Bibliographie

[1] C. Bessaga and A. Pełczyński, *Spaces of continuous functions*, Studia Math. 19 (1960), pp. 53–62.
[2] J. Lindenstrauss and A. Pełczyński, *Contribution to the theory of classical Banach spaces*, J. Functional Analysis 8 (1971), pp. 225–249.
[3] A. Pełczyński, *On C(S) subspaces of separable Banach spaces*, Studia Math. 31 (1968), pp. 513–532.
[4] — *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. (Rozprawy Matematyczne) 58 (1968).
[5] D. Alspach and Y. Benyamini, *Primariness of spaces of continuous functions on ordinals*, Israel J. Math. 27 (1977).

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The Marcinkiewicz interpolation theorem extends to weighted spaces

by

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Abstract. It is shown that under the hypotheses of the Marcinkiewicz interpolation theorem the strong type result extends to an estimate involving certain weight functions. These functions depend on the weak type parameters of the operator.

The Marcinkiewicz theorem concerning the interpolation of operations states that if T is a sublinear operator of weak type (p_i, q_i), $1 \leqslant p_i \leqslant q_i \leqslant \infty$ $i = 0, 1$; $p_0 < p_1$, $q_0 \neq q_1$, then

$$\|Tf\|_q \leqslant A\,\|f\|_p,$$

where $1/p = \theta/p_0 + (1-\theta)/p_1$, $1/q = \theta/q_0 + (1-\theta)/q_1$, $0 < \theta < 1$, and A is a constant independent of f.

The purpose of this note is to extend this result to certain weighted norms. The proof is seen to be a consequence of a generalization of Hardy's inequality (Lemma 1) and an inequality of Calderón (Lemma 2).

For notation and additional information we refer to [4], Chapter V. The constant A appearing throughout the paper will depend on the parameters only, but may be different at different appearances.

LEMMA 1. *If w is a non-negative, non-increasing function defined on $(0, \infty)$ and $g(x) \geqslant 0$, then for $0 < r < \infty$ and $p \geqslant 1$*

$$\Big\{\int_0^\infty w(x)x^{-r-1}\Big(\int_0^x g(y)\,dy\Big)^p dx\Big\}^{1/p} \leqslant p/r\Big\{\int_0^\infty w(x)x^{-r-1}[xg(x)]^p\,dx\Big\}^{1/p}.$$

Proof. The obvious changes of variables and Minkowski's integral inequality yields

$$\Big\{\int_0^\infty w(x)x^{-r-1}\Big(\int_0^x g(y)\,dy\Big)^p dx\Big\}^{1/p} = \Big\{\int_0^\infty w(x)x^{-r-1}\Big(\int_0^1 g(xy)\,dy\Big)^p x^p\,dx\Big\}^{1/p}$$

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$$\leqslant \int_0^1 \Big\{\int_0^\infty w(x)x^{-r-1}g(xy)^p x^p\,dx\Big\}^{1/p} dy$$
$$= \int_0^1 \Big\{\int_0^\infty w(x/y)\,y^{r-p}x^{-r+p-1}g(x)^p\,dx\Big\}^{1/p} dy$$
$$\leqslant \int_0^1 y^{r/p-1}dy\Big\{\int_0^\infty w(x)x^{-r-1}[xg(x)]^p\,dx\Big\}^{1/p},$$

from which the result follows.

A dual inequality of this type is proved in a similar way, only then w must be taken to be non-decreasing. However, we shall not need it in the sequel.

LEMMA 2. *If g is a non-negative, non-increasing function defined on $(0,\infty)$ and a real, then for $0 < p \leqslant q$*

$$\Big\{\int_0^\infty [x^a g(x)]^q x^{-1}dx\Big\}^{1/q} \leqslant A\Big\{\int_0^\infty [x^a g(x)]^p x^{-1}dx\Big\}^{1/p}.$$

Proof. Since

$$\Big\{\int_0^\infty [x^a g(x)]^q x^{-1}dx\Big\}^{p/q} \leqslant \Big\{\sum_{k=-\infty}^{\infty} g(2^{k-1})^q \int_{2^{k-1}}^{2^k} x^{aq-1}dx\Big\}^{p/q}$$
$$\leqslant A\Big\{\sum_{k=-\infty}^{\infty} [g(2^{k-1})2^{ak}]^q\Big\}^{p/q}$$
$$\leqslant A\sum_{k=-\infty}^{\infty} [g(2^{k-1})2^{ak}]^p$$
$$\leqslant A\sum_{k=-\infty}^{\infty}\int_{2^{k-2}}^{2^{k-1}} [x^a g(x)]^p x^{-1}dx$$
$$\leqslant A\int_0^\infty [x^a g(x)]^p x^{-1}dx,$$

the result follows.

The weight functions we consider in the next theorem satisfy the following: Let w be a non-negative function defined on $(0,\infty)$. $w \in W^0(a)$, $a > 0$, if w is non-increasing and satisfies the estimate

$$\int_0^x t^{a-1}w(t)\,dt \leqslant Ax^a w(x).$$

Also, $w \in W_0(a)$, $a < 0$, if w is non-decreasing and

$$\int_x^\infty t^{a-1}w(t)\,dt \leqslant Ax^a w(x).$$

THEOREM. *Suppose T is a sublinear operator of weak type (p_i, q_i), $1 \leqslant p_i \leqslant q_i \leqslant \infty$, $i = 0, 1$; $p_0 < p_1$, $q_0 \neq q_1$. If σ is the slope of the line segment from $(1/p_0, 1/q_0)$ to $(1/p_1, 1/q_1)$ and*

$$1/q = \theta/q_0 + (1-\theta)/q_1, \quad 1/p = \theta/p_0 + (1-\theta)/p_1, \quad 0 < \theta < 1,$$

then

$$\Big\{\int_0^\infty [w(t)(Tf)^*(t)]^q dt\Big\}^{1/q} \leqslant A\Big\{\int_0^\infty [w(x^{1/\sigma})f^*(x)]^p dx\Big\}^{1/p},$$

for all $w \in W^0(1/q - 1/q_1)$ if $q_0 < q_1$ and all $w \in W_0(1/q - 1/q_1)$ whenever $q_1 < q_0$.

Proof. Note that

$$\sigma = \frac{1/q_0 - 1/q}{1/p_0 - 1/p} = \frac{1/q - 1/q_1}{1/p - 1/p_1} = \frac{1/q_0 - 1/q_1}{1/p_0 - 1/p_1}$$

is positive if $q_0 < q_1$ and negative for $q_1 < q_0$.

If $t > 0$, define $f^t(x) = f(x)$ if $|f(x)| > f^*(t^\sigma)$ and $f^t(x) = 0$ otherwise, and $f_t(x) = f(x) - f^t(x)$. Then

$$(f^t)^*(x) \leqslant \begin{cases} f^*(x) & \text{if } 0 < x \leqslant t^\sigma, \\ 0 & \text{if } t^\sigma < x, \end{cases}$$

and

$$(f_t)^*(x) \leqslant \begin{cases} f^*(t^\sigma) & \text{if } x \leqslant t^\sigma, \\ f^*(x) & \text{if } t^\sigma < x. \end{cases}$$

Since T is sublinear and satisfies the weak type conditions, it follows that

$$(1)\quad (Tf)^*(t) \leqslant 2[(Tf^t)^*(t/2) + (Tf_t)^*(t/2)]$$
$$\leqslant A[t^{-1/q_0}\|f^t\|_{p_0} + t^{-1/q_1}\|f_t\|_{p_1}]$$
$$\leqslant A\Big[t^{-1/q_0}\Big(\int_0^{t^\sigma} f^*(y)^{p_0}dy\Big)^{1/p_0} +$$
$$+ t^{-1/q_1}\Big(\int_0^{t^\sigma} f^*(t^\sigma)^{p_1}dy + \int_{t^\sigma}^\infty f^*(y)^{p_1}dy\Big)^{1/p_1}\Big].$$

We now consider the case $q_0 < q_1$. By hypotheses, $1 \leqslant p \leqslant q$, so that Lemma 2 and Minkowski's inequality imply

$$\Big\{\int_0^\infty [w(t)(Tf)^*(t)]^q dt\Big\}^{1/q} \leqslant A\Big\{\int_0^\infty t^{p/q-1}[w(t)(Tf)^*(t)]^p dt\Big\}^{1/p}$$
$$\leqslant A\Big\{\Big[\int_0^\infty w(t)^p t^{p/q-p/q_0-1}\Big(\int_0^{t^\sigma} f^*(y)^{p_0}dy\Big)^{p/p_0}dt\Big]^{1/p} +$$
$$+ \Big[\int_0^\infty w(t)^p t^{p/q-p/q_1-1}\Big(\int_0^{t^\sigma} f^*(t^\sigma)^{p_1}dy + \int_{t^\sigma}^\infty f^*(y)^{p_1}dy\Big)^{p/p_1}dt\Big]^{1/p}\Big\}$$
$$= A[I_1 + I_2],$$

respectively. The change of variable $t\to x^{1/\sigma}$ and an application of Lemma 1 shows that

$$I_1^p = \frac{1}{\sigma}\int\limits_0^\infty w(x^{1/\sigma})^p x^{-p/p_0}\Big(\int\limits_0^x f^*(y)^{p_0}dy\Big)^{p/p_0}dt$$
$$\leqslant \frac{1}{\sigma}\Big(\frac{p}{p-p_0}\Big)^{p/p_0}\int\limits_0^\infty [w(x^{1/\sigma})f^*(x)]^p dx.$$

The same variable change yields

$$I_2^p \leqslant \int\limits_0^\infty [w(t)f^*(t^\sigma)]^p t^{p/q-p/q_1+\sigma p/p_1-1}dt +$$
$$+ \int\limits_0^\infty w(t)^p t^{p/q-p/q_1-1}\Big(\int\limits_{t^\sigma}^\infty f^*(y)^{p_1}dy\Big)^{p/p_1}dt$$
$$= \frac{1}{\sigma}\int\limits_0^\infty [w(x^{1/\sigma})f^*(x)]^p dx + \frac{1}{\sigma}\int\limits_0^\infty w(x^{1/\sigma})^p x^{-p/p_1}\Big(\int\limits_x^\infty f^*(y)^{p_1}dy\Big)^{p/p_1}dx$$

and the last integral can be written as

$$\int\limits_0^\infty w(x^{1/\sigma})^p x^{-p/p_1}\Big(\int\limits_0^\infty [y^{p/p_1}f^*(x+y)^p]^{p_1/p}y^{-1}dy\Big)^{p/p_1}dx$$
$$\leqslant A\int\limits_0^\infty w(x^{1/\sigma})^p x^{-p/p_1}\Big(\int\limits_0^x+\int\limits_x^\infty\Big)y^{p/p_1-1}f^*(x+y)^p\,dy\,dx$$
$$\leqslant A\int\limits_0^\infty [w(x^{1/\sigma})f^*(x)]^p dx + A\int\limits_0^\infty w(x^{1/\sigma})^p x^{-p/p_1}\Big(\int\limits_x^\infty y^{p/p_1-1}f^*(y)^p dy\Big)dx.$$

Here we applied Lemma 2 to the inner integral and used the fact that f^* is non-increasing. Interchanging the order of integration of the second integral, then applying Lemma 2 again, we have

$$\int\limits_0^\infty y^{p/p_1-1}f^*(y)^p\Big(\int\limits_0^y [w(x^{1/\sigma})x^{-1/p_1+1/p}]^p x^{-1}dx\Big)dy$$
$$\leqslant A\int\limits_0^\infty y^{p/p_1-1}f^*(y)^p\Big(\int\limits_0^y w(x^{1/\sigma})x^{-1/p_1+1/p-1}dx\Big)^p dy.$$

Now the change $x\to t^\sigma$ and the condition $w\in W^0(1/q-1/q_1)$ show that this integral is also dominated by

(2) $$\int\limits_0^\infty [w(y^{1/\sigma})f^*(y)]^p dy.$$

Next, we consider the case $q_1<q_0$. Now $w\in W_0(1/q-1/q_1)$ and $\sigma<0$. By (1) and Minkowski's inequality,

$$\Big(\int\limits_0^\infty [w(t)(Tf)^*(t)]^q dt\Big)^{1/q} \leqslant A\Big\{\Big[\int\limits_0^\infty w(t)^q t^{-q/q_0}\Big(\int\limits_0^{t^\sigma} f^*(y)^{p_0}dy\Big)^{q/p_0}dt\Big]^{1/q} +$$
$$+\Big[\int\limits_0^\infty w(t)^q t^{-q/q_1}\Big(\int\limits_0^{t^\sigma} f^*(t^\sigma)^{p_1}dy + \int\limits_{t^\sigma}^\infty f^*(y)^{p_1}dy\Big)^{q/p_1}dt\Big]^{1/q}\Big\}$$
$$\equiv A[J_1+J_2],$$

respectively. As in the estimate of I_1, the change $t\to x^{1/\sigma}$ and an application of Lemma 1 and Lemma 2 yields

$$J_1^q \leqslant -\frac{1}{\sigma}\Big(\frac{p}{p-p_0}\Big)^{q/p_0}\int\limits_0^\infty [w(x^{1/\sigma})f^*(x)]^q x^{q/p-1}dx$$
$$\leqslant A\Big(\int\limits_0^\infty [w(x^{1/\sigma})f^*(x)]^p dx\Big)^{q/p}.$$

Observe that now $p_1<q$, hence we may apply Minkowski's inequality to $J_2^{p_1}$ to get

$$J_2^{p_1} \leqslant \Big\{\int\limits_0^\infty [w(t)f^*(t^\sigma)]^q t^{-q/q_1+\sigma q/p_1}dt\Big\}^{p_1/q} +$$
$$+\Big\{\int\limits_0^\infty w(t)^q t^{-q/q_1}\Big(\int\limits_{t^\sigma}^\infty f^*(y)^{p_1}dy\Big)^{q/p_1}dt\Big\}^{p_1/q}.$$

The change $t\to x^{1/\sigma}$ and the use of Lemma 2 show that the first integral is dominated by (2). The second integral is by Lemma 2 domainated by

$$A\Big\{\int\limits_0^\infty w(x^{1/\sigma})^p x^{-p/p_1}\Big(\int\limits_x^\infty f^*(y)^{p_1}dy\Big)^{p/p_1}dx\Big\}^{p_1/p}$$
$$= A\Big\{\int\limits_0^\infty w(x^{1/\sigma})^p x^{-p/p_1}\Big(\int\limits_0^\infty [y^{1/p_1}f^*(x+y)]^{p_1}y^{-1}dy\Big)^{p/p_1}dx\Big\}^{p_1/p}$$
$$\leqslant A\Big\{\int\limits_0^\infty w(x^{1/\sigma})^p x^{-p/p_1}\Big[\Big(\int\limits_0^x+\int\limits_x^\infty\Big)y^{p/p_1-1}f^*(x+y)^p dy\Big]dx\Big\}^{p_1/p}$$
$$\leqslant A\Big\{\int\limits_0^\infty [w(x^{1/\sigma})f^*(x)]^p dx + \int\limits_0^\infty w(x^{1/\sigma})^p x^{-p/p_1}\Big(\int\limits_x^\infty y^{p/p_1-1}f^*(y)^p dy\Big)dx\Big\}^{p_1/p}.$$

The interchange of order of integration, the use of Lemma 2 and the fact that $w\in W_0(1/q-1/q_1)$ show as before that the last integral is also dominated by (2).

This concludes the proof for finite parameters.

If $q_0 = \infty$ and $q_1 < \infty$ or $q_1 = \infty$ and $q_0 < \infty$, the result follows in the same way by taking $1/q_0 = 0$, respectively $1/q_1 = 0$.

If $p_1 = \infty$, it follows directly that I_2 is dominated by (2) and

$$J_2 \leqslant A\Big(\int_0^\infty [w(y^{1/\sigma})f^*(y)]^q y^{q/p-1}\,dy\Big)^{1/q} \leqslant A\Big(\int_0^\infty [w(y^{1/\sigma})f^*(y)]^p\,dy\Big)^{1/p}$$

by Lemma 2.

An examination of the proof shows that the result extends to $0 < p_i \leqslant q_i \leqslant \infty$, $i = 0, 1$, provided that for $0 < p < 1$, $w \in W^0(p/q - p/q_1)$ if $q_0 < q_1$ and $w \in W_0(p/q - p/q_1)$ whenever $q_1 < q_0$.

We single out a specific weight in the following

COROLLARY. (a) *If* $q_0 < q_1$ *and* $0 < \beta < (1/q - 1/q_1)/(1/q_0 - 1/q)$, *then*

$$\Big(\int_0^\infty x^{-\beta(q/q_0-1)}(Tf)^*(x)^q\,dx\Big)^{1/q} \leqslant A\Big(\int_0^\infty x^{-\beta(p/p_0-1)}f^*(x)^p\,dx\Big)^{1/p}.$$

(b) *If* $q_1 < q_0$ *and* $(1/q - 1/q_1)/(1/q - 1/q_0) < \beta < 0$, *then*

$$\Big(\int_0^\infty x^{\beta(q/q_0-1)}(Tf)^*(x)^q\,dx\Big)^{1/q} \leqslant A\Big(\int_0^\infty x^{\beta(p/p_0-1)}f^*(x)^p\,dx\Big)^{1/p}.$$

EXAMPLE. Let T be defined by $Tf = \hat f$, where $\hat f$ is the is Fourier transform of f. Since T is of type $(1, \infty)$ and (2.2) we obtain for $w \in W_0(1/2 - 1/p)$, $1 < p < 2$,

$$\Big\{\int_0^\infty [w(x)\hat f^*(x)]^q\,dx\Big\}^{1/q} \leqslant A\Big\{\int_0^\infty [w(1/x)f^*(x)]^p\,dx\Big\}^{1/p},$$

$1/q = 1 - 1/p$, an extension of the Hausdorff–Young inequality.

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References

[1] E. M. Stein, *Interpolation of linear operators*, Trans. Amer. Math. Soc. 83 (1956), pp. 482–492.

[2] — *Singular integrals and differentiability properties of functions*, Princeton 1970.

[3] E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc. 87 (1958), pp. 159–172.

[4] —, — *Introduction to Fourier analysis on Euclidean spaces*, Princeton 1971.

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Suites concordantes d'espaces normés et leurs applications I

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Résumé. Nous exposons dans ce travail une méthode pour l'"approximation" des éléments d'un espace de Banach (ou de Hilbert) donné par des suites concordantes d'espaces normés (dans les applications pratiques ces espaces ont un nombre fini de dimensions). Nous indiquons aussi une réalisation numérique de ces considérations dans les problèmes variationnels aux dérivées partielles qui se présentent dans la théorie de l'élasticité ([3]).

1. Construction d'un espace de suites concordantes. Soit une suite d'espaces normés $(X_n, \|\cdot\|_n)$ $(n = 1, 2, \ldots)$ et supposons données les applications linéaires continues $\varphi_{n,m}\colon X_n \to X_m$ $(m \geqslant n = 1, 2, \ldots)$.

Par $\boldsymbol{P}(X_n)$ nous désignerons l'ensemble de toutes les suites $\{x_n\}$ telles que $x_n \in X_n$ $(n = 1, 2, \ldots)$ et satisfaisant à la condition

$$(1.1)\qquad p(\{x_n\}) \overset{\text{df}}{=} \overline{\lim}\, \|x_n\|_n < \infty.$$

Nous admettons les définitions suivantes:

DÉFINITION (1.1). Les éléments de l'ensemble

$$(1.2)\qquad C = [\{x_n\} \in \boldsymbol{P}(X_n) \bigwedge_{\varepsilon>0} \bigvee_{n_\varepsilon} \bigwedge_{m \geqslant n \geqslant n_\varepsilon} \|x_m - \varphi_{n,m}x_n\|_m \leqslant \varepsilon].$$

seront appelés *suites concordantes*.

DÉFINITION (1.2). Les éléments de l'ensemble

$$(1.3)\qquad C_s = [\{x_n\} \in \boldsymbol{P}(X_n)\colon \bigvee_{n_0} \bigvee_{x_{n_0} \in X_{n_0}} x_n = \varphi_{n_0,n}x_{n_0} \text{ pour } n \geqslant n_0]$$

seront appelés *suites presque constantes*.

Remarque. Nous identifierons les suites qui ne diffèrent que par un nombre fini de coordonnées.

LEMME (1.1). *Si les applications* $\varphi_{n,m}$ *satisfont à l'hypothèse*

$$\bigwedge_{n_0} \bigwedge_{x_{n_0} \in X_{n_0}} \sup_{n \geqslant n_0} \|\varphi_{n_0,n}x_{n_0}\|_n < \infty$$