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Harmonic analysis on the group of rigid motions of the Euclidean plane*

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Abstract. Aspects of Fourier analysis on $M(2)$ relevant to the study of L^p multiplier operators are developed. Relations between multiplier operators on $M(2)$ and $SO(3)$ or $SU(2)$ are studied. Applications are given to transplantation results for Bessel functions.

Introduction. The idea of considering the real line to be a limit of circles with increasingly large radii has long been used to relate Fourier analysis on the line, R, to Fourier analysis on the torus, T. In the study of multiplier operators, this idea leads to the following classical theorem: Let m be a continuous function on R. Suppose that for each $\lambda > 0$, there exists an operator M_λ acting continuously on $L^p(T)$, given by

$$M_\lambda f(x) = \sum_{n=-\infty}^{\infty} m\left(\frac{n}{\lambda}\right) a_n e^{inx},$$

where a_n is the nth Fourier coefficient of f. Assume that the operator norms $\|M_\lambda\|$ are uniformly bounded. Then m defines a bounded multiplier operator M on $L^p(R)$ ([3], p. 264).

We wish to generalize this result by replacing the torus, which may be identified with $SO(2)$, with the non-abelian group $SO(3)$, or with its universal covering group $SU(2)$, which is naturally identifiable with the unit sphere in two-dimensional complex space. By a limiting process analagous to the classical passage from the circle to the line, the group $SO(3)$ can be shown to tend to a non-compact non-abelian group: the group of rigid motions of the Euclidean plane, denoted by $M(2)$.

In this paper, we shall show how Fourier analysis on $M(2)$ is closely

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related to Fourier analysis on the plane, R^2, and to properties of the Hankel transform on $(0, \infty)$. We then establish an analogue of the classical multiplier theorem presented above, relating multipliers on $SO(3)$ (or $SU(2)$) to those on $M(2)$.

As an application of this work, we prove a transplantation result for Bessel functions.

The multiplier theorem we study in this paper may be compared to a theorem of Bonami and Clerc, [6], in which they relate zonal multipliers on the unit ball of R^{n+1}, Σ_n, to radial multipliers on R^n. By replacing R^3 with $M(2)$ and Σ_3 with $SO(3)$, we obtain a correspondence between multipliers on $SO(3)$ and $M(2)$ which does not require either the radial or zonal restrictions.

§ 1. Fourier analysis on $M(2)$. Detailed discussions of the Lie group structure of $M(2)$ and of the associated representation theory may be found in Vilenkin [4] or Bingen [1]. A rigid motion of the plane, C, is a map (x, φ): $C \to C$ of the form $(x, \varphi)(z) = e^{i\varphi}z + x$ where $x \in C$, $\varphi \in T$. We shall consider φ as a real number defined modulo 2π. $M(2)$ is the set of these motions, together with the operation of composition of motions, which may be written $(x, \varphi)\cdot(y, \psi) = (e^{i\varphi}y + x, \varphi + \psi)$. Algebraically, $M(2)$ is the semi-direct product of R^2 with $SO(2)$.

We shall make the natural identification of $SO(2)$ with the one-dimensional torus, T. One may topologize $M(2)$ so as to make it homeomorphic to $R^2 \times T$. In fact, $M(2)$ may be considered to be a three-dimensional connected non-abelian non-compact Lie group. The (normalized) Haar measure on $M(2)$ is given by

$$\int_{M(2)} f(g)\,dg = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{R^2} f(x, \varphi)\,dx\,d\varphi.$$

Consider the representation, A, of $M(2)$ acting on functions on R^2 given by $[A(g)h](z) = h(g^{-1}z) = h\big(e^{-i\varphi}(z - x)\big)$, where $z \in R^2$, $g = (x, \varphi) \in M(2)$. We shall use A to introduce the Fourier transform of a function on $M(2)$. Let $z \in C$, $y = Re^{i\theta} \in C$ and $y \cdot z = Re(y\bar{z})$. We write (formally) the Fourier inversion formula using polar coordinates as

$$f(z) = \frac{1}{2\pi} \int_{R^2} \check{f}(y) e^{-iz\cdot y}\,dy = \frac{1}{2\pi} \int_0^{\infty} \int_{-\pi}^{\pi} \check{f}(Re^{i\theta}) e^{-iz\cdot Re^{i\theta}}\,d\theta\,R\,dR,$$

where

$$\check{f}(y) = \int_{R^2} f(w) e^{iy\cdot w}\,dw.$$

This polar form motivates us to introduce the following vector spaces of functions on C:

$$H_R = \Big\{h\colon h(z) = \int_{-\pi}^{\pi} g(\theta) e^{-iz\cdot Re^{i\theta}}\,d\theta,\ g \in L^2(T)\Big\}.$$

Defining $(h_1, h_2) \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} g_1(\theta)\overline{g_2(\theta)}\,d\theta$, we convert H_R into a Hilbert space; moreover, the map p: $H_R \to L^2(T)$ defined by $Ph = g$ where $h(z) = \int_{-\pi}^{\pi} g(\theta) e^{-z\cdot Re^{i\theta}}\,d\theta$ is an isometry of H_R onto $L^2(T)$.

It is easy to check that H_R is invariant under the action of A. First, observe that for $h \in H_R$ and $(x, \varphi) \in M(2)$,

$$[A(x, \varphi)](h) = \int_{-\pi}^{\pi} g(\theta) e^{-i(e^{-i\varphi}(z-x)\cdot Re^{i\theta})}\,d\theta$$
$$= \int_{-\pi}^{\pi} e^{-z\cdot Re^{i\theta}} \{g(\theta - \varphi) e^{ix\cdot Re^{i\theta}}\}\,d\theta.$$

The invariance of H_R follows from the fact that

$$[L^R(x, \varphi)g](\theta) = e^{ix\cdot Re^{i\theta}} g(\theta - \varphi)$$

defines a map L^R: $M(2) \to \mathrm{Aut}\big(L^2(T)\big)$ which is a unitary representation of $M(2)$. It is proved in Vilenkin [4] that L^R is irreducible for $R \neq 0$. Moreover, since $PAP^{-1} = L^R$, the restriction of A to H_R and L^R are equivalent representations. Clearly, H_R is also invariant under the adjoint of A; $A^* = P^{-1}(L^R)^*P$.

We are now in a position to define the Fourier transform of a function $f \in L^1\big(M(2)\big)$. Consider the integral operator F_f defined on H_R by

$$(F_f h)(z) = \frac{1}{2\pi} \int_{R^2} \int_{-\pi}^{\pi} f(x, \varphi)[A^*(x, \varphi)h](z)\,d\varphi\,dx$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\cdot Re^{i\theta}} \Big\{\int_{R^2} \int_{-\pi}^{\pi} f(x, \varphi)[L^{R^*}(x, \varphi)Ph](\theta)\,d\varphi\,dx\Big\}\,d\theta.$$

Since $L^R(x, \varphi)$ is a unitary operator on $L^2(T)$,

$$\Big\| \int_{R^2} \int_{-\pi}^{\pi} f(x, \varphi)[L^{R^*}(x, \varphi)Ph](\theta)\,d\varphi\,dx \Big\|_{2,T} \leqslant \|f\|_{1,M(2)} \|Ph\|_{2,T}.$$

Consequently, F_f maps H_R into itself.

Let $T_f(R)$ denote the operator defined on $L^2(T)$ by

$$[T_f(R)g](\theta) = \frac{1}{2\pi}\int_{R^2}\int_{-\pi}^{\pi} f(x,\varphi)[L^{R^*}(x,\varphi)g](\theta)\,d\varphi\,dx.$$

The argument just presented shows that $T_f(R)$ maps $L^2(T)$ into itself. Moreover, it is clear that $PF_f = T_f(R)P$. We shall call $T_f(R)$ the *Fourier transform of f evaluated at* $R>0$.

It is not hard to express $T_f(R)$ in terms of the Fourier transform associated with R^2. In fact, if

$$\hat f(y,\varphi) = \frac{1}{2\pi}\int_{R^2} f(x,\varphi)e^{-iy\cdot x}\,dx \quad \text{for} \quad y\in R^2,\ \varphi\in[-\pi,\pi),$$

then

$$[T_f(R)g](\varphi) = \int_{-\pi}^{\pi}\hat f(Re^{i(\varphi+\theta)},\theta)g(\varphi+\theta)\,d\theta.$$

The matricial Fourier transform of f evaluated at R is the (infinite) matrix of $T_f(R)$ with respect to the basis $\{e^{in\theta}\}$ of $L^2(T)$; the j,k entry of this matrix is:

$$(1)\qquad T_f(R,j,k) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\hat f(Re^{i\theta},\psi)e^{i(k-j)\theta}e^{ij\psi}\,d\theta\,d\psi, \qquad j,k=0,\pm1,\ldots$$

This matricial Fourier transform can also be expressed in terms of the Bessel function

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}e^{ix\sin\theta}e^{-in\theta}\,d\theta, \qquad n=0,\pm1,\ldots$$

In fact,

$$T_f(R,j,k) = (-i)^{k-j}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left\{\frac{1}{2\pi}\int_0^{\infty}f(re^{i\theta},\psi)J_{k-j}(rR)\,r\,dr\right\}e^{i[(k-j)\varphi+j\psi]}\,d\psi\,d\varphi.$$

The expression with in the brackets is known as the $k-j$-th *Hankel transform of $f(re^{i\varphi},\psi)$* regarded as a function of r.

If convolution of two integrable functions on $M(2)$ is defined by $f*g(u) = \int_{M(2)} f(v)g(uv^{-1})\,dv$, then $T_{f*g}(R) = T_f(R)T_g(R)$.

The Fourier transform can be defined for functions which are not necessarily in $L^1(M(2))$. For example, if $f\in L^2(M(2))$, then $f(x,\varphi)$ belongs to $L^2(R^2)$ for a.e. $\varphi\in[-\pi,\pi)$. Consequently, $\hat f(x,\varphi)$ is almost everywhere defined and, making use of Plancherel's theorem, we obtain a natural extension of formula (1) to functions in $L^2(M(2))$.

Exploiting the fact that the Fourier transform on $L^2(M(2))$ is defined in terms of the Fourier or Hankel transforms on $L^2(R^2)$ or $L^2((0,\infty), R\,dR)$, it is easy to use the L^2 theories of these transforms to establish the corresponding L^2 theory of the Fourier transform on $M(2)$. Adopting the notation $\operatorname{tr}A$ to denote the trace of a matrix, $|||A|||$ to denote the Hilbert–Schmidt norm of A ($|||A|||^2 = \sum_{i,j}|a_{ij}|^2$), and $J(\varphi,r,\psi)$ for the matrix whose j,k entry is $i^{j-k}J_{j-k}(r)e^{-i[(j-k)\varphi+k\psi]}$, we summarize the basic facts of this theory in the following theorem.

THEOREM A. (i) *If f is any measurable function on $M(2)$, then*

$$\|f\|^2_{2,M(2)} = \frac{1}{2\pi}\int_0^{2\pi}|||T_f(r)|||^2\,r\,dr$$

in the sense that the equality must hold if the expressions on either side are finite, while if either expression is infinite, so is the other.

(ii) *If $f\in L^2(M(2))$, then*

$$f(re^{i\varphi},\psi) = \frac{1}{2\pi}\int_0^{\infty}\operatorname{tr}[T_f(s)J(\varphi,sr,\psi)]\,s\,ds \qquad (\textit{equality in } L^2).$$

(iii) *If $A(s)$ is a countably infinite matrix for each $s>0$ with j,k entry $A(s,j,k)$, and if $\int_0^{\infty}|||A(s)|||^2 s\,ds<\infty$, then the function*

$$f(re^{i\varphi},\psi) = \frac{1}{2\pi}\int_0^{\infty}\operatorname{tr}[A(r)J(\varphi,sr,\psi)]\,s\,ds$$

is in $L^2(M(2))$ and $T_f(s)=A(s)$.

We call the formula given in (ii) of Theorem A the *Fourier expansion of f.*

Given a countably infinite matrix-valued function $M(R)$ on the positive real numbers, we define the *left multiplier operator M on $L^p(M(2))$* induced by $M(R)$ to be the operator which maps $f\in L^p(M(2))\cap L^2(M(2))$ to the function

$$Mf(re^{i\varphi},\psi) = \frac{1}{2\pi}\int_0^{\infty}\operatorname{tr}[M(R)T_f(R)J(\varphi,rR,\psi)]\,R\,dR.$$

We say that a multiplier operator, M, *is bounded on $L^p(M(2))$* if it is a bounded map of $L^p(M(2))\cap L^2(M(2))$ into $L^p(M(2))$.

Using Theorem A it is not difficult to characterize bounded multiplier operators on $L^2(M(2))$. In fact, a straightforward argument proves the following corollary:

Corollary A. *A multiplier $M(R)$ defines a bounded multiplier operator on $L^2(M(2))$ if and only if the norm of the matrix $M(R)$ considered as an operator on $l^2(C)$ is essentially bounded on $(0, \infty)$.*

One of our goals in this paper is to study how the L^p boundedness of multiplier operators on $M(2)$ may be derived from the L^p boundedness of related operators on $SO(3)$ or $SU(2)$. In order to accomplish this goal we need to know some basic facts about $L^p(M(2))$.

We begin by describing a class of functions whose role in Fourier analysis on $M(2)$ is analogous to the role of the trigonometric polynomials on T.

Set

$$k_s^l(re^{i\varphi}, \psi) = \frac{2}{\pi(l+1)}\left[\frac{\sin[(1/2)(l+1)\psi]}{2\sin[(1/2)\psi]}\right]^2 K(sr)s^2,$$

where $K(r) = 4r^{-2}J_2(r)$. A standard argument using the observation that $\int\limits^{\infty} K(r)rdr = 1$ and the properties of the Fejér kernel shows that

$$\lim_{\substack{l\to\infty\\ s\to\infty}} \|k_s^l * f - f\|_{p,M(2)} = 0.$$

Moreover,

$$(2)\qquad T_{k_s^l * f * k_s^l}(R, j, k)$$
$$= \begin{cases} \left(1-\dfrac{R^2}{s^2}\right)^2\left(1-\dfrac{|j|}{l}\right)\left(1-\dfrac{|k|}{l}\right)T_f(R, j, k) & \text{if } |j|, |k| < l;\ 0 < R \leqslant s, \\ 0 & \text{otherwise.}\end{cases}$$

It is useful to restate these observations in the following manner.

Property A. *If $1 \leqslant p < \infty$, the set of all functions in $L^p(M(2)) \cap L^2(M(2))$ whose matricial Fourier transforms, $T_f(R)$, consist of finite matrices supported on a finite interval of $(0, \infty)$ is dense in $L^p(M(2))$.*

The importance of the functions $k_s^l * f * k_s^l$ for Fourier analysis on $M(2)$ is based on the following property.

Property B. *Let $f \in L^p(M(2))$, $1 < p < \infty$. Then the Fourier expansion of $k_s^l * f * k_s^l$ equals $k_s^l * f * k_s^l$ almost everywhere in $M(2)$.*

Proof. Using standard estimates for J_2 and the Fejér kernel it is easy to check that $k_s^l \in L^q(M(2))$, $1 \leqslant q \leqslant \infty$. It follows that $k_s^l * f$ is a continuous function which vanishes at infinity and therefore that $k_s^l * f * k_s^l \in L^q(M(2))$, $1 \leqslant q < \infty$. Moreover, $k_s^l * f * k_s^l$ is infinitely differentiable on $C - \{0\} \times (0, 2\pi)$ since k_s^l is. Applying these observations, we see that $k_s^l * f * k_s^l(re^{i\varphi}, \psi)$ is absolutely integrable and locally of bounded variation as a function of r. Thus, we may use the classical inversion theorem for the Hankel transform, [5], p. 456, to conclude that

$$k_s^l * f * k_s^l(re^{i\varphi}, \psi) = \int_0^\infty i^m \Big[\int_0^\infty k_s^l * f * k_s^l(Re^{i\varphi}, \psi)(-i)^m J_m(Rt)RdR\Big] J_m(tr)\,tdt$$

for any integer m and almost every $(re^{i\varphi}, \psi) \in M(2)$.

Similarly, since $k_s^l * f * k_s^l(re^{i\varphi}, \psi)$ is infinitely differentiable in both φ and ψ, we may apply Fourier inversion to $k_s^l * f * k_s^l$ considered as a function of φ and ψ on $(0, 2\pi) \times (0, 2\pi)$. After a small amount of manipulation we obtain the equality

$$k_s^l * f * k_s^l(re^{i\varphi}, \psi) = \frac{1}{2\pi}\int_0^\infty \sum_{\substack{|n|\leqslant l\\ |m|\leqslant l}} i^m T_{k_s^l * f * k_s^l}(r, n, m+n) J_m(tr) e^{-in\psi} e^{-im\varphi}\,tdt$$
$$= \frac{1}{2\pi}\int_0^\infty \operatorname{tr}[T_{k_s^l}(r) J(\varphi, tr, \psi)]\,tdt$$

for almost every $(re^{i\varphi}, \psi) \in M(2)$. ∎

Properties A and B show that a function $f \in L^p(M(2))$, $1 < p < \infty$, is determined by its Fourier transform.

§ 2. $SO(3)$ and $SU(2)$. The group of rotations of three-dimensional Euclidean space, $SO(3)$, and its universal covering group, $SU(2)$, have been widely studied. In particular, Coifman and Weiss have studied singular integrals and multiplier operators on these groups (cf. [2]).

In the remainder of this paper we shall be concerned with establishing relations between multiplier operators on $SO(3)$ or $SU(2)$ and $M(2)$. We shall use the notation and results of Coifman and Weiss as they appear in [2]. Thus, we shall describe points in $SU(2)$ by their Euler angles, (φ, θ, ψ), where $0 \leqslant \varphi \leqslant 2\pi$, $-2\pi \leqslant \psi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \pi$. Haar measure on $SU(2)$ will be normalized as $\dfrac{1}{16\pi^2}\sin\theta\, d\varphi\, d\theta\, d\psi$. $t_{jk}^n(\varphi, \theta, \psi)$ will denote the j, k matrix entry of the irreducible unitary representations of $SU(2)$ described in [2]. (Here, $2n$ is a non-negative integer; $j-n$, $k-n$ are integers; and $|j|, |k| \leqslant n$.) The Fourier expansion of functions in $L^2(SU(2))$ will be written in terms of these matrix entries by letting $T^n(\varphi, \theta, \psi)$ denote the matrix with j, k entry $t_{jk}^n(\varphi, \theta, \psi)$, so that we obtain

$$f(\varphi, \theta, \psi) = \sum_{2n=0}^{\infty}(2n+1)\operatorname{tr}[\hat f(n) T^n(\varphi, \theta, \psi)]$$

where

$$\hat{f}(n) = \int_{-2\pi}^{2\pi}\int_{0}^{\pi}\int_{0}^{2\pi} f(\varphi, \theta, \psi) T^{n}\big((\varphi, \theta, \psi)^{-1}\big) \sin\theta \, d\varphi \, d\theta \, d\psi .$$

Fourier analysis on $SO(3)$ is easily related to analysis on $SU(2)$ if we describe points in $SO(3)$ by their Euler angles, (φ, θ, ψ), where $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \psi \leqslant 2\pi$. Then Haar measure on $SO(3)$ is given by $\frac{1}{8\pi^2}\sin\theta d\varphi d\theta d\psi$. Moreover, the functions $t_{jk}^{n}(\varphi, \theta, \psi)$ with n a non-negative integer and ψ restricted to $0 \leqslant \psi \leqslant 2\pi$ may be interpreted as the matrix entries of a complete system of irreducible unitary representations of $SO(3)$. The formulae for the Fourier transform and the Fourier expansion of a function, f, in $L^2\big(SO(3)\big)$ is identical to that given above for $f \in L^2\big(SU(2)\big)$, except that in this context, n ranges over non-negative integers, and $0 \leqslant \psi \leqslant 2\pi$.

Since we are interested in relating Fourier analysis on $SO(3)$ and $SU(2)$ to Fourier analysis on $M(2)$, the following lemma which establishes a correspondence between the matrix entries t_{kj}^{n}, which appear in the Fourier transform on $SO(3)$ or $SU(2)$, and Bessel functions, which appear in the Fourier transform on $M(2)$, is important.

Lemma A. *$t_{kj}^{n}(0, \theta, 0) = i^{j-k} J_{j-k}\big(\theta(n^2-j^2)^{1/2}\big) + O(\theta)$ provided $|j|, |k| < L$, $n \neq 0$, and $n\theta \leqslant N$. The bound corresponding to $O(\theta)$ depends on L and N.*

Proof. We begin by studying the integral representation:

$$t_{kj}^{n}(0, \theta, 0) = (2\pi)^{-1}\left(\frac{(n+k)!(n-k)!}{(n+j)!(n-j)!}\right)^{1/2} \times$$
$$\times \int_{-\pi}^{\pi}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}e^{-i\varphi}\right)^{n-j}\left(i\sin\frac{\theta}{2}e^{i\varphi} - \cos\frac{\theta}{2}\right)^{n+j} \exp\big((k-j)i\varphi\big) d\varphi ,$$

cf. [2], p. 33. Setting $1+x \equiv \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}e^{-i\varphi}$, which implies $x = O(\theta)$, and using the fact that if $0 < |1+x| < 1$, $(1+x)^{n-j} = e^{(n-j)x}\big(1+ + (n-j)O(x^2)\big)$, we see that

$$\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}e^{-i\varphi}\right)^{n-j} = \exp\left[(n-j)\left(i\frac{\theta}{2}e^{-i\varphi} + O(\theta^3)\right)\right]\big(1+(n-j)O(\theta^2)\big).$$

for $\varphi \neq l\pi/2$ where l is an integer and $0 < \theta < \pi$. Similarly,

$$\left(i\sin\frac{\varphi}{2}e^{i\varphi} + \cos\frac{\varphi}{2}\right)^{n+j} = \exp\left[(n+j)\left(i\frac{\theta}{2}e^{i\varphi} + O(\theta^3)\right)\right]\big(1+(n-j)O(\theta^2)\big).$$

We conclude that

$$t_{kj}^{n}(0, \theta, 0) = (2\pi)^{-1}\left(\frac{(n+k)!(n-k)!}{(n+j)!(n-j)!}\right)^{1/2} \times$$
$$\times \int_{-\pi}^{\pi} \exp\left(i\frac{\theta}{2}[(n+j)e^{i\varphi} - (j-n(e^{-i\varphi}]\right) \exp\big(i(k-j)\big)\varphi\big(1+O(n)O(\theta^2)\big) d\varphi .$$

The relationship between this expression and the Bessel functions may be seen by studying the generating function for Bessel functions,

$$\exp\left(\frac{1}{2}z(r-r^{-1})\right) = \sum_{m=-\infty}^{\infty} r^m J_m(z).$$

Substituting $z = -a\big((s-t)(s+t)\big)^{1/2}$, $r = -i(s+t)^{1/2}(s-t)^{-1/2}\exp(i\varphi)$, into this formula gives the relation:

$$\exp\left(i\frac{a}{2}\left[(s+t)e^{i\varphi} - (t-s)e^{-i\varphi}\right]\right) = \sum_{m=-\infty}^{\infty} J_m\big(a(s^2-t^2)^{1/2}\big) i^m \left(\frac{s+t}{s-t}\right)^{m/2} e^{im\varphi}.$$

Assume that $|j|, |k| < n$. Combining the above formulae and using the hypothesis $n\theta < N$, we obtain the representation:

$$(2)\quad t_{kj}^{n}(0, \theta, 0) = \left(\frac{(n+k)!(n-k)!}{(n+j)!(n-j)!}\right)^{1/2}\left(J_{j-k}\big(\theta(n^2-j^2)^{1/2}\big)\left(-\frac{n+j}{n-j}\right)^{(j-k)/2} + O(\theta)\right).$$

In order to simplify this expression, first assume that $j+k \leqslant 0$, $j-k < 0$, $|j|, |k| < n$. Let L be the number given in the statement of the lemma. Then $\frac{(n+k)!(n-k)!}{(n+j)!(n-j)!} < 1$, and if $n > 3L$:

$$\frac{(n+k)!(n-k)!}{(n+j)!(n-j)!}\left(\frac{n+j}{n-j}\right)^{j-k} = \prod_{a=1}^{k-j}\left(l+\frac{a}{n+j}\right)\left(1-\frac{a-l}{n-j}\right)^{-1} = 1+O\left(\frac{1}{n}\right)$$

where $O\left(\frac{1}{n}\right)$ depends on L. The hypotheses of the lemma imply that $\theta(n^2 - -j^2) \leqslant N$, which implies that

$$i^{k-j}J_{k-j}\big(\theta(n^2-j^2)^{1/2}\big) = O(\theta)(n^2-j^2)^{1/2}.$$

Thus,

$$i^{k-j}J_{k-j}\big(\theta(n^2-j^2)^{1/2}\big)O\left(\frac{1}{n}\right) = O(\theta).$$

Combining these estimates, we see that in this case,

$$(2) = i^{j-k}J_{j-k}\big(\theta(n^2-j^2)^{1/2}\big)+O(\theta).$$

If $n \leqslant 3L$, $\left(\dfrac{n+j}{n-j}\right)^{(j-k)/2} = O(1)$. Thus

$$\left(\frac{(n+k)!\,(n-k)!}{(n+j)!\,(n-j)!}\right)^{1/2} J_{j-k}\big(\theta(n^2-j^2)\big)\left(-\frac{n+j}{n-j}\right)^{(j-k)/2} - \\ - i^{j-k}J_{j-k}\big(\theta(n^2-j^2)\big) = O(\theta).$$

We conclude that if $j+k \leqslant 0$, $j-k<0$, $|j|$, $|k|<n$, then

$$(3)\qquad t^n_{kj}(0,\theta,0) = i^{j-k}J_{j-k}\big(\theta(n^2-j^2)^{1/2}\big)+O(\theta).$$

The facts that $t^n_{-j\,-k}(0,\theta,0) = (-1)^{j-k}t^n_{jk}(0,\theta,0)$, [2], p. 109, and $J_{-j+k} = (-1)^{j-k}J_{j-k}$ imply that (3) holds for $j+k \geqslant 0$, $k-j<0$, $|j|<n$, $|k|<n$. Similarly, the formula $t^n_{jk}(0,\theta,0) = t^n_{kj}(0,\theta,0)$, [2], p. 109, implies that $t^n_{kj}(0,\theta,0) = i^{j-k}J_{j-k}\big(\theta(n^2-k^2)^{1/2}\big)+O(\theta)$ when $|j|$, $|k|<n$ and either $j+k\leqslant 0$, $j-k>0$, or $j+k\geqslant 0$, $k-j>0$. Furthermore, $J_{j-k}\big(\theta(n^2-k^2)^{1/2}\big)-J_{j-k}\big(\theta(n^2-j^2)^{1/2}\big) = O(\theta)$ if $|j|$, $|k|\leqslant L$ and $\theta n \leqslant N$, where $O(\theta)$ depends on L and N. This can be verified by noting that

(i) $J'_{j-k}(x) = O(1)$ which implies

$$J_{j-k}\big(\theta(n^2-k^2)^{1/2}\big)-J_{j-k}\big(\theta(n^2-j^2)^{1/2}\big) = O\Big(\big(\theta(n^2-j^2)^{1/2}\big)-(n^2-k^2)^{1/2}\Big),$$

and

(ii) $|j|$, $|k|<n$, $|j|$, $|k|<L$ imply that $(n^2-k^2)^{1/2}-(n^2-j^2)^{1/2} = O(1)$.

Using these estimates, we see that (3) holds whenever $|j|$, $|k|<L$, $|j|$, $|k|<n$ and $j\neq k$. When $j=k$ and $|j|<L$, (3) follows immediately from (2).

The remaining cases when $|j| = n$ or $|k| = n$ may be verified by using the symmetry properties of the t^n_{kj} already discussed and the facts that:

(i) in these cases $n<L$,

(ii) $t^n_{kn}(0,\theta,0) = \dfrac{(2n)!}{(n-k)!\,(n+k)!}\, i^{n-k}\left(\sin\dfrac{\theta}{2}\right)^{n-k}\left(\cos\dfrac{\theta}{2}\right)^{n+k}$,

(iii) $J_m(0) = 0$ if $m\neq 0$. ■

§ 3. Multiplier operators. We introduced multiplier operators on $M(2)$ in Section 1. In this section we shall show how the L^p boundedness of such operators is related to properties of operators on $SO(3)$ or $SU(2)$.

If $m(n)$ is a $2n+1$ by $2n+1$ matrix for each non-negative integer n, then we define the multiplier operator m on $L^p\big(SO(3)\big)$ induced by $\{m(n)\}_{n=0}^{\infty}$ to be the operator which maps $f\in L^p\big(SO(3)\big)\cap L^2\big(SO(3)\big)$ to the function $mf(\varphi,\theta,\psi) = \sum_{n=0}^{\infty}(2n+1)\operatorname{tr}[m(n)\hat{f}(n)T^n(\varphi,\theta,\psi)]$. Multiplier operators on

$SU(2)$ are defined analogously. If $\|mf\|_p \leqslant C\|f\|_p$ for $f\in L^2\cap L^p$, the multiplier operator m is said to be *bounded on* L^p.

Let R denote the canonical two-to-one map of $SU(2)$ onto $SO(3)$ (cf. [2], p. 105, for details). If f is a complex-valued function on $SO(3)$, define f_0 on $SU(2)$ by setting $f_0(u) = f(Ru)$, $u\in SU(2)$. It is easy to check that $\|f_0\|_{p,SU(2)} = \|f\|_{p,SO(3)}$. Moreover, $\hat{f}_0(n) = 0$ if n is a half-integer, while $\hat{f}_0(n) = \hat{f}(n)$ if n is an integer.

Using these observations it is not hard to prove the following lemma:

LEMMA B. *If* $\{m(n)\}_{2n=0}^{\infty}$ *defines a bounded multiplier operator on* $L^p\big(SU(2)\big)$, *then* $\{m(n)\}_{n=0}^{\infty}$ *defines a bounded multiplier operator on* $L^p\big(SO(3)\big)$ *with the same operator norm.*

Next we show how multiplier operators defined on $M(2)$ induce multiplier operators defined on $SO(3)$. Let $M(R)$ be a countably infinite matrix-valued function on the positive real numbers with j,k entry $M(R,j,k)$. Let M be the corresponding multiplier operator on $M(2)$. For each $\lambda>0$, M induces a multiplier operator on $SO(3)$, denoted m_λ, by the following process. Let $m_\lambda(n)$ be the $2n+1$ by $2n+1$ matrix whose j,k entry is $M(n/\lambda,j,k)$, $-n\leqslant j,k\leqslant n$. Then m_λ is the multipler operator defined by $\{m_\lambda(n)\}_{n=0}^{\infty}$.

We are now in a position to state the fundamental theorem of this paper.

THEOREM B. *Let M be a bounded multiplier operator on $L^2\big(M(2)\big)$. Suppose that the matrix entries $M(R,j,k)$ of the function $M(R)$ defining M are continuous functions of R for each j,k. For $\lambda>0$, let m_λ be the multiplier operator on $SO(3)$ induced by M. If the operator norms of m_λ on $L^p\big(SO(3)\big)$, $1<p<\infty$ satisfy $\liminf_{\lambda\to\infty}\|m_\lambda\|_p<\infty$, then M is a bounded operator on $L^p\big(M(2)\big)$.*

Proof. Define $M_{sl}f = k^l_s * Mf$. If we show that

$$(4)\qquad \|M_{sl}f\|_{p,M(2)} \leqslant A\,\|f\|_{p,M(2)}$$

whenever $f\in L^p\big(M(2)\big)\cap L^2\big(M(2)\big)$ with A independent of s and l, the theorem will be proved. To see this, note that using the proof of Property A, $\lim_{\substack{l\to\infty\\ s\to\infty}}\|k^l_s * Mf - Mf\|_{2,M(2)} = 0$.

Thus

$$\lim_{\substack{l\to\infty\\ s\to\infty}}\int_{M(2)} k^l_s * Mg(u)h(u)\,du = \int_{M(2)} Mg(u)h(u)\,du$$

for $h\in C_0^\infty\big(M(2)\big)$, the infinitely differentiable functions with compact support on $M(2)$. Hölder's inequality and (4) imply that if $1/p+1/q = 1$, $\Big|\int_{M(2)} Mg(u)h(u)\,du\Big| \leqslant A\,\|g\|_{p,M(2)}$ for all $h\in C_0^\infty\big(M(2)\big)$ such that $\|h\|_{q,M(2)} = 1$. An application of the converse to Hölder's inequality proves our claim: the hypotheses of Theorem B and (4) imply $\|Mg\|_{p,M(2)} \leqslant A\|g\|_{p,M(2)}$.

A computation shows that M_{sl} is the multiplier operator induced by the matrices $M_{sl}(R)$, $R > 0$, with j, k entries

$$M_{sl}(R, j, k) = \begin{cases} \left(1-\dfrac{R^2}{s^2}\right)\left(1-\dfrac{|j|}{l}\right) M(R, j, k) & \text{if} \quad j \leqslant l,\ 0 < R < s, \\ 0 & \text{otherwise.} \end{cases}$$

For $\lambda > 0$, let $m_{sl\lambda}$ be the multiplier operator on $SO(3)$ induced by M_{sl}. We claim that $\liminf\limits_{\lambda\to\infty} \|m_{sl\lambda}\|_p < \infty$. To see this, note that the multiplier operator $m_{s\lambda}$ defined by $\left\{\left(1-\dfrac{n}{(s\lambda)}2\right)\right\}_{2n=0}^{s}$ is bounded on $L^p(SU(2))$, $1<p<\infty$, with bound independent of λ and s. This follows easily by applying a theorem of Coifman and Weiss: [2], p. 87. Use Lemma B to conclude that the multiplier operator $m_{s\lambda}$ defined by $\left\{\left(1-\dfrac{n}{(s\lambda)}2\right)\right\}_{n=0}^{s\lambda}$ is bounded on $L^p(SO(3))$, $1 < p < \infty$, with bound independent of λ and s.

Observe that if

$$F_l(\gamma) \equiv \frac{2}{\pi(l+1)}\left[\frac{\sin\frac{1}{2}(l+1)\gamma}{2\sin\frac{1}{2}\gamma}\right]^2,$$

then

$$m_{sl\lambda} f(\varphi, \theta, \psi) = \int\limits_0^{2\pi} F_l(\gamma) m_{sl} m_\lambda g(\varphi, \theta, \psi-\gamma)\, d\gamma.$$

It follows that from the standard properties of the Fejér kernel that $\|m_{sl\lambda} f\|_{p,SO(3)} \leqslant D\|f\|_{p,SO(3)}$ with D independent of s, l, λ. We have reduced the proof of Theorem B to showing that $\liminf\limits_{\lambda\to\infty} \|m_{sl\lambda}\|_p < \infty$ implies $\|M_{sl} f\|_{p,M(2)} \leqslant A\|f\|_{p,M(2)}$ for $f \in L^2(M(2)) \cap L^p(M(2))$ with A independent of l and S.

We require a further reduction. If $f \in L^p(M(2))$, set

$$f^l(\varphi, r, \psi) = \int\limits_0^{2\pi}\int\limits_0^{2\pi} F^l(\gamma) F^l(\beta) f(\varphi+\gamma, r, \psi+\beta)\, d\gamma\, d\beta$$

with F^l as defined above. It is easy to check that $\|f^l\|_{p,M(2)} \leqslant \|f\|_{p,M(2)}$ independent of l. Similarly, considering

$$f^l(\varphi, \theta, \psi) = \int\limits_0^{2\pi}\int\limits_0^{2\pi} F^l(\gamma) F^l(\beta) f(\varphi+\gamma, \theta, \psi+\beta)\, d\gamma\, d\beta$$

to be a function on $SO(3)$, $\|f^l\|_{p,SO(3)} \leqslant \|f\|_{p,SO(3)}$ independent of l.

The standard properties of the Fejér kernel imply that for $f \in L^p(Q)$ $\lim\limits_{l\to\infty} \|f^l - f\|_{p,Q} = 0$ if $Q = M(2)$ or $SO(3)$. Since $C_0^\infty(M(2))$ is dense in $L^p(M(2))$, $1 \leqslant q < \infty$, Theorem B will be proved if we show that $\liminf\limits_{\lambda\to\infty} \|m_{sl\lambda}\|_p < \infty$ implies $\|M_{sl} f^l\|_{p,M(2)} \leqslant A\|f^l\|_{p,M(2)}$ for $f \in C_0^\infty(M(2))$.

Let $f \in C_0^\infty(M(2))$ have support in $(0, 2\pi) \times (0, \mu] \times (0, 2\pi)$. Define $f_\lambda(\varphi, \theta, \psi) \equiv f(\varphi, \lambda\theta, \psi)$. For λ sufficiently large, the support of f_λ will be contained in $(0, 2\pi) \times (0, \pi) \times (0, 2\pi)$ so that we may consider f_λ to be a function in $C_0^\infty(SO(3))$. It will be convenient to study $f_\lambda^+(\varphi, \theta, \psi) \equiv f(\varphi, \theta, \psi+\varphi)$. Clearly, $\|f_\lambda^+\|_{p,SO(3)} = \|f_\lambda\|_{p,SO(3)}$. The hypothesis $\liminf\limits_{\lambda\to\infty} \|m_{sl\lambda}\|_p < \infty$ implies that for infinitely many λ:

$$\Big\|\sum_{n=0}^{\infty} (2n+1)\operatorname{tr}\left[m_{sl\lambda}(n)\left[(f_\lambda^l)^+\right]^\wedge(n) T^n(\varphi, \theta, \psi)\right]\Big\|_{p,SO(3)} \leqslant \|m_{sl\lambda}\|_p \|(f_\lambda^l)^+\|_{p,SO(3)}.$$

Thus,

$$\Big\|\sum_{n=0}^{\infty} (2n+1)\operatorname{tr}\left[m_{sl\lambda}(n)\left[(f_\lambda^l)^+\right]^\wedge(n) T^n(\varphi, \theta, \psi-\varphi)\right]\Big\|_{p,SO(3)} \leqslant \|m_{sl\lambda}\|_p \|f_\lambda^l\|_{p,SO(3)}.$$

Note that

$$\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_0^{2\pi} |f^l(\varphi, \lambda\theta, \psi)|^p \sin\theta\, d\varphi\, d\theta\, d\psi$$
$$= \lambda^{-2} \int\limits_0^{2\pi}\int\limits_0^{\lambda\pi}\int\limits_0^{2\pi} |f^l(\varphi, r, \psi)|^p \left(2\lambda\sin\frac{r}{2\lambda}\right)\cos\frac{r}{2\lambda}\, d\varphi\, dr\, d\psi$$

and

$$8\pi^2 \Big\|\sum_{n=0}^{\infty} (2n+1)\operatorname{tr}\left[m_{sl\lambda}(n)\left[(f_\lambda^l)^+\right]^\wedge(n) T^n(\varphi, \theta, \psi-\varphi)\right]\Big\|_{p,SO(3)}^p$$
$$= \lambda^{-2} \int\limits_0^{2\pi}\int\limits_0^{\lambda\pi}\int\limits_0^{2\pi} \Big|\sum_{n=0}^{\infty} (2n+1)\operatorname{tr}\left[m_{sl\lambda}(n)\left[(f_\lambda^l)^+\right]^\wedge(n) T^n(\varphi, \theta, \psi-\varphi)\right]\Big|^p \times$$
$$\times \left(2\lambda\sin\frac{r}{2\lambda}\right)\cos\frac{r}{2\lambda}\, d\varphi\, dr\, d\psi.$$

These equalities together with the fact that $\lim\limits_{\lambda\to\infty}\left(2\sin\dfrac{r}{2\lambda}\right)\cos\dfrac{r}{2\lambda} = r$ and Fatou's lemma show that

(5)
$$\frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^{\infty}\int\limits_0^{2\pi} \liminf_{\lambda\to\infty} \Big|\sum_{n=0}^{\infty} (2n+1)\operatorname{tr}\left[m_{sl\lambda}(n)\left[(f_\lambda^l)^+\right]^\wedge(n) T^n\left(\varphi, \frac{r}{\lambda}, \psi-\varphi\right)\right]\Big|^p r\, d\varphi\, dr\, d\psi$$
$$\leqslant \liminf_{\lambda\to\infty} \|m_{sl\lambda}\|_p \|f^l\|_{p,M(2)}.$$

Define

$$G_{sl\lambda}(\varphi, r, \psi) = \sum_{n=0}^{\infty} (2n+1)\operatorname{tr}\left(m_{sl\lambda}(n)\,[(f_\lambda^l)^+]\hat{}\,(n)\,T^n\left(\varphi, \frac{r}{\lambda}, \psi-\varphi\right)\right).$$

If we show $\lim\limits_{\lambda\to\infty} G_{sl\lambda}(\varphi, r, \psi) = M_{sl} f^l(\varphi, r, \psi)$, then (5) implies that $\|M_{sl}f^l\|_{p,M(2)} \leqslant A\|f^l\|_{p,M(2)}$ and thus, by our previous reductions, Theorem B will be proved. Computing $[(f_\lambda^l)^+]\hat{}\,(n)$, we find

$$G_{sl\lambda}(\varphi, r, \psi) = \sum_{n=0}^{s\lambda} (2n+1)\left(1-\frac{n^2}{(s\lambda)^2}\right)\times$$

$$\times \sum_{\substack{j=-n\\|j|<l}}^{n} \sum_{\substack{k=-n\\|a-k|<l}}^{n} \sum_{\substack{a=-n\\|a|<l}}^{n} M\left(\frac{n}{\lambda}, j, a\right)\left(1-\frac{|j|}{l}\right)\left(1-\frac{|k-a|}{l}\right)\left(1-\frac{|a|}{l}\right)(8\pi^2)^{-1}\times$$

$$\times \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f_\lambda(\Phi, \theta, \Psi)\, e^{i[(k-a)\Phi+a\Psi]}\overline{t_{ka}^n(0, \theta, 0)}\sin\theta\, d\Phi\, d\theta\, d\Psi\;\; t_{kj}^n\left(\varphi, \frac{r}{\lambda}, \psi-\varphi\right).$$

We shall use Lemma A to study this expression.

We begin by fixing $X > 0$ and making the following restrictions: $0 \leqslant r \leqslant X$; $|j|, |k| \leqslant n$; $|j|, |k| < L$; $0 < n \leqslant s\lambda$. These imply that $n\dfrac{r}{\lambda} \leqslant SX \equiv N$. Using Lemma A, we obtain the equality

$$(6)\qquad t_{kj}^n\left(\varphi, \frac{r}{\lambda}, \psi-\varphi\right) = e^{-i(k-j)\varphi}e^{-ij\Psi}i^{j-k}J_{j-k}\left(\frac{r}{\lambda}(n^2-j^2)^{1/2}\right) + O\left(\frac{r}{\lambda}\right)$$

where the bound on $O\left(\dfrac{r}{\lambda}\right)$ depends only on l, s, and X, hence only on N and L. It is not difficult to check that $J_{j-k}\left(\dfrac{r}{\lambda}n\right) - J_{j-k}\left(\dfrac{r}{\lambda}(n^2-j^2)^{1/2}\right) = O\left(\dfrac{r}{\lambda}\left(n-(n^2-j^2)^{1/2}\right)\right)$. The above restrictions give $\dfrac{r}{\lambda}\left(n-(n^2-j^2)^{1/2}\right) \leqslant n\dfrac{r}{\lambda} = N$.

We conclude that $J_{j-k}\left(\dfrac{r}{\lambda}(n^2-j^2)^{1/2}\right) = J_{j-k}\left(\dfrac{r}{\lambda}n\right) + O(1)$, where the bound arising from the $O(1)$ depends only on L and N. Combining this estimate with the fact that $\lambda \geqslant 1$ implies $\dfrac{r}{\lambda} \leqslant X$, we see that (6) implies

$$(7)\qquad (2n+1)t_{kj}^n\left(\varphi, \frac{r}{\lambda}, \psi-\varphi\right) = 2ne^{-i[(k-j)\varphi+j\psi]}i^{j-k}J_{j-k}\left(\frac{r}{\lambda}n\right) + (2n+1)O(1)$$

where the bound on $O(1)$ depends only on N and L. When $n = 0$, (7) holds trivially, therefore we have established equality (7) whenever $0 \leqslant r \leqslant X$; $|j|, |k| \leqslant n$; $|j|, |k| < L$; $0 \leqslant n \leqslant s\lambda$.

Next, suppose that $0 < n \leqslant s\lambda$; $|a|, |k| \leqslant n$; $|j|, |k| < L$. Since $f \in C_0^\infty(M(2))$, we see that

$$\hat{f}_\lambda(n)$$

$$= (8\pi^2\lambda^2)^{-1}\int_0^{2\pi}\int_0^{\lambda\pi}\int_0^{2\pi} f(\varphi, r, \psi)\, e^{i[(k-a)\varphi+a\psi]}\overline{t_{ka}^n\left(0, \frac{r}{\lambda}, 0\right)}\left(2\lambda\sin\frac{r}{2\lambda}\right)\cos\frac{r}{2\lambda}\, d\varphi\, dr\, d\psi$$

$$= (8\pi^2\lambda^2)^{-1}\int_0^{2\pi}\int_0^{\lambda\pi}\int_0^{2\pi} f(\varphi, r, \psi)\, e^{i[(k-a)\varphi+a\psi]}\overline{t_{ka}^n\left(0, \frac{r}{\lambda}, 0\right)}\, r\, d\varphi\, dr\, d\psi + \lambda^{-2}O(1).$$

f has support in $(0, 2\pi)\times(0, \mu]\times(0, 2\pi)$; thus we may assume that $0 < r < \mu$, which implies $nr/\lambda \leqslant \mu s$. Applying Lemma A to this expression and simplifying in the manner described above, we see that

$$(8)\qquad \hat{f}_\lambda(n)$$

$$= (8\pi^2\lambda^2)^{-1}\int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi} f(\varphi, r, \psi)\, e^{i[(k-a)\varphi+a\psi]}\, i^{a-k}J_{a-k}\left(\frac{r}{\lambda}n\right) r\, d\varphi\, dr\, d\psi + \lambda^{-2}O(1)$$

where the bound on $O(1)$ depends on L, s and μ (hence f). The fact that $t_{00}^0(\varphi, r/\lambda, \psi) = J_0(0)$ implies (8) holds for $n = 0$.

Combining estimates (7) and (8), we have shown

$$G_{sl\lambda}(\varphi, r, \psi)$$

$$= \sum_{n=0}^{s\lambda}\left(1-\frac{n^2}{(s\lambda)^2}\right)\sum_{\substack{j=-n\\|j|<l}}^{n} \sum_{\substack{k=-n\\|a-k|<l}}^{n} \sum_{\substack{a=-n\\|a|<l}}^{n} M\left(\frac{n}{\lambda}, j, a\right)\left(1-\frac{|j|}{l}\right)\left(1-\frac{|k-a|}{l}\right)\left(1-\frac{|a|}{l}\right)\times$$

$$\times\left[(4\pi\lambda^2)^{-1}T_f\left(\frac{n}{\lambda}, a, k\right) + \lambda^{-2}O(1)\right]\times$$

$$\times\left[2ne^{-i[(k-j)\varphi+j\psi]}i^{k-j}J_{k-j}\left(\frac{r}{\lambda}n\right)(2n+1)O(1)\right].$$

Noting that the bounds corresponding to the "O" terms depend only on L, s, and f, that $\sup\left\{\left|M\left(\dfrac{n}{\lambda}, j, a\right)\right| : 0 \leqslant n \leqslant s\lambda;\ |j|, |a| < l\right\} < \infty$, and that $n/\lambda \leqslant s$, an easy computation shows $\lim\limits_{\lambda\to\infty} G_{sl\lambda}(\varphi, r, \psi) = M_{sl}f^l(\varphi, r, \psi)$. As we noted above, this proves Theorem B. ∎

Combining Theorem B with Lemma B, the following corollary is immediate.

COROLLARY B. *Let M be a bounded multiplier operator on $L^2(M(2))$. Suppose that the function $M(R)$ defining M is continuous on $(0,\infty)$. For $\lambda>0$, let m be the multiplier operator on $SU(2)$ defined by the sequence $\{m_\lambda(n)\}_{2n=0}^{\infty}$, where the j,k entry of m_λ is $M\left(\frac{n}{\lambda},j,k\right)$, $-2n\leqslant j,k\leqslant 2n$, $0\leqslant 2n<\infty$. If the operator norms of the m_λ on $L^p(SU(2))$, $1<p<\infty$, satisfy $\liminf_{\lambda\to\infty}\|m_\lambda\|_p<\infty$, then M is a bounded operator on $L^p(M(2))$.*

§ 4. An application. In this section we show how the results we have obtained may be used to prove a transplantation result for Bessel functions. Consider the multiplier operator, M, defined on $M(2)$ by the matrices, $M(R)$, with j,k entry equal to one if $j=l$, $k=0$, and zero otherwise. An easy application of Corollary A shows that this operator is bounded on $L^2(M(2))$. For $\lambda>0$, let m_λ be the multiplier operator on $SU(2)$ induced from M by the procedure described in Corollary B. Since the entries of $M(R)$ are independent of R, the matrix entries of m_λ are independent of λ. The operators m_λ were studied by Coifman and Weiss ([2], pp. 136–138) who showed that these operators are bounded on $L^p(SU(2))$, $1<p<\infty$. Applying Corollary B to M, we find that it is bounded on $L^p(M(2))$, $1<p<\infty$. Let $f\colon(0,\infty)\to C$ be such that the Hankel transform $\int_0^\infty f(R)J_u(rR)R\,dR$ belongs to $L^p((0,\infty),r\,dr)$ for some integer u and some p greater than one.

Define

$$F(\Phi,r,\Psi)=(2\pi)^{-1}\int_0^\infty f(R)J_u(rR)re^{-iu\Phi}R\,dR.$$

$F(\Phi,r,\Psi)\in L^p(M(2))$. Applying M to F shows that $\|MF\|_{p,M(2)}\leqslant A\|F\|_{p,M(2)}$, which implies that

$$\int_0^\infty\left|\int_0^\infty f(R)J_{u-1}(rR)R\,dR\right|^p r\,dr\leqslant A\int_0^\infty\left|\int_0^\infty f(R)J_u(rR)R\,dR\right|^p r\,dr.$$

By iterating this procedure and by considering the multiplier operator on $M(2)$ defined by the matrices with j,k entry equal to one if $j=-l$, $k=0$ and zero otherwise, we obtain the following corollary:

COROLLARY C. *If $f\colon(0,\infty)\to C$ is such that for some integer u, $\int_0^\infty f(R)J_u(Rr)R\,dR\in L^p((0,\infty),r\,dr)$, then given any integer v, there exists a number A_v such that*

$$\int_0^\infty\left|\int_0^\infty f(R)J_v(rR)R\,dR\right|^p r\,dr\leqslant A_v\int_0^\infty\left|\int_0^\infty f(R)J_u(rR)R\,dR\right|^p r\,dr.$$

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