepsilon_\nu} - G^{\varepsilon_\mu}$ tends to zero boundedly, we have that $G^{\varepsilon_\nu}F$ is a Cauchy sequence in the complete space M; and therefore it tends in H^2 to an element H of M. Now let $t \in \Omega$ be arbitrary but fixed, then $\|(G^{\varepsilon_\nu}F)_t - H_t\|_{L^2(B)} \leqslant \|G^{\varepsilon_\nu}F - H\|_{H^2}$ and therefore $(G^{\varepsilon_\nu}F)_t|_B \mapsto H_t|_B$ in $L^2(B)$ norm. On the other hand, $G^{\varepsilon_\nu}_t|_B \to 1$ everywhere on B. Consequently, $F_t|_B = H_t|_B$ because both functions are continuous. Since $t \in \Omega$ was arbitrary, it follows that $F = H$, and hence, that $M = H^2$.

To prove (iv) let $(w, \omega) \in D$ and $F \in H^2$. By the assertions of the theorem already proved, we have

$$F(w, \omega) = \int_{\Omega'} e^{2\pi i\langle\lambda, w\rangle}\hat{F}(\lambda, \omega)\,d\lambda \tag{15}$$

where $\hat{F} = \boldsymbol{U}^{-1}F \in \hat{H}^2$. Since $F(\lambda, \cdot) \in \mathscr{H}^\lambda$ for almost every $\lambda \in \Omega'$, we have $\hat{F}(\lambda, \omega) = \langle\hat{F}(\lambda, \cdot)\,|\,K^\lambda_\omega\rangle_\lambda$ for almost every $\lambda \in \Omega'$. Introducing this into (15) and rewriting the integral as a double integral, we have (only formally, so far)

$$F(w, \omega) = \int_{\Omega'\times V} e^{-\pi H_\lambda(\zeta,\zeta)}\hat{F}(\lambda, \zeta)\{\overline{e^{-2\pi i\langle\lambda, \bar{w}\rangle}e^{\pi H_\lambda(\zeta,\omega)}\varrho(\lambda)}\}\,d\lambda d\zeta. \tag{16}$$

Denote the quantity in curly brackets by $T_{(w,\omega)}(\lambda, \zeta)$. A straightforward check verifies that $(\lambda, \zeta) \mapsto T_{(w,\omega)}(\lambda, \zeta)$ belongs to $\hat{H}^2$. Therefore, the double integral in (16) is absolutely convergent (this justifies the passage from (15) to (16)) and equal to $\langle\hat{F}\,|\,T_{(w,\omega)}\rangle_{\hat{H}^2}$. Consequently,

$$F(w, \omega) = \langle\hat{F}\,|\,T_{(w,\omega)}\rangle_{\hat{H}^2}. \tag{17}$$

Now a simple calculation shows that $UT_{(w,\omega)}$ is the function $S_{(w,\omega)}$ defined by (2), hence in view of the fact that U is unitary, (17) yields

$$F(w,\omega) = \langle F \mid S_{(w,\omega)} \rangle_{H^2}.$$

But this is equation (3) in assertion (iv) of the theorem whose proof is now complete.

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On an integral representation of antisymmetric operations in Hilbert spaces
I. Bounded operations

by

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Abstract. In this note we give the representation of the bounded and antysymmetric operation A defined and valued in the Hilbert space H (real or complex) in the following form:

$$Ax = \int_{\alpha}^{\beta} \lambda dQ_{\lambda} x,$$

where $Q \in I(H) = \{Q \in L(H): Q^3 = -Q \text{ and } Q^* = -Q\}$.

Moreover, we give the properties of the operation of the class $I(H)$ and some form of the solution of the equation

$$\frac{d}{dt}x(t) = Ax(t), \quad \text{where } A \text{ is antisymmetric}$$

with the initial condition $x(0) = x_0$.

1. Introduction. In this paper we give the spectral representation of bounded antisymmetric operations in Hilbert spaces; the case of unbounded operations will be presented in the next paper.

In our theory we formally give an effective solution of the equation

$$\frac{d}{dt}x(t) = Ax(t) \tag{1}$$

with the antisymmetric and bounded operation A.

2. Class of operations $I(H)$. Let H denote a Hilbert space and let $L(H)$ denote a linear space of all linear and bounded operations in H. The class of operations $I(H)$ is defined as follows:

$$Q \in I(H) \equiv Q \in L(H) \quad \text{and} \quad Q^3 = -Q = Q^*, \tag{2}$$

where Q^* denotes the conjugate operation with Q.