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# On pairs of commuting operators

by

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**Abstract.** The aim of this paper is to give a characterization of the joint spectrum, in the sense of J. L. Taylor of a pair of commuting operators on Hilbert spaces. An application to tensor products of operators in then presented.

**1.** In a previous paper we characterized the joint spectrum of a finite commuting system of linear continuous operators by means of the non-invertibility of a certain operator, acting on a direct sum of copies of the initial space [6]. In the sequel we intend to obtain a similar result for pairs of commuting operators, which is not a direct consequence of the abovementioned characterization. The present statement is more specific and it may be used in some problems of spectral theory, as we shall exemplify at the end of this paper.

Let $H$ be a complex Hilbert space and $B(H)$ the algebra of all linear continuous operators on $H$. Let $a = (a_1, a_2) \subset B(H)$ be a pair of commuting operators.

Consider the sequence

$$0 \to H \xrightarrow{\delta_a^0} H \oplus H \xrightarrow{\delta_a^1} H \to 0, \tag{1.1}$$

where $\delta_a^0(x) = a_1 x \oplus a_2 x$ $(x \in H)$ and $\delta_a^1(x_1 \oplus x_2) = a_1 x_2 - a_2 x_1$ $(x_1, x_2 \in H)$. It is clear that $\delta_a^1 \cdot \delta_a^0 = 0$. We recall that $a$ is said to be *nonsingular* [4] if the sequence (1.1) is exact. The (*joint*) *spectrum* $\sigma(a, H)$ of $a$ on $H$ is, by definition, the complement of the set of all $z = (z_1, z_2) \in \boldsymbol{C}^2$ such that $z - a = (z_1 - a_1, z_2 - a_2)$ is nonsingular on $H$.

The main result of this paper is the following

1.1. THEOREM. *Let $a = (a_1, a_2) \in B(H)$ be a commuting pair. Then $a$ is nonsingular on $H$ if ond only if the operator*

$$\alpha(a) = \begin{bmatrix} a_1 & a_2 \\ -a_2^* & a_1^* \end{bmatrix} \tag{1.2}$$

*is invertible on $H \oplus H$.*

This theorem can be used to solve some special problems concerning the tensor product of two linear operators; in particular, one can obtain a new proof of the result of Brown and Pearcy concerning the spectrum of the tensor product of the operators [1].

**2.** In this section we give the proof of Theorem 1.1.

Suppose first that $a = (a_1, a_2) \subset B(H)$ is nonsingular on $H$. Consider the dual sequence of (1.1), namely

$$0 \to H \xrightarrow{\delta_a^{1*}} H \oplus H \xrightarrow{\delta_a^{0*}} H \to 0, \tag{2.1}$$

where $\delta_a^{1*}(x) = -a_2^* x \oplus a_1^* x$ $(x \in H)$ and $\delta_a^{0*}(x_1 \oplus x_2) = a_1^* x_1 + a_2^* x_2$ $(x_1, x_2 \in H)$. It is easy to see that the pair $a^* = (a_1^*, a_2^*)$ is nonsingular on $H$ if and only if the sequence (2.1) is exact, and thus if and only if the pair $a = (a_1, a_2)$ is nonsingular on $H$.

2.1. Lemma. *If $a = (a_1, a_2)$ is nonsingular on $H$, then both $a_1 a_1^* + a_2 a_2^*$ and $a_1^* a_1 + a_2^* a_2$ are invertible on $H$.*

Indeed, let us show that $a_1 a_1^* + a_2 a_2^*$ is injective and surjective on $H$. If $(a_1 a_1^* + a_2 a_2^*)x = 0$ for a certain $x \in H$, then $-a_2^* x \oplus a_1^* x \in \ker \delta_a^{1*} = (\operatorname{im} \delta_a^1)^{\perp}$. But $-a_2^* x \oplus a_1^* x \in \operatorname{im} \delta_a^{1*}$; hence $a_1^* x = a_2^* x = 0$. Since $\ker \delta_a^{1*} = 0$, we have $x = 0$. Take an arbitrary $y \in H$ and let us find an $x \in H$ such that $y = a_1 a_1^* x + a_2 a_2^* x$. We infer that $\delta_a^1\colon (\ker \delta_a^1)^{\perp} \to H$ is an isomorphism, and therefore $y = \delta_a^1(y_1 \oplus y_2)$ with $y_1 \oplus y_2 \in (\ker \delta_a^1)^{\perp} = \operatorname{im} \delta_a^{1*}$; hence $y_1 \oplus y_2 = -a_2^* x \oplus a_1^* x$.

Analogously, the operator $a_1^* a_1 + a_2^* a_2$ is invertible and this completes the proof of the lemma.

Let us return to the proof of Theorem 1.1. According to Lemma 2.1, it is clear that the operator

$$\begin{bmatrix} a_1^*(a_1 a_1^* + a_2 a_2^*)^{-1} & -a_2(a_1^* a_1 + a_2^* a_2)^{-1} \\ a_2^*(a_1 a_1^* + a_2 a_2^*)^{-1} & a_1(a_1^* a_1 + a_2^* a_2)^{-1} \end{bmatrix} \tag{2.2}$$

is a left inverse for the operator $\alpha(a)$ given by (1.2); hence $\alpha(a)$ is surjective on $H \oplus H$. Let us also notice that $\alpha(a)$ is injective too. Indeed, if $\alpha(a)(x_1 \oplus x_2) = 0$, then $x_2 \oplus (-x_1) \in \ker \delta_a^1 \cap \ker \delta_a^{0*} = \ker \delta_a^1 \cap (\ker \delta_a^1)^{\perp} = \{0\}$, and hence $x_1 = x_2 = 0$.

Conversely, suppose that $\alpha(a)$ is invertible on $H \oplus H$. Then $\alpha(a)^*$ is invertible; therefore

$$\alpha(a)\alpha(a)^* = \begin{bmatrix} a_1 a_1^* + a_2 a_2^* & 0 \\ 0 & a_1^* a_1 + a_2^* a_2 \end{bmatrix}$$

is invertible, and hence $(a_1 a_1^* + a_2 a_2^*)^{-1}$ and $(a_1^* a_1 + a_2^* a_2)^{-1}$ are operators from $B(H)$.



Let us prove that the sequence (1.1) is exact. Indeed, if $\delta_a^0(x) = a_1 x \oplus a_2 x = 0$, then $(a_1^* a_1 + a_2^* a_2)x = 0$, whence $x = 0$.

Assume now that $\delta_a^1(x_1 \oplus x_2) = a_1 x_2 - a_2 x_1 = 0$. If $y = a_2^* x_2 + a_1^* x_1$, then $\alpha(a)\big(x_2 \oplus (-x_1)\big) = 0 \oplus (-y)$; hence $x_2 \oplus (-x_1) = \alpha(a)^{-1}\big(0 \oplus (-y)\big)$, and thus on account of (2.2) we obtain

$$x_1 = a_1(a_1^* a_1 + a_2^* a_2)^{-1} y,$$

$$x_2 = a_2(a_1^* a_1 + a_2^* a_2)^{-1} y,$$

i.e. the exactness of (1.1) at the second step.

Finally, if $y \in H$ is arbitrary, then $x_j = a_j^*(a_1 a_1^* + a_2 a_2^*)^{-1} y$ $(j = 1, 2)$ satisfy the equation $a_1 x_1 + a_2 x_2 = y$, and the proof of Theorem 1.1 is complete.

2.2. Corollary. *If $a = (a_1, a_2) \subset B(H)$ is a commuting pair, then the spectrum $\sigma(a, H)$ of $a$ on $H$ is given by the set*

$$\boldsymbol{C}^2 \setminus \{z \in \boldsymbol{C}^2;\ \big(\alpha(z) - \alpha(a)\big)^{-1} \in B(H \oplus H)\}.$$

As is known, the set $\sigma(a, H)$ is compact and nonempty [4] (see also [6] for Hilbert spaces).

Let us remark that the set of matrices $\{\alpha(z);\ z \in \boldsymbol{C}^2\}$ can be identified with the algebra of quaternions and that the map $z \to \alpha(z)$ is an $\boldsymbol{R}$-linear isometric isomorphism [6].

Notice also that $a = (a_1, a_2) \subset B(H)$ is nonsingular if and only if the matrix

$$\beta(a) = \begin{bmatrix} a_1 & -a_2^* \\ a_2 & a_1^* \end{bmatrix} = \alpha(a^*)^* \tag{2.3}$$

is invertible on $H \oplus H$.

2.3. Corollary. *If $a = (a_1, a_2)$ is nonsingular on $H$, then we have the following commuting relations:*

$$\begin{aligned} a_1^*(a_1 a_1^* + a_2 a_2^*)^{-1} a_1 + a_2(a_1^* a_1 + a_2^* a_2)^{-1} a_2^* &= 1, \\ a_2^*(a_1 a_1^* + a_2 a_2^*)^{-1} a_2 + a_1(a_1^* a_1 + a_2^* a_2)^{-1} a_1^* &= 1, \\ a_1^*(a_1 a_1^* + a_2 a_2^*)^{-1} a_2 - a_2(a_1^* a_1 + a_2^* a_2)^{-1} a_1^* &= 0. \end{aligned} \tag{2.4}$$

Formulas (2.4) can be obtained by using the fact that (2.2) provides also a right inverse for $\alpha(a)$.

**3.** In this section we shall give an application of Theorem 1.1. If $H_1, H_2$ are Hilbert spaces, then we denote by $H_1 \otimes H_2$ the tensor product of $H_1$ and $H_2$, complete for the canonical norm.

3.1. Theorem. *Let $H_j$ $(j = 1, 2)$ be Hilbert spaces, $a_j \in B(H_j)$, $H = H_1 \otimes H_2$, $\tilde{a}_1 = a_1 \otimes 1$, $\tilde{a}_2 = 1 \otimes a_2$ and $\tilde{a} = (\tilde{a}_1, \tilde{a}_2) \subset B(H)$. Then we* *have*

$$\sigma(\tilde{a}, H) = \sigma(a_1, H_1) \times \sigma(a_2, H_2). \tag{3.1}$$

Proof. It is known that $\sigma(\tilde{a}_j, H) = \sigma(a_j, H_j)$ (see, for example, [2]); hence

$$\sigma(\tilde{a}, H) \subset \sigma(\tilde{a}_1, H) \times \sigma(\tilde{a}_2, H) = \sigma(a_1, H_1) \times \sigma(a_2, H_2).$$

In order to prove the reverse inclusion, let us introduce some notations. Namely, for any $b \in B(H)$ let $\pi(b)$ be the approximate point spectrum, $\gamma(b) = \sigma(b, H) \setminus \pi(b)$, and $\sigma_p(b)$ the point spectrum of $b$.

It will be sufficient to show that if $\tilde{a}$ is nonsingular on $H$, then $(0, 0) \notin \sigma(a_1, H_1) \times \sigma(a_2, H_2)$. We shall show that $(0, 0) \in \sigma(a_1, H_1) \times \sigma(a_2, H_2)$ with $\tilde{a}$ nonsingular leads to a contradiction. We have to eliminate certain possibilities.

(1) $0 \in \pi(a_1) \cap \pi(a_2)$. We then have two sequences $x_n \in H_1$, $y_n \in H_2$, $\|x_n\| = \|y_n\| = 1$, $a_1 x_n \to 0$, $a_2 y_n \to 0$ $(n \to \infty)$. Notice that

$$\beta(\tilde{a}) \begin{bmatrix} x_n \otimes y_n \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \otimes 1 & -1 \otimes a_2^* \\ 1 \otimes a_2 & a_1^* \otimes 1 \end{bmatrix} \begin{bmatrix} x_n \otimes y_n \\ 0 \end{bmatrix} \to 0 \qquad (n \to \infty),$$

while $\|(x_n \otimes y_n) \oplus 0\| = \|x_n \otimes y_n\| = 1$, which is not possible since $\tilde{a}$ is nonsingular.

(2) $0 \in \gamma(a_1) \cap \pi(a_2)$. As $\gamma(a_1) \subset \sigma_p(a_1^*) \subset \pi(a_1^*)$ and $\sigma(a_2, H_2) = \sigma(a_2^*, H_2)$, if $0 \in \pi(a_2^*)$ then $0 \in \pi(a_1^*) \cap \pi(a_2^*)$ and we proceed as above $\tilde{a}^*$ being also nonsingular.

If $0 \in \gamma(a_2^*) \subset \sigma_p(a_2^{**}) = \sigma_p(a_2)$, then $0 \in \sigma_p(a_1^*) \cap \sigma_p(a_2)$; hence $a_1^* x^* = 0$, $a_2 y = 0$ with $x^* \neq 0$, $y \neq 0$, and therefore

$$\begin{bmatrix} a_1 \otimes 1 & 1 \otimes a_2 \\ -1 \otimes a_2^* & a_1^* \otimes 1 \end{bmatrix} \begin{bmatrix} 0 \\ x^* \otimes y \end{bmatrix} = 0$$

and $0 \oplus (x^* \otimes y) \neq 0$, which is not possible since $\alpha(\tilde{a})$ is invertible.

The case $0 \in \pi(a_1) \cap \gamma(a_2)$ is similar.

(3) $0 \in \gamma(a_1) \cap \gamma(a_2)$. Then we have $0 \in \sigma_p(a_1^*) \cap \sigma_p(a_2^*)$ and it is easy to construct a vector $\xi \in H \oplus H$, $\xi \neq 0$, such that $\beta(\tilde{a})\xi = 0$, which is again a contradiction.

In this way we have eliminated all possibilities and the proof is complete.

As a consequence of Theorem 3.1 we obtain the well-known result of Brown and Pearcy [1]:

3.2. Corollary. *With the notations of Theorem* 3.1 *we have*

$$\sigma(a_1 \otimes a_2, H) = \sigma(a_1, H_1) \times \sigma(a_2, H_2).$$



Proof. It is sufficient to apply the spectral mapping theorem ([5] Th. 4.8) to the commuting pair $\tilde{a} = (\tilde{a}_1, \tilde{a}_2)$ and to the function $f(z_1, z_2) = z_1 z_2$.

Analogously, if $f(z_1, z_2)$ is any analytic function in a neighbourhood of $\sigma(a_1, H_1) \times \sigma(a_2, H_2)$, then $f(\tilde{a}_1, \tilde{a}_2)$ makes sense and we have

$$\sigma\big(f(\tilde{a}_1, \tilde{a}_2), H\big) = f\big(\sigma(a_1, H_1), \sigma(a_1, H_2)\big), \tag{3.2}$$

by applying again the spectral mapping theorem from [5] (see also [3] for similar topics).

Theorem 3.1 is a partial answer to a problem raised by D. Voiculescu within the seminar of Operator Theory, Institute of Mathematics, Bucharest, 1976.

It is plausible that Theorem 3.1 can be extended to the case of $n$ operators ($n \geqslant 2$ arbitrary), with a similar proof, by using the characterization of the nonsingularity of a finite commuting system given in [6].

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