On pairs of commuting operators

by

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Abstract. The aim of this paper is to give a characterization of the joint spectrum, in the sense of J. L. Taylor of a pair of commuting operators on Hilbert spaces. An application to tensor products of operators is then presented.

1. In a previous paper we characterized the joint spectrum of a finite commuting system of linear continuous operators by means of the non-invertibility of a certain operator, acting on a direct sum of copies of the initial space [5]. In the sequel we intend to obtain a similar result for pairs of commuting operators, which is not a direct consequence of the above-mentioned characterization. The present statement is more specific and it may be used in some problems of spectral theory, as we shall exemplify at the end of this paper.

Let $H$ be a complex Hilbert space and $B(H)$ the algebra of all linear continuous operators on $H$. Let $a = (a_1, a_2) \in B(H)$ be a pair of commuting operators.

Consider the sequence

\[ 0 \rightarrow H \xrightarrow{d_0} H \otimes H \xrightarrow{d_1} H \rightarrow 0, \]

where $d_0^0(x) = a_1 x \otimes e_2 (x \in H)$ and $d_0^1(x_1 \otimes e_2) = a_1 x_1 - a_2 x_1 (x_1, x_2 \in H)$.

It is clear that $d_0^0 d_0^1 = 0$. We recall that $a$ is said to be non-singular [4] if the sequence (1.1) is exact. The (joint) spectrum $\sigma(a, H)$ of $a$ on $H$ is, by definition, the complement of the set of all $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ such that $\lambda - a = (\lambda_1 - a_1, \lambda_2 - a_2)$ is non-singular on $H$.

The main result of this paper is the following:

1.1. Theorem. Let $a = (a_1, a_2) \in B(H)$ be a commuting pair. Then $a$ is non-singular on $H$ if and only if the operator

\[ a(a) = \begin{bmatrix} a_1 & a_2 \\ -a^*_2 & a^*_1 \end{bmatrix} \]

is invertible on $H \otimes H$.
This theorem can be used to solve some special problems concerning the tensor product of two linear operators; in particular, one can obtain a new proof of the result of Brown and Pearcy concerning the spectrum of the tensor product of the operators [1].

2. In this section we give the proof of Theorem 1.1.

Suppose first that \( a = (a_1, a_2) \in B(H) \) is nonsingular on \( H \). Consider the dual sequence of (1.1), namely

\[
0 \to \mathbb{H}^* \xrightarrow{\delta^*} H \otimes H \xrightarrow{\delta^*} H \to 0,
\]

where \( \delta^*(x) = a_1^* x \otimes a_2^* x \) (\( x \in H \)) and \( \delta^*(x) = a_1^* x \otimes a_2^* x \) (\( x_1, x_2 \in H \)). It is easy to see that the pair \( a^* = (a_1^*, a_2^*) \) is nonsingular on \( H \) if and only if the sequence (2.1) is exact, and thus if and only if the pair \( a = (a_1, a_2) \) is nonsingular on \( H \).

2.1. Lemma. If \( a = (a_1, a_2) \) is nonsingular on \( H \), then both \( a_1 a_2^* + a_2 a_1^* \) and \( a_1 a_2^* + a_2 a_1^* \) are invertible on \( H \).

Indeed, let us show that \( a_1 a_2^* + a_2 a_1^* \) is injective and surjective on \( H \). If \( (a_1 a_2^* + a_2 a_1^*) x = 0 \) for a certain \( x \in H \), then \( -a_1 a_2^* x \otimes a_2 a_1^* x \in \ker \delta^* = \ker \delta^* = (\ker \delta^*)^* \). But \( a_1 a_2^* x \otimes a_2 a_1^* x \in \ker \delta^* \); hence \( a_1 a_2^* x = a_2 a_1^* x = 0 \). Since \( \ker \delta^* \) is closed, we have \( x = 0 \). Take an arbitrary \( y \in H \) and let us find an \( x \in H \) such that \( y = a_1 a_2^* x + a_2 a_1^* x \). We infer that \( \delta^* (\ker \delta^*)^* \to H \) is an isomorphism, and therefore \( y = \delta^* (y_1 \otimes y_2) \) with \( y_1 \otimes y_2 \in (\ker \delta^*)^* \); hence \( y_1 \otimes y_2 = -a_1 a_2^* (a_1 a_2^* + a_2 a_1^*) \).

Analogously, the operator \( a_1 a_2^* + a_2 a_1^* \) is invertible and this completes the proof of the lemma.

Let us return to the proof of Theorem 1.1. According to Lemma 2.1, it is clear that the operator

\[
\begin{pmatrix}
-a_1 (a_1 a_2^* + a_2 a_1^*)^{-1} & -a_2 (a_1 a_2^* + a_2 a_1^*)^{-1} \\
0 & a_1 (a_1 a_2^* + a_2 a_1^*)^{-1}
\end{pmatrix}
\]

is a left inverse for the operator \( a (a) \) given by (1.2); hence \( a (a) \) is surjective on \( H \otimes H \). Let us also note that \( a (a) \) is injective too. Indeed, if \( a (a) (x_1 \otimes x_2) = 0 \), then \( x_1 \otimes (-x_2) \in \ker a (a) \cap \ker \delta^* = \ker \delta^* \cap (\ker \delta^*)^* = \{0\} \), and hence \( x_1 = x_2 = 0 \).

Conversely, suppose that \( a (a) \) is invertible on \( H \otimes H \). Then \( a (a)^* \) is invertible; therefore

\[
\begin{pmatrix}
a_1 a_1^* + a_2 a_2^* & 0 \\
0 & a_1 a_1^* + a_2 a_2^*
\end{pmatrix}
\]

is invertible, and hence \( a_1 a_1^* + a_2 a_2^* \) and \( a_1 a_1^* + a_2 a_2^* \) are operators from \( B(H) \).

Let us prove that the sequence (1.1) is exact. Indeed, if \( \delta^* (x) = a_1 x \otimes a_2 x = 0 \), then \( (a_1 a_1^* + a_2 a_2^*) x = 0 \), whence \( x = 0 \).

Assume now that \( \delta^* (a_1 \otimes a_2) = a_1 a_2 - a_2 a_1 = 0 \). If \( y = a_1 a_2 + a_2 a_1 \), then \( a (a) (a_1 \otimes a_2) = 0 \); hence \( \delta^* (a_1 \otimes a_2) = a (a) \), and thus on account of (2.2) we obtain

\[
\begin{align*}
a_1 &= a_1 (a_1 a_2^* + a_2 a_1^*)^{-1} y, \\
a_2 &= a_2 (a_1 a_2^* + a_2 a_1^*)^{-1} y,
\end{align*}
\]

i.e., the exactness of (1.1) at the second step.

Finally, if \( y \in H \) is arbitrary, then \( x_j = a_j (a_1 a_2^* + a_2 a_1^*)^{-1} y \) \( (j = 1, 2) \) satisfy the equation \( a_1 x_1 + a_2 x_2 = y \), and the proof of Theorem 1.1 is complete.

2.2. Corollary. If \( a = (a_1, a_2) \in B(H) \) is a commuting pair, then the spectrum \( c(a, H) \) of \( a \) on \( H \) is given by the set

\[
C^\times \setminus \{x \in C^\times \mid [a(a) - a]^* = 0 \}
\]

As is known, the set \( c(a, H) \) is compact and nonempty [4] (see also [6] for Hilbert spaces).

Let us remark that the set of matrices \( (a_1 a_2 + a_2 a_1) \) can be identified with the algebra of quaternions and that the map \( x \mapsto a (a) \) is a \( R \)-linear isometric isomorphism [6].

Notice also that \( a = (a_1, a_2) \in B(H) \) is nonsingular if and only if the matrix

\[
\begin{pmatrix}
a_1 & -a_2^* \\
-a_1^* & a_2
\end{pmatrix}
\]

is invertible on \( H \otimes H \).

2.3. Corollary. If \( a = (a_1, a_2) \) is nonsingular on \( H \), then we have the following commuting relations:

\[
\begin{align*}
a_1 (a_1 a_2^* + a_2 a_1^*)^{-1} a_1 + a_2 (a_1 a_2^* + a_2 a_1^*)^{-1} a_2 &= 1, \\
a_2 (a_1 a_2^* + a_2 a_1^*)^{-1} a_1 + a_1 (a_1 a_2^* + a_2 a_1^*)^{-1} a_2 &= 1, \\
a_2 (a_1 a_2^* + a_2 a_1^*)^{-1} a_1 - a_1 (a_1 a_2^* + a_2 a_1^*)^{-1} a_2 &= 0.
\end{align*}
\]

Formulas (2.4) can be obtained by using the fact that (2.2) provides also a right inverse for \( a \).

3. In this section we shall give an application of Theorem 1.1. If \( H_1, H_2 \) are Hilbert spaces, then we denote by \( H \) the tensor product of \( H_1 \) and \( H_2 \), complete for the canonical norm.

3.1. Theorem. Let \( H \) be a Hilbert space, \( a \in B(H) \), \( \delta_1 = a_1 \otimes 1, \delta_2 = 1 \otimes a_2 \), and \( \delta = (\delta_1, \delta_2) \in H \). Then we
have

\[ \sigma(\tilde{a}, H) = \sigma(a_1, H_1) \times \sigma(a_2, H_2). \]

Proof. It is known that \( \sigma(\tilde{a}, H) = \sigma(a_2, H_2) \) (see, for example, [2]); hence

\[ \sigma(\tilde{a}, H) = \sigma(a_1, H_1) \times \sigma(a_2, H_2). \]

In order to prove the reverse inclusion, let us introduce some notations. Namely, for any \( b \in B(H) \) let \( \pi(b) \) be the approximate point spectrum, \( \gamma(b) = \sigma(b, H) \setminus \pi(b) \), and \( \sigma_p(b) \) the point spectrum of \( b \).

It will be sufficient to show that if \( \tilde{a} \) is nonsingular on \( H \), then \( (0, 0) \notin \sigma(a_1, H_1) \times \sigma(a_2, H_2) \). We shall show that \( (0, 0) \notin \sigma(a_1, H_1) \times \sigma(a_2, H_2) \) with \( \tilde{a} \) nonsingular leads to a contradiction. We have to eliminate certain possibilities.

(1) \( 0 \in \pi(a_1) \cap \pi(a_2) \). We then have two sequences \( a_n \in H_1, y_n \in H_1, \|y_n\| = 1, a_n \to 0, y_n \to 0 \) as \( n \to \infty \). Notice that

\[ \beta(\tilde{a}) \begin{bmatrix} x_n \otimes y_n \\ 0 \end{bmatrix} = \begin{bmatrix} a_n \otimes 1 \\ 1 \otimes a_n \end{bmatrix} \begin{bmatrix} x_n \otimes y_n \\ 0 \end{bmatrix} \to 0 \quad (n \to \infty), \]

while \( \|x_n \otimes y_n\| = 1 \), which is not possible since \( \tilde{a} \) is nonsingular.

(2) \( 0 \in \gamma(a_1) \cap \gamma(a_2) \). As \( \gamma(a_1) \cup \gamma(a_2) \in \sigma(a_1) \cap \sigma(a_2) \) and \( \sigma(a_1, H_1) = \sigma(a_2, H_2) \), if \( 0 \in \sigma(a_1) \cap \sigma(a_2) \), we proceed as above \( \tilde{a}^* \) being also nonsingular.

If \( 0 \in \sigma(a_1) \cap \sigma(a_2) \), then \( 0 \in \sigma(a_1) \cap \sigma(a_2) \); hence \( \tilde{a}^* \in \sigma(a_1) \cap \sigma(a_2) \), and therefore

\[ \begin{bmatrix} a_n \otimes 1 \\ 1 \otimes a_n \end{bmatrix} \begin{bmatrix} x_n \otimes y_n \\ 0 \end{bmatrix} = 0 \]

and \( 0 \otimes (a^* \otimes y) \neq 0 \), which is not possible since \( a(\tilde{a}) \) is invertible.

The case \( 0 \in \pi(a_1) \cap \gamma(a_2) \) is similar.

(3) \( 0 \in \gamma(a_1) \cap \gamma(a_2) \). Then we have \( 0 \in \pi(a_1) \cap \pi(a_2) \) and it is easy to construct a vector \( \xi \in H \oplus H \), \( \xi \neq 0 \), such that \( \beta(\tilde{a}) \xi = 0 \), which is again a contradiction.

In this way we have eliminated all possibilities and the proof is complete.

As a consequence of Theorem 3.1 we obtain the well-known result of Brown and Pearcy [1]:

3.2. Corollary. With the notations of Theorem 3.1 we have

\[ \sigma(a_1 \otimes a_2, H) = \sigma(a_1, H_1) \times \sigma(a_2, H_2). \]

Proof. It is sufficient to apply the spectral mapping theorem ([5] Th. 4.8) to the commuting pair \( \tilde{a} = (a_1, a_2) \) and to the function \( f : (a_1, a_2) = a_2 \).

Analogously, if \( f(a_1, a_2) \) is any analytic function in a neighbourhood of \( \sigma(a_1, H_1) \times \sigma(a_2, H_2) \), then \( f(\tilde{a}, a_2) \) makes sense and we have

\[ \sigma(f(\tilde{a}, a_2), H) = f(\sigma(a_1, H_1), \sigma(a_2, H_2)), \]

by applying again the spectral mapping theorem from [5] (see also [3] for similar topics).

Theorem 3.1 is a partial answer to a problem raised by D. Voiculescu within the seminar of Operator Theory, Institute of Mathematics, Bucharest, 1976.

It is plausible that Theorem 3.1 can be extended to the case of \( n \gg 2 \) arbitrary, with a similar proof, by using the characterization of the nonsingularity of a finite commuting system given in [6].

References


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