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A multilinear interpolation theorem

by

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Abstract. We prove a general multilinear interpolation theorem for the "espaces de moyenne" of Lions and Peetre. This result is used to obtain a multilinear Marcinkiewicz interpolation theorem.

Let (B^0, B^1) be an interpolation pair of Banach spaces, and denote by $(B^0, B^1)_{s,p}$ (or more simply $B_{s,p}$), $0 < s < 1, 1 \leqslant p \leqslant \infty$, the real interpolation spaces of Lions and Peetre (see [1], Chapter 3, and [8]). In this paper, we obtain a multilinear interpolation theorem for the spaces $B_{s,p}$ which generalizes a result of the aforementioned authors. As a consequence, we obtain results concerning the real interpolation of operator spaces. In the context of L_p spaces, our results, combined with methods closely related to the technique of reiteration (see [1], Chapter 3), yield a multilinear version of the Marcinkiewicz interpolation theorem. These results are then applied to obtain a Marcinkiewicz-type interpolation theorem for the H^p spaces, $0 < p < \infty$.

We begin our discussion with some notations and definitions.

1. Let (B^0, B^1) be a pair of complex Banach spaces continuously embedded in a topological linear space $\mathscr{V}$. Then (B^0, B^1) is called an *interpolation pair*. If $x \in B^j$, we denote its norm by $\|x\|_j$ or $\|x\|_{B^j}$, $j = 0, 1$. Under the norm $\|x\|_{B^0 \cap B^1} = \max(\|x\|_0, \|x\|_1)$, $B^0 \cap B^1$ becomes a Banach space continuously embedded in $\mathscr{V}$. The algebraic sum of B^0 and B^1, defined by $\{y+z \mid y \in B^0, z \in B^1\}$ and denoted by $B^0 + B^1$, becomes a Banach space continuously embedded in $\mathscr{V}$, when furnished with the norm $\|x\| = \inf\{\|y\|_0 + \|z\|_1 \mid x = y+z \text{ and } y \in B^0, z \in B^1\}$.

For any Banach space X, we denote by $L_*^p(X)$ the Banach space of all strongly measurable functions f with domain $(0, \infty)$ and with values in X for which

$$\|f\|_{L_*^p(X)} = \left(\int_0^\infty \|f(t)\|_X^p \frac{dt}{t} \right)^{1/p} < \infty \quad \text{if} \quad 1 \leqslant p < \infty$$

and

$$\|f\|_{L_*^\infty(X)} = \operatorname*{ess\,sup}_{t>0} \|f(t)\|_X < \infty,$$

where, as usual, we identify functions agreeing almost everywhere. If X = complex numbers, we write $L^r_*(X) = L^r_*$.

If (B^0, B^1) is an interpolation pair, we define the *real interpolation space* $(B^0, B^1)_{s,p}$ (or $B_{s,p}$) to be the collection of all elements $x \in B^0+B^1$ for which there exists a strongly measurable function u with values in $B^0 \cap B^1$ such that

$$x = \int_0^\infty u(t)\frac{dt}{t} \tag{1}$$

and

$$t^{-s}u(t) \in L^p_*(B^0) \quad \text{and} \quad t^{1-s}u(t) \in L^p_*(B^1). \tag{2}$$

It follows by condition (2) that $u \in L^1_*(B^0+B^1)$ so that the integral in (1) is well defined. For $0 < s < 1$ and $1 \leqslant p \leqslant \infty$, the space $(B^0, B^1)_{s,p}$ becomes a Banach space continuously embedded in $\mathscr{V}$ under the norm

$$\|x\|_{s,p} = \inf\{\max(\|t^{-s}u\|_{L^p_*(B^0)}, \|t^{1-s}u\|_{L^p_*(B^1)})\},$$

the infimum taken over all u for which $x = \int_0^\infty u(t)\frac{dt}{t}$ and the condition (2) above is satisfied.

We recall that there are several other equivalent definitions for the spaces $B_{s,p}$ (see Chapter 3 of [1], [8] and [9]). We use the above definition since it simplifies certain computations in Section 2. The basic properties of these spaces can be found in [1], [8] and [9], and will be used freely throughout.

We now turn to the Lorentz spaces. Let (Ω, Σ, μ) be a σ-finite measure space, and let $\mathscr{M}$ denote all complex-valued measurable functions on Ω. Then $\mathscr{M}$ becomes a linear topological space under the topology of convergence in measure on all sets of finite measure in Σ.

For any $f \in \mathscr{M}$, we let f^* denote the non-increasing, right continuous, rearrangement of f (see [6], Section 1). We define the *Lorentz space* $L_{p,q}(\mu)$ to be the collection of all $f \in \mathscr{M}$ so that $\|f\|^*_{p,q} < \infty$, where

$$\|f\|^*_{p,q} = \begin{cases} \left(\int_0^\infty (t^{1/p}f^*(t))^q \frac{dt}{t}\right)^{1/q} & \text{if} \quad 0 < p,\ q < \infty, \\ \sup_{t>0} t^{1/p}f^*(t) & \text{if} \quad 0 < p < \infty,\ q = \infty. \end{cases}$$

We define $L_{\infty,q}(\mu) = L_\infty(\mu)$, $0 < q \leqslant \infty$.

In general, $\|\cdot\|^*_{p,q}$ is not a norm. However, $L_{p,q}(\mu)$ is a metrizable linear topological space (see [6], Section 2). Moreover, if $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, then $\|\cdot\|^*_{p,q}$ is equivalent to a norm. Specifically, let $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$. Define

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty (t^{1/p}f^{**}(t))^q \frac{dt}{t}\right)^{1/q} & \text{if} \quad 1 < p < \infty,\ 1 \leqslant q < \infty, \\ \sup_{t>0} t^{1/p}f^{**}(t) & \text{if} \quad 1 < p < \infty,\ q = \infty. \end{cases}$$

Then there exists a constant C (depending only on p and q) so that $\|f\|^*_{p,q} \leqslant \|f\|_{p,q} \leqslant C\|f\|^*_{p,q}$, for all $f \in L_{p,q}(\mu)$; the space $L_{p,q}(\mu)$ becomes a Banach space under the norm $\|\cdot\|_{p,q}$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$ (see [3], [6] or [11], Chapter 5, Section 3). The $L_{p,q}(\mu)$ spaces play an important role in the theory of interpolation of operators. In fact, the weak type condition of Marcinkiewicz is equivalent to a boundedness condition between appropriate Lorentz spaces. We refer the reader to [1] and [11] for this theory. The relation of the $L_{p,q}$ spaces to the abstract interpolation theory of Lions and Peetre may be found in [1], Chapter 3, in case $1 < p < \infty$, and in [7], for the full range $0 < p < \infty$, $0 < q \leqslant \infty$. We will assume these results throughout.

2. In this section, we obtain a multilinear interpolation theorem for the real interpolation spaces $B_{s,p}$. This generalizes a theorem of Lions and Peetre (see [8], Chapter 1, Theorem 4.1). Our result, combined with interpolation properties of Lorentz spaces will yield a general multilinear Marcinkiewicz interpolation theorem (see Theorem 2.9). We begin with some lemmas.

LEMMA 2.1. *Let (B^0_k, B^1_k), (C^0, C^1) be interpolation pairs continuously embedded in the topological linear spaces $\mathscr{V}_k$ and $\mathscr{W}$, respectively, $1 \leqslant k \leqslant n$. Let T be a multilinear operator from $\bigoplus_{k=1}^n B^0_k \cap B^1_k$ into $C^0 \cap C^1$ such that*

$$\|T(x_1, x_2, \ldots, x_n)\|_{C^j} \leqslant M_j \prod_{k=1}^n \|x_k\|_{B^j_k}, \tag{1}$$

$j = 0, 1$, for all $(x_1, x_2, \ldots, x_n) \in \bigoplus_{k=1}^n B^0_k \cap B^1_k$. Fix i, $1 \leqslant i \leqslant n$. Then for all $(x_1, x_2, \ldots, x_n) \in \bigoplus_{k=1}^n B^0_k \cap B^1_k$ we have

$$\|T(x_1, x_2, \ldots, x_n)\|_{C^0+C^1} \leqslant M\Big(\prod_{\substack{k=1\\k\neq i}}^n \|x_k\|_{B^0_k\cap B^1_k}\Big)\|x_i\|_{B^0_i+B^1_i},$$

where $M = \max(M_0, M_1)$.

The proof is a direct consequence of (1), the definitions involved, and by noting that if $x \in B^0_i \cap B^1_i$ and $x = a+b$ where $a \in B^0_i$ and $b \in B^1_i$, then both a and b are in $B^0_i \cap B^1_i$.

LEMMA 2.2. *Assume the notation of the preceding lemma. Let* $v_k \in L^1_*(B^0_k \cap B^1_k)$, $1 \leqslant k \leqslant n-1$, *and let* $v_n \in L^1_*(B^0_n + B^1_n)$ *such that*

(a) v_n *is* $B^0_n \cap B^1_n$*-continuous;*

(b) $$\int_0^\infty v_n(t)\frac{dt}{t} \in B^0_n \cap B^1_n.$$

Let $a_k = \int_0^\infty v_k(t)\frac{dt}{t}$, $1 \leqslant k \leqslant n$. *Then*

$$T(a_1, \ldots, a_n) = \int_0^\infty \ldots \int_0^\infty T\big(v_1(t_1), \ldots, v_n(t_n)\big)\frac{dt_1}{t_1}\ldots\frac{dt_n}{t_n},$$

the n-fold multiple integral converging in $C^0 + C^1$.

Proof. By (a), (b), and the corollary to Theorem 3.7.4 of [5], we see that there exists a sequence $\{f_{n,m}\}$ of finitely valued, strongly $B^0_n \cap B^1_n$ measurable functions such that

$$(1) \qquad \int_0^\infty \|v_n(t) - f_{n,m}(t)\|_{B^0_n + B^1_n}\frac{dt}{t} \to 0 \quad \text{as} \quad m \to \infty.$$

Also, since $v_k \in L^1_*(B^0_k \cap B^1_k)$, $1 \leqslant k \leqslant n-1$, we see that there exists a sequence $\{f_{k,m}\}$ of finitely valued, strongly $B^0_k \cap B^1_k$-measurable functions such that

$$(2) \qquad \int_0^\infty \|v_k(t) - f_{k,m}(t)\|_{B^0_k \cap B^1_k}\frac{dt}{t} \to 0 \quad \text{as} \quad m \to \infty.$$

The lemma now follows by (1), (2), Lemma 2.1, and since the result evidently holds for functions of the form $f_{k,m}$, $1 \leqslant k \leqslant n$, $1 \leqslant m < \infty$.

The following lemma can be obtained by transforming Lemma 3.1, Chapter 1 of [8] from $(-\infty, \infty)$ to $(0, \infty)$.

LEMMA 2.3. *Let* (B^0, B^1) *be an interpolation pair, and let* $0 < s < 1$, $1 \leqslant p \leqslant \infty$. *Then if* $x \in B_{s,p}$,

$$\|x\|_{s,p} = \inf\left(\|t^{-s}u\|^{1-s}_{L^p_*(B^0)}\|t^{1-s}u\|^s_{L^p_*(B^1)}\right),$$

the infimum taken over all strongly measurable $B^0 \cap B^1$*-valued functions u satisfying* $t^{-s}u \in L^p_*(B^0)$, $t^{1-s}u \in L^p_*(B^1)$, *and* $x = \int_0^\infty u(t)\frac{dt}{t}$.

We now turn to our general multilinear theorem.

THEOREM 2.4. *Let* (B^0_k, B^1_k), (C^0, C^1) *be interpolation pairs,* $1 \leqslant k \leqslant n$.

Let T be a multilinear operator from $\bigoplus_{k=1}^n B^0_k \cap B^1_k$ *into* $C^0 \cap C^1$ *such that*

$$\|T(x_1, \ldots, x_n)\|_{C^j} \leqslant M_j \prod_{k=1}^n \|x_k\|_{B^j_k},$$

$j = 0, 1$, *for all* $(x_1, x_2, \ldots, x_n) \in \bigoplus_{k=1}^n B^0_k \cap B^1_k$. *Let* $0 < s < 1$, $1 \leqslant p_k \leqslant \infty$, *and suppose* $1/r = \sum_{k=1}^n 1/p_k - n + 1 \geqslant 0$.

Then

$$\|T(x_1, \ldots, x_n)\|_{(C^0,C^1)_{s,r}} \leqslant M_0^{1-s}M_1^s \prod_{k=1}^n \|x_k\|_{(B^0_k,B^1_k)_{s,p_k}},$$

for all $(x_1, \ldots, x_n) \in \bigoplus_{k=1}^n B^0_k \cap B^1_k$. *In particular, if* $p_k < \infty$, $1 \leqslant k \leqslant n$, *then T has a unique extension as a bounded multilinear operator from* $\bigoplus_{k=1}^n (B_k, B^1_k)_{s,p_k}$ *into* $(C^0, C^1)_{s,r}$ *of norm* $\leqslant M_0^{1-s}M_1^s$.

In the special case $B^0_k \subseteq B^1_k$, $1 \leqslant k \leqslant n$, this result may be found in [8], Chapter 1, Theorem 4.1.

Proof. For simplicity, we write $(C^0, C^1)_{s,r} = C_{s,r}$ and use the somewhat abusive notation $(B^0_k, B^1_k)_{s,p_k} = B_{s,p_k}$. Since $1/r = \sum_{k=1}^n \frac{1}{p_k} - n + 1 \geqslant 0$ and $1 \leqslant p_k \leqslant \infty$, we see that there exists at most one k such that $p_k = \infty$. Hence, without loss of generality, we may assume $1 \leqslant p_k < \infty$, $1 \leqslant k \leqslant n-1$ and $1 \leqslant p_n \leqslant \infty$.

Let $0 < \gamma < 1$ be fixed, and let $(x_1, \ldots, x_n) \in \bigoplus_{k=1}^n B^0_k \cap B^1_k$. Clearly, we may assume $x_k \neq 0$ for all k. By [8], Chapter 1, Lemma 2.1, transformed from $(-\infty, \infty)$ to $(0, \infty)$, we see that there exist functions u_k with values in $B^0_k \cap B^1_k$, and continuous with respect to the norm of $B^0_k \cap B^1_k$ such that

$$t^{-s}u_k \in L^{p_k}_*(B^0_k), \qquad t^{1-s}u_k \in L^{p_k}_*(B^1_k),$$

$$(1) \qquad x_k = \int_0^\infty u_k(t)\frac{dt}{t},$$

$$\max\left(\|t^{-s}u_k\|_{L^{p_k}_*(B^0_k)}, \|t^{1-s}u_k\|_{L^{p_k}_*(B^1_k)}\right) < (1+\gamma)\|x_k\|_{B_{s,p_k}}.$$

Now for $0 < \varepsilon < 1$, and $1 \leqslant k \leqslant n-1$ define

$$(2) \qquad u_{k,\varepsilon}(t) = \begin{cases} u_k(t) & \text{if} \quad \varepsilon \leqslant t \leqslant 1/\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Define

$$(3)\qquad x_\varepsilon^k = \int_0^\infty u_{k,\varepsilon}(t)\frac{dt}{t}.$$

Since u_k is $B_k^0\cap B_k^1$-continuous, it follows by (2) that the integral defining x_ε^k converges in the norm of $B_k^0\cap B_k^1$, $1\leqslant k\leqslant n-1$.

It is evident that for $0<\varepsilon<1$, $1\leqslant k\leqslant n-1$, and $j=0,1$,

$$(4)\qquad \|u_{k,\varepsilon}(t)\|_{B_k^j}\leqslant \|u_k(t)\|_{B_k^j}\quad \text{for all } t\in(0,\infty).$$

Thus since $p_k<\infty$, $1\leqslant k\leqslant n-1$, we see by the dominated convergence theorem that for $1\leqslant k\leqslant n-1$,

$$\max\big[\|t^{-s}(u_k-u_{k,\varepsilon})\|_{L_*^{p_k}(B_k^0)},\|t^{1-s}(u_k-u_{k,\varepsilon})\|_{L_*^{p_k}(B_k^1)}\big]\to 0\quad\text{as}\quad \varepsilon\to 0,$$

$$(5)\qquad \|x_k-x_\varepsilon^k\|_{B_{s,p_k}}\to 0\quad\text{as}\quad \varepsilon\to 0.$$

Moreover, by (4) we have

$$(6)\qquad \begin{aligned}\|t^{-s}u_{k,\varepsilon}\|_{L_*^{p_k}(B_k^0)}&\leqslant \|t^{-s}u_k\|_{L_*^{p_k}(B_k^0)},\\ \|t^{1-s}u_{k,\varepsilon}\|_{L_*^{p_k}(B_k^1)}&\leqslant \|t^{1-s}u_k\|_{L_*^{p_k}(B_k^1)}.\end{aligned}$$

Choose ε_k with $0<\varepsilon_k<1$, $1\leqslant k\leqslant n-1$. To simplify notation we write

$$(7)\qquad \begin{aligned} x_{\varepsilon_k}^k &= a_k,\ 1\leqslant k\leqslant n-1\quad\text{and}\quad x_n=a_n,\\ u_{k,\varepsilon_k}&=v_k,\ 1\leqslant k\leqslant n-1\quad\text{and}\quad u_n=v_n.\end{aligned}$$

Since, by (2) and (7), the integral defining a_k converges in the norm of $B_k^0\cap B_k^1$, $1\leqslant k\leqslant n-1$, and since the integral defining a_n converges in the norm of $B_n^0+B_n^1$ and satisfies the hypotheses of Lemma 2.2, we have

$$(8)\qquad T(a_1,a_2,\ldots,a_n) = \int_0^\infty\ldots\int_0^\infty T\big(v_1(t_1),v_2(t_2),\ldots,v_n(t_n)\big)\frac{dt_1}{t_1}\frac{dt_2}{t_2}\ldots\frac{dt_n}{t_n},$$

the n-fold multiple integral converging in C^0+C^1.

Define a function w as the $(n-1)$-fold multiple integral

$$(9)\qquad w(t) = \int_0^\infty\ldots\int_0^\infty T\left(v_1\left(\frac{t}{t_1}\right),v_2\left(\frac{t_1}{t_2}\right),\ldots,v_{n-1}\left(\frac{t_{n-2}}{t_{n-1}}\right),v_n(t_{n-1})\right)\frac{dt_1}{t_1}\ldots\frac{dt_{n-1}}{t_{n-1}}.$$

Then by (9), the assumptions concerning the multilinear character and boundedness of T, and Young's inequality on convolutions, we obtain

$$(10)\qquad \begin{aligned}\|t^{-s}w\|_{L_*^r(C^0)} &\leqslant M_0\left\|t^{-s}\int_0^\infty\ldots\int_0^\infty\left\|v_1\left(\frac{t}{t_1}\right)\right\|_{B_1^0}\ldots\left\|v_{n-1}\left(\frac{t_{n-2}}{t_{n-1}}\right)\right\|_{B_{n-1}^0}\|v_n(t_{n-1})\|_{B_n^0}\frac{dt_1}{t_1}\ldots\frac{dt_{n-1}}{t_{n-1}}\right\|_{L_*^r}\\ &\leqslant M_0\prod_{k=1}^n\|t^{-s}v_k\|_{L_*^{p_k}(B_k^0)}<\infty.\end{aligned}$$

Similarly,

$$(11)\qquad \|t^{1-s}w\|_{L_*^r(C^1)}\leqslant M_1\prod_{k=1}^n\|t^{1-s}v_k\|_{L_*^{p_k}(B_k^1)}<\infty.$$

Also, it follows by (9), Lemma 2.2, and the translation invariance of the Haar measure $\frac{dt}{t}$ on $(0,\infty)$ that

$$(12)\qquad T(a_1,a_2,\ldots,a_n) = \int_0^\infty w(t)\frac{dt}{t}.$$

Hence $T(a_1,a_2,\ldots,a_n)\in C_{s,r}$ by (10), (11), and (12). Moreover, by Lemma 2.3, (10), and (11) we have

$$(13)\qquad \begin{aligned}\|T(a_1,a_2,\ldots,a_n)\|_{C_{s,r}} &\leqslant \big(\|t^{-s}w\|_{L_*^r(C^0)}\big)^{1-s}\big(\|t^{1-s}w\|_{L_*^r(C^1)}\big)^s\\ &\leqslant\left(M_0\prod_{k=1}^n\|t^{-s}v_k\|_{L_*^{p_k}(B_k^0)}\right)^{1-s}\left(M_1\prod_{k=1}^n\|t^{1-s}v_k\|_{L_*^{p_k}(B_k^1)}\right)^s\\ &= M_0^{1-s}M_1^s\prod_{k=1}^n\big(\|t^{-s}v_k\|_{L_*^{p_k}(B_k^0)}\big)^{1-s}\big(\|t^{1-s}v_k\|_{L_*^{p_k}(B_k^1)}\big)^s.\end{aligned}$$

Therefore, by (1), (6) and (7), it follows that

$$\|T(a_1,\ldots,a_n)\|_{C_{s,r}}\leqslant M_0^{1-s}M_1^s(1+\gamma)^n\prod_{k=1}^n\|x_k\|_{B_{s,p_k}}.$$

Thus by (7) and the above inequality,

$$(14)\qquad \|T(x_{\varepsilon_1}^1,x_{\varepsilon_2}^2,\ldots,x_{\varepsilon_{n-1}}^{n-1},x_n)\|_{C_{s,r}}\leqslant M_0^{1-s}M_1^s(1+\gamma)^n\prod_{k=1}^n\|x_k\|_{B_{s,p_k}},$$

the above inequality valid for $0<\varepsilon_k<1$, $1\leqslant k\leqslant n-1$.

Now if $1 \leqslant i \leqslant n-1$ and $0 < \alpha, \beta < 1$, we define

$$\nu(i, \alpha, \beta) = \max[\|t^{-s}(u_{i,\alpha}-u_{i,\beta})\|_{L_*^{p_i}(B_i^0)}, \|t^{1-s}(u_{i,\alpha}-u_{i,\beta})\|_{L_*^{p_i}(B_i^1)}]. \tag{15}$$

Note that by (5) we have

$$\nu(i, \alpha, \beta) \to 0 \quad \text{as} \quad \alpha, \beta \to 0. \tag{16}$$

By an argument similar to that leading to inequality (14) (see, in particular (13), where now the ith term is treated separately from the other terms in the product), and by (15), we see that for $0 < \varepsilon_k < 1$ and $0 < \alpha, \beta < 1$,

$$\|T(x_{\varepsilon_1}^1, \ldots, x_\alpha^i - x_\beta^i, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n)\|_{C_{s,r}} \leqslant N\nu(i, \alpha, \beta) \prod_{\substack{k=1\\k\neq i}}^{n} \|x_k\|_{B_{s,p_k}} \tag{17}$$

for $1 \leqslant i \leqslant n-1$, where N may be chosen as $2^n M_0^{1-s} M_1^s$ (recall $0 < \gamma < 1$).

Choose $R > 0$ such that $\|x_k\|_{B_{s,p_k}} \leqslant R$ for $1 \leqslant k \leqslant n$. Then by (17),

$$\|T(x_{\varepsilon_1}^1, \ldots, x_\alpha^i - x_\beta^i, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n)\|_{C_{s,r}} \leqslant K\nu(i, \alpha, \beta), \tag{18}$$

for $0 < \varepsilon_k < 1$, $1 \leqslant k \leqslant n-1$, $k \neq i$ and $0 < \alpha, \beta < 1$, where $K = NR^{n-1}$.

We now show how (14), (16), and (18) yield our theorem. First take $i = 1$ in (18). By (16) and (18) we see that $\{T(x_\alpha^1, x_{\varepsilon_2}^2, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n)\}$ converges in $C_{s,r}$ as $\alpha \to 0$, and so converges in $C^0 + C^1$ as $\alpha \to 0$.

But by (5), $x_\alpha^1 \to x_1$ in B_{s,p_1}, so $x_\alpha^1 \to x_1$ in $B_1^0 + B_1^1$. By Lemma 2.1,

$$T(x_\alpha^1, x_{\varepsilon_2}^2, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n) \to T(x_1, x_{\varepsilon_2}^2, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n)$$

in $C^0 + C^1$ as $\alpha \to 0$. Therefore,

$$T(x_\alpha^1, x_{\varepsilon_2}^2, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n) \to T(x_1, x_{\varepsilon_2}^2, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n) \tag{19}$$

as $\alpha \to 0$, the convergence in $C_{s,r}$.

Note that (19) is valid for any fixed, but arbitrarily chosen ε_k with $0 < \varepsilon_k < 1$, $2 \leqslant k \leqslant n-1$. Hence by (14) and (19) we obtain

$$\|T(x_1, x_{\varepsilon_2}^2, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n)\|_{C_{s,r}} \leqslant M_0^{1-s} M_1^s (1+\gamma)^n \prod_{k=1}^{n} \|x_k\|_{B_{s,p_k}}. \tag{20}$$

We now fix $i \geqslant 2$ in (18), fix $0 < \varepsilon_k < 1$ for $2 \leqslant k \leqslant n-1$, $k \neq i$, and fix $0 < \alpha, \beta < 1$. By (19),

$$\begin{aligned} &T(x_{\varepsilon_1}^1, x_{\varepsilon_2}^2, \ldots, x_\alpha^i - x_\beta^i, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n) \\ &= T(x_{\varepsilon_1}^1, \ldots, x_\alpha^i, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n) - T(x_{\varepsilon_1}^1, \ldots, x_\beta^i, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n) \\ &\to T(x_1, x_{\varepsilon_2}^2, \ldots, x_\alpha^i - x_\beta^i, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n) \quad \text{as} \quad \varepsilon_1 \to 0, \end{aligned} \tag{21}$$

the convergence in $C_{s,r}$.

Hence, letting $\varepsilon_1 \to 0$ in (18), we see by (21) that

$$\|T(x_1, x_{\varepsilon_2}^2, \ldots, x_\alpha^i - x_\beta^i, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n)\|_{C_{s,r}} \leqslant K\nu(i, \alpha, \beta), \tag{22}$$

for all $0 < \varepsilon_k < 1$, $2 \leqslant i \leqslant n-1$, $2 \leqslant k \leqslant n-1$, and $0 < \alpha, \beta < 1$.

We now repeat the procedure which led from (14) to (20), and from (18) to (22), now using (22) in place of (18) and (20) in place of (14). The result is

$$\|T(x_1, x_2, x_{\varepsilon_3}^3, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n)\|_{C_{s,r}} \leqslant M_0^{1-s} M_1^s (1+\gamma)^n \prod_{k=1}^{n} \|x_k\|_{B_{s,p_k}} \tag{23}$$

and

$$\|T(x_1, x_2, x_{\varepsilon_3}^3, \ldots, x_\alpha^i - x_\beta^i, \ldots, x_{\varepsilon_{n-1}}^{n-1}, x_n)\|_{C_{s,r}} \leqslant K\nu(i, \alpha, \beta), \tag{24}$$

for all $0 < \varepsilon_k < 1$, $3 \leqslant k \leqslant n-1$, $3 \leqslant i \leqslant n-1$, and $0 < \alpha, \beta < 1$.

It is clear that after $n-1$ steps we obtain

$$\|T(x_1, x_2, \ldots, x_n)\|_{C_{s,r}} \leqslant M_0^{1-s} M_1^s (1+\gamma)^n \prod_{k=1}^{n} \|x_k\|_{B_{s,p_k}}. \tag{25}$$

The theorem follows by (25), the fact that $\gamma > 0$ was arbitrary, and by recalling that $B_{s,p_k} = (B_k^0, B_k^1)_{s,p_k}$.

Comment. The preceding proof could be simplified if we knew the validity of the following result: The elements x_ε^k (see equation (3) in the proof of 2.4) satisfy

$$\|x_\varepsilon^k\|_{B_{s,p_k}} = \inf[\|t^{-s}v\|_{L_*^{p_k}(B_k^0)}^{1-s} \|t^{1-s}v\|_{L_*^{p_k}(B_k^1)}^{s}],$$

the infimum taken over all $B_k^0 \cap B_k^1$-valued functions v with *compact support* in $(0, \infty)$, which are continuous on their support, and which satisfy the additional two properties $x_\varepsilon^k = \int_0^\infty v(t)\frac{dt}{t}$ and $t^{j-s}v \in L_*^{p_k}(B_k^j)$, $j = 0, 1$. (Essentilly this requires the "smooth elements" x_ε^k to attain their norm over "smooth functions".) We do not know whether this, in fact, occurs, and anyway, this result does not seem to be of great intrinsic interest.

We remark that Theorem 2.4 plays a crucial role in the construction of multiplier transformations which are of weak type (p, p), but which are not bounded on L_p. Here $1 < p < 2$ (see [12]).

Corollary 2.5. *Assume the notations of Theorem 2.4. Suppose that* $0 \leqslant 1/q \leqslant \sum_{k=1}^{n} 1/p_k - n + 1$. *Then there exists a constant* $K > 0$, *depending only on the* p_k, q, n, *and* s, *such that for all* $(x_1, \ldots, x_n) \in \bigoplus_{k=1}^{n} B_k^0 \cap B_k^1$ *we have*

$$\|T(x_1, x_2, \ldots, x_n)\|_{(C^0, C^1)_{s,q}} \leqslant K M_0^{1-s} M_1^s \prod_{k=1}^{n} \|x_k\|_{(B_k^0, B_k^1)_{s,p_k}}.$$

Proof. Let $1/r = \sum_{k=1}^{n} 1/p_k - n + 1$. The result follows by Theorem 2.4, since for $r \leqslant q$, $(C^0, C^1)_{s,r} \subseteq (C^0, C^1)_{s,q}$, the embedding being continuous (see [1], Corollary 3.2.13).

We turn to some applications of Theorem 2.4. As in 2.4, let (B_k^0, B_k^1), $1 \leqslant k \leqslant n$, (C^0, C^1) be interpolation pairs. Let $\mathcal{M}$ denote the space of bounded multilinear mappings T of $\bigoplus_{k=1}^{n} B_k^0 \cap B_k^1$ into $C^0 + C^1$ with the norm

$$\|T\| = \sup \|T(x_1, \ldots, x_n)\|_{C^0+C^1},$$

the supremum taken over all $(x_1, \ldots, x_n) \in \bigoplus_{k=1}^{n} B_k^0 \cap B_k^1$ such that $\|x_k\|_{B_k^0 \cap B_k^1} \leqslant 1$, $1 \leqslant k \leqslant n$. We further assume $B_k^0 \cap B_k^1$ is dense in B_k^j, $j = 0, 1$, $1 \leqslant k \leqslant n$, and define $\mathcal{M}_j$ to be the collection of all bounded multilinear mappings of $\bigoplus_{k=1}^{n} B_k^j$ into C^j with the norm $\|T\| = \sup \|T(x_1, \ldots, x_n)\|_{C^j}$, the supremum taken over all $(x_1, \ldots, x_n) \in \bigoplus_{k=1}^{n} B_k^0 \cap B_k^1$ such that $\|x_k\|_{B_k^j} \leqslant 1$, $1 \leqslant k \leqslant n$, $j = 0, 1$.

Then $\mathcal{M}$ is a Banach space and $\mathcal{M}_0, \mathcal{M}_1$ are Banach spaces continuously embedded in $\mathcal{M}$ (see [2], Section 10.2). We now have the following theorem, which is the analogue for real interpolation spaces of a result of Calderón (see [2], Section 10.2).

THEOREM 2.6. *Let (B_k^0, B_k^1), (C^0, C^1) be interpolation pairs such that $B_k^0 \cap B_k^1$ is dense in B_k^j, $j = 0, 1$, $1 \leqslant k \leqslant n$. Let $\mathcal{M}$, $\mathcal{M}_0$ and $\mathcal{M}_1$ be as in the preceding paragraph. Let $0 < s < 1$, $1 \leqslant q \leqslant \infty$, $1 \leqslant l < \infty$, and $1 \leqslant p_k \leqslant \infty$. Suppose $0 \leqslant 1/q \leqslant \sum_{k=1}^{n} 1/p_k + 1/l - n$. Then if $T \in (\mathcal{M}_0, \mathcal{M}_1)_{s,l}$, there exists a constant $K > 0$, depending only on the p_k, q, l, n, and s, such that*

$$(1) \qquad \|T(x_1, \ldots, x_n)\|_{(C^0,C^1)_{s,q}} \leqslant K \|T\|_{(\mathcal{M}_0,\mathcal{M}_1)_{s,l}} \prod_{k=1}^{n} \|x_k\|_{(B_k^0,B_k^1)_{s,p_k}},$$

for all $(x_1, \ldots, x_n) \in \bigoplus_{k=1}^{n} B_k^0 \cap B_k^1$. In particular, if $p_k < \infty$ for all k, then T has a unique extension (again denoted by T) so that (1) remains valid for all $(x_1, \ldots, x_n) \in \bigoplus_{k=1}^{n} (B_k^0, B_k^1)_{s,p_k}$.

Proof. Let $L(x_1, x_2, \ldots, x_n, T) = T(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n, T) \in \bigoplus_{k=1}^{n} (B_k^0 \cap B_k^1) \oplus (\mathcal{M}_0 \cap \mathcal{M}_1)$. We now argue as in the proof of 10.2 of [2], using Corollary 2.5.

We now obtain a multilinear analogue of the classical interpolation theorem of Marcinkiewicz. We will adhere to the following notation.

NOTATION 2.7. Let $(\Omega_k, \Sigma_k, \mu_k)$ and (Ω, Σ, μ) denote σ-finite measure spaces, $1 \leqslant k \leqslant n$. We denote by $\mathcal{S}_k$ the integrable simple functions on Ω_k, and by $\mathcal{M}$ the measurable functions on Ω. Let $1 \leqslant p_0^k \neq p_1^k \leqslant \infty$, $1 \leqslant k \leqslant n$, $1 \leqslant q_0 \neq q_1 \leqslant \infty$, and $0 < s < 1$. Define $1/p_k = (1-s)/p_0^k + s/p_1^k$ and $1/q = (1-s)/q_0 + s/q_1$. Finally, we recall the definition $L_{\infty,q} = L_\infty$, $1 \leqslant q \leqslant \infty$.

We require the following preliminary lemma.

LEMMA 2.8. *Assume the notations of 2.7 (and Section 1). Let T be a multilinear operator from $\bigoplus_{k=1}^{n} \mathcal{S}_k$ into $\mathcal{M}$ such that*

$$\|T(f_1, \ldots, f_n)\|^*_{L_{q_j,\infty}(\mu)} \leqslant M_j \prod_{k=1}^{n} \|f_k\|^*_{L_{p_j^k,1}(\mu_k)},$$

for all $(f_1, \ldots, f_n) \in \bigoplus_{k=1}^{n} \mathcal{S}_k$, $j = 0, 1$. Then

$$\|T(f_1, \ldots, f_n)\|_{L_{q,\infty}(\mu)} \leqslant K M_0^{1-s} M_1^s \prod_{k=1}^{n} \|f_k\|_{L_{p_k,1}(\mu_k)},$$

for all $(f_1, \ldots, f_n) \in \bigoplus_{k=1}^{n} \mathcal{S}_k$, where K is a constant depending only on s, n, p_j^k and q_j, $1 \leqslant k \leqslant n$, $j = 0, 1$.

Proof. We begin by noting that if $1 \leqslant r \leqslant \infty$, $1 \leqslant k \leqslant n$, and if $E \in \Sigma_k$ has finite, non-zero μ_k-measure, then

$$(1) \qquad \begin{aligned} \|\chi_E\|^*_{r,1} &= r\mu_k(E)^{1/r} \quad \text{if} \quad r < \infty, \\ \|\chi_E\|^*_{\infty,1} &= 1, \end{aligned}$$

where χ_E is the characteristic function of the set E. (Recall we define $L_{\infty,1}$ to be L_∞.)

Suppose first $p_j^k < \infty$ for all k and j. Let E_k have finite non-zero μ_k-measure, $1 \leqslant k \leqslant n$. By (1), it follows easily that

$$(2) \qquad T(\chi_{E_1}, \ldots, \chi_{E_n})^*(t) \leqslant C_j M_j t^{-1/q_j} \prod_{k=1}^{n} \mu_k(E_k)^{1/p_j^k},$$

where C_j is a constant depending only on the p_j^k and n. (Note this holds even if q_0 or $q_1 = \infty$, provided we define $t^{-1/\infty} = 1$.)

Therefore, by (2) we obtain

$$\begin{aligned} T(\chi_{E_1}, \ldots, \chi_{E_n})^*(t) &= [T(\chi_{E_1}, \ldots, \chi_{E_n})^*(t)]^{1-s} [T(\chi_{E_1}, \ldots, \chi_{E_n})^*(t)]^s \\ &\leqslant C M_0^{1-s} M_1^s t^{-1/q} \prod_{k=1}^{n} \mu_k(E_k)^{1/p_k}, \end{aligned}$$

where $C = C_0^{1-s}C_1^s$. It follows by (1) that

(3) $$\|T(\chi_{E_1}, \ldots, \chi_{E_n})\|^*_{q,\infty} \leqslant K_0 M_0^{1-s} M_1^s \prod_{k=1}^n \|\chi_{E_k}\|^*_{p_k,1},$$

where K_0 depends only on p_j^k and p_k, $1 \leqslant k \leqslant n$, $j = 0, 1$. It is also clear that a similar argument applies in case $p_j^k = \infty$ for some k and j, for an appropriate constant K_0.

Since $1 < q < \infty$, we see by [3], Theorem 6, that there exists $K_1 > 0$, depending only on q and K_0, so that

(4) $$\|T(\chi_{E_1}, \ldots, \chi_{E_n})\|_{L_{q,\infty}(\mu)} \leqslant K_1 M_0^{1-s} M_1^s \prod_{k=1}^n \|\chi_{E_k}\|_{L_{p_k,1}(\mu_k)}$$

for all $E_k \in \Sigma_k$ with finite, positive, μ_k-measure.

We show that (4) yields the general result by an argument which is well known in the case of linear operators (see [3], Theorem 7). Let $f_1 \in \mathcal{S}_1$ with $f_1 \geqslant 0$ on Ω_1. We write $f_1 = \sum_{m=1}^N a_m \chi_{A_m}$, where $A_m \in \Sigma_1$ has finite, non-zero μ_1-measure, $a_m > 0$, $1 \leqslant m \leqslant N$, and $f_1^* = \sum_{m=1}^N a_m \chi^*_{A_m}$. Then $f_1^{**} = \sum_{m=1}^N a_m \chi^{**}_{A_m}$ and $\|f_1\|_{L_{p_1,1}(\mu_1)} = \sum_{m=1}^N a_m \|\chi_{A_m}\|_{L_{p_1,1}(\mu_1)}$. Then by (4),

$$\|T(f_1, \chi_{E_2}, \ldots, \chi_{E_n})\|_{L_{q,\infty}(\mu)} \leqslant \sum_{m=1}^N a_m \|T(\chi_{A_m}, \chi_{E_2}, \ldots, \chi_{E_n})\|_{L_{q,\infty}(\mu)}$$

(5) $$\leqslant K_1 M_0^{1-s} M_1^s \Big(\prod_{k=2}^n \|\chi_{E_k}\|_{L_{p_k,1}(\mu_k)}\Big)\Big(\sum_{m=1}^N a_m \|\chi_{A_m}\|_{L_{p_1,1}(\mu_1)}\Big)$$

$$= K_1 M_0^{1-s} M_1^s \|f_1\|_{L_{p_1,1}(\mu_1)} \prod_{k=2}^n \|\chi_{E_k}\|_{L_{p_k,1}(\mu_k)},$$

where $E_k \in \Sigma_k$ has finite, non-zero μ_k-measure.

Now let $f_2 \in \mathcal{S}_2$ with $f_2 \geqslant 0$ on Ω_2. By the argument leading to (5), now using (5) in place of (4), we obtain

(6) $$\|T(f_1, f_2, \chi_{E_3}, \ldots, \chi_{E_n})\|_{L_{q,\infty}(\mu)} \leqslant K_1 M_0^{1-s} M_1^s \|f_1\|_{L_{p_1,1}(\mu_1)} \|f_2\|_{L_{p_2,1}(\mu_2)} \prod_{k=3}^n \|\chi_{E_k}\|_{L_{p_k,1}(\mu_k)},$$

whenever $E_k \in \Sigma_k$, with $0 < \mu_k(E_k) < \infty$, $3 \leqslant k \leqslant n$. After n steps we obtain

(7) $$\|T(f_1, \ldots, f_n)\|_{L_{q,\infty}(\mu)} \leqslant K_1 M_0^{1-s} M_1^s \prod_{k=1}^n \|f_k\|_{L_{p_k,1}(\mu_k)},$$

for all $f_k \in \mathcal{S}_k$ with $f_k \geqslant 0$, $1 \leqslant k \leqslant n$.

The lemma follows easily by (7).

THEOREM 2.9. *Assume the notations of 2.7 (and Section 1). Moreover, suppose* $0 \leqslant 1/q \leqslant \sum_{k=1}^n 1/p_k - n + 1$. *Let* T *be a multilinear operator from* $\bigoplus_{k=1}^n \mathcal{S}_k$ *into* $\mathcal{M}$ *such that*

$$\|T(f_1, \ldots, f_n)\|^*_{L_{q_j,\infty}(\mu)} \leqslant M_j \prod_{k=1}^n \|f_k\|^*_{L_{p_j^k,1}(\mu_k)},$$

for all $(f_1, \ldots, f_n) \in \bigoplus_{k=1}^n \mathcal{S}_k$. *Then*

(*) $$\|T(f_1, \ldots, f_n)\|_{L_q(\mu)} \leqslant K M_0^{1-s} M_1^s \prod_{k=1}^n \|f_k\|_{L_{p_k}(\mu_k)},$$

for all $(f_1, \ldots, f_n) \in \bigoplus_{k=1}^n \mathcal{S}_k$, *where* K *is a constant depending only on the* p_j^k, q_j, n *and* s. *In particular,* T *has a unique extension to* $\bigoplus_{k=1}^n L_{p_k}(\mu_k)$ *satisfying* (*).

Proof. We assume $1 \leqslant q_1 < q_0 \leqslant \infty$, so that $q_1 < q < q_0$. Choose q_0', q_1' such that $1 \leqslant q_1 < q_1' < q < q_0' < q_0 \leqslant \infty$. Choose s_0 and s_1 satisfying $0 < s_0 < s < s_1 < 1$ and

(1) $$\frac{1}{q_j'} = \frac{1-s_j}{q_0} + \frac{s_j}{q_1},$$

$j = 0, 1$. Let $1 < r_j^k < \infty$ be defined by

(2) $$\frac{1}{r_j^k} = \frac{1-s_j}{p_0^k} + \frac{s_j}{p_1^k},$$

$j = 0, 1$, $1 \leqslant k \leqslant n$.

Let $0 < \theta < 1$ so that $s = (1-\theta)s_0 + \theta s_1$. Then

(3) $$\frac{1}{p_k} = \frac{1-\theta}{r_0^k} + \frac{\theta}{r_1^k}, \quad 1 \leqslant k \leqslant n \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0'} + \frac{\theta}{q_1'}.$$

By Lemma 2.8, (1) and (2) we obtain

(4) $$\|T(f_1, \ldots, f_n)\|_{L_{q_j',\infty}(\mu)} \leqslant K_1 M_0^{1-s_j} M_1^{s_j} \prod_{k=1}^n \|f_k\|_{L_{r_j^k,1}(\mu_k)},$$

$j = 0, 1$, for all $(f_1, \ldots, f_n) \in \bigoplus_{k=1}^n \mathcal{S}_k$, where K_1 is an appropriate constant. It is easy to see that T has a unique extension so that (4) remains valid for all $(f_1, \ldots, f_n) \in \bigoplus_{k=1}^n L_{r_0^k,1} \cap L_{r_1^k,1}$. (Note that this holds even if r_0^k or r_1^k equals ∞.)

Since $L_{q_j',\infty}(\mu)$ and $L_{r_j^k,1}(\mu_k)$ are Banach spaces, Theorem 2.4 (and Corollary 2.5) are directly applicable. Thus by (4), the interpolation properties of Lorentz spaces (see [1], Chapter 3, Theorem 3.3.10), and by the definition of θ we obtain

$$\|T(f_1,\ldots,f_n)\|_{L_q(\mu)} \leqslant KM_0^{1-s}M_1^s \prod_{k=1}^n \|f_k\|_{L_{p_k}(\mu_k)},$$

for all $(f_1,\ldots,f_n) \in \bigoplus_{k=1}^n \mathscr{S}_k$, where K is as in the statement of the theorem. This completes the proof.

3. In this section, we obtain a Marcinkiewicz-type interpolation theorem for the H^p spaces, $0<p<\infty$. Our result is analogous to the Riesz-type theorem for these spaces, as obtained by Calderón and Zygmund (see [13], Chapter 12, Section 3). In the theorem of these aforementioned authors, the point of departure is a multilinear analogue of the Riesz convexity theorem. Similarly, the basis of our result is the multilinear theory of Section 2. Our principal result, Theorem 3.2, was obtained some years ago. In the intervening period, this theorem has been subsumed by a much stronger result of Fefferman, Riviere and Sagher [4]. We include a proof primarily as an application of the results of Section 2, and we will thus be brief in our presentation.

We begin by recalling the definition of the H^p spaces. If $0<p<\infty$, we define H^p as the collection of all functions f, analytic in the unit disc, for which

$$\|f\|_{H^p} = \sup_{0\leqslant r<1}\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} < \infty.$$

In case $p \geqslant 1$, H^p is isometric and isomorphic to $\{f\in L_p \mid \hat{f}(n) = 0 \text{ for } n<0\}$. The basic facts concerning these spaces may be found in Chapter 7 of [13].

NOTATION 3.1. Let (Ω, Σ, μ) denote a σ-finite measure space, and let $\mathscr{M}$ denote the measurable functions on Ω. Denote by $\mathscr{P}$ the class of all complex polynomials, restricted to the unit disc. For all real numbers $a>0$, we let $[a]$ denote the greatest integer less than or equal to a.

THEOREM 3.2. *Assume the notations of* 3.1. *Let* $0<p_0<p_1<\infty$ *and* $1\leqslant q_0 \neq q_1 \leqslant \infty$. *Let T be a linear operator from $\mathscr{P}$ into $\mathscr{M}$ such that*

$$\|T(f)\|^*_{q_j,\infty} \leqslant M_j\|f\|_{H^{p_j}},$$

for all $f\in\mathscr{P}$, $j=0,1$. *Let* $0<s<1$, *and let* $1/p = (1-s)/p_0 + s/p_1$, $f1/q = (1-s)/q_0+s/q_1$. *Suppose* $1/p$ *is not a positive integer and that* $1/q \leqslant 1/p-[1/p]$. *Then*

$$\|T(f)\|_{L_q(\mu)} \leqslant KM_0^{1-s}M_1^s\|f\|_{H^p}$$

or all $f\in\mathscr{P}$, *where K is a constant depending only on* p_j, q_j, *and* s, $j=0,1$.

We will require three preliminary lemmas.

LEMMA 3.3. *Assume the notations of* 2.7. *Let* $1\leqslant p_0^k, p_1^k \leqslant \infty$, $1\leqslant q_0 \neq q_1 \leqslant\infty$, *and let* $0<\theta<1$. *Define* $1/p_k = (1-\theta)/p_0^k + \theta/p_1^k$, $1\leqslant k\leqslant n$, *and* $1/q = (1-\theta)/q_0+\theta/q_1$. *Let T be a multilinear operator from* $\bigoplus_{k=1}^n \mathscr{S}_k$ *into* $\mathscr{M}$ *such that*

$$\|T(f_1,\ldots,f_n)\|^*_{q_j,\infty} \leqslant M_j \prod_{k=1}^n \|f_k\|_{L_{p_j^k}(\mu_k)},$$

$j=0,1$, *for all* $(f_1,\ldots,f_n)\in\bigoplus_{k=1}^n\mathscr{S}_k$. *Then*

$$\|T(f_1,\ldots,f_n)\|_{L_{q,\infty}(\mu)} \leqslant KM_0^{1-\theta}M_1^\theta \prod_{k=1}^n \|f_k\|_{L_{p_k}(\mu_k)}$$

for all $(f_1,\ldots,f_n)\in\bigoplus_{k=1}^n\mathscr{S}_k$, *where K is a constant depending only on* p_j^k, q_j *and* θ, $1\leqslant k\leqslant n$, $j=0,1$.

The proof of this result is similar to that of the strong type interpolation theorem as proved by Hunt in [6], pp. 266–269. In fact, the proof of our lemma is easier since we consider multilinear rather than sublinear operators, and since the representation (3.5), p. 266 of [6] becomes elementary under our hypotheses. Hunt's proof will also yield more general versions of Lemma 3.3, but we do not pursue these generalizations.

LEMMA 3.4. *Assume the notations of* 3.1. *Let* $0<p_0, p_1<\infty$ *and* $1\leqslant q_0\neq q_1\leqslant\infty$. *Let T be a linear operator from $\mathscr{P}$ into $\mathscr{M}$ such that*

$$\|T(f)\|^*_{q_j,\infty} \leqslant M_j\|f\|_{H^{p_j}},$$

for all $f\in\mathscr{P}$, $j=0,1$. *Let* $0<\theta<1$, *and let* $1/p=(1-\theta)/p_0+\theta/p_1$, $1/q = (1-\theta)/q_0+\theta/q_1$. *Then*

$$\|T(f)\|_{L_{q,\infty}(\mu)} \leqslant KM_0^{1-\theta}M_1^\theta\|f\|_{H^p},$$

for all $f\in\mathscr{P}$, *where K is a constant depending only on* p_j, q_j, *and* θ, $j=0,1$.

Proof. The proof of this result is very similar to that of Theorem 3.9, Chapter 12 of [13]. We now use Lemma 3.3 in place of the multilinear Riesz convexity theorem (see Theorem 3.3, Chapter 12 of [13]).

LEMMA 3.5. *Assume the notations of* 3.1. *Let* $0<p_0<p_1<\infty$ *and* $1\leqslant q_0\neq q_1\leqslant\infty$. *Let T be a linear operator from $\mathscr{P}$ into $\mathscr{M}$ such that*

$$\|T(f)\|^*_{q_j,\infty} \leqslant M_j\|f\|_{H^{p_j}}, \tag{1}$$

for all $f\in\mathscr{P}$, $j=0,1$. *Let* $0<\theta<1$, *and let* $1/p = (1-\theta)/p_0+\theta/p_1$, $1/q=(1-\theta)/q_0+\theta/q_1$. *Suppose* $1/q\leqslant 1/p-[1/p]$ *and that there exists a positive integer n such that* $n-1<1/p_1<1/p_0<n$. *Then*

$$\|T(f)\|_{L_q(\mu)} \leqslant KM_0^{1-\theta}M_1^\theta\|f\|_{H^p}$$

for all $f\in\mathscr{P}$, *where K is a constant depending only on* p_j, q_j, *and* θ, $j=0,1$.

Proof. Again the proof is similar to that of Theorem 3.9, Chapter 12 of [13]. We first extend T uniquely to H^{p_1} ($\subseteq H^{p_0}$) so that (1) remains valid. Define the multilinear mapping T^* by

(2) $$T^*(g_1, \ldots, g_n) = T(F_1 F_2 \cdots F_n),$$

where

(3) $$F_k(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} g_k(t)\, dt$$

for all complex-valued, integrable simple functions g_k on $(0, 2\pi)$.

By the inequality of M. Riesz, it is easy to see that

(4) $$\|F_k\|_{H^r} \leqslant A_r \|g_k\|_{L_r}$$

for $1 < r < \infty, 1 \leqslant k \leqslant n$. In particular, $F_k \in H^{np_1}, 1 \leqslant k \leqslant n$, and $\prod_{k=1}^{n} F_k \in H^{p_1}$. Thus the operator T^* is well-defined. Moreover, by Hölder's inequality, (1) and (4) we see that

$$\|T^*(g_1, \ldots, g_n)\|^*_{q_j,\infty} \leqslant CM_j \prod_{k=1}^{n} \|g_k\|_{L_{np_j}},$$

$j = 0, 1$, for all integrable simple functions g_k on $(0, 2\pi)$, $1 \leqslant k \leqslant n$. Here C is a constant depending only on p_j and $n, j = 0, 1$.

We note also that $1/q \leqslant \sum_{k=1}^{n} 1/np - n + 1$ (since this latter term is just $1/p - [1/p]$). Thus Theorem 2.9 implies that

(5) $$\|T^*(g_1, \ldots, g_n)\|_{L_q(\mu)} \leqslant KM_0^{1-\theta} M_1^{\theta} \prod_{k=1}^{n} \|g_k\|_{L_{np}}$$

for all integrable simple functions g_k on $(0, 2\pi)$. Here K depends only on p_j, q_j, and $\theta, j = 0, 1$.

As in the proof of Theorem 3.9, Chapter 12 of [13], we may extend T^* to $\bigoplus_{k=1}^{n} L_{np_0} \cap L_{np_1}$ so that T^* is still defined by (2) with the functions F_k and g_k related by (3), and so that (5) remains valid. The remainder of the proof is identical to the argument in the aforementioned theorem.

We are finally in a position to prove the principal result of this section.

Proof of Theorem 3.2. Since $1/p$ is not a positive integer, we may choose a positive integer n so that

$$n - 1 < 1/p < n.$$

Choose p_0' and p_1' with

(1) $$n - 1 < \frac{1}{p_1'} < \frac{1}{p} < \frac{1}{p_0'} < n$$

and

(2) $$\frac{1}{p_1} < \frac{1}{p_1'} < \frac{1}{p_0'} < \frac{1}{p_0}.$$

We now pick θ_j satisfying

$$\frac{1}{p_j'} = \frac{1-\theta_j}{p_0} + \frac{\theta_j}{p_1},$$

$j = 0, 1$. Note that $0 < \theta_0 < s < \theta_1 < 1$. Define q_j' by the equations

$$\frac{1}{q_j'} = \frac{1-\theta_j}{q_0} + \frac{\theta_j}{q_1},$$

$j = 0, 1$. By Lemma 3.4, we obtain

(3) $$\|T(f)\|_{L_{q_j',\infty}(\mu)} \leqslant K_1 M_0^{1-\theta_j} M_1^{\theta_j} \|f\|_{H^{p_j'}}$$

for all $f \in \mathscr{P}, j = 0, 1$, where K_1 depends only on the p_j, q_j, and θ_j.

Now choose θ with $s = (1-\theta)\theta_0 + \theta\theta_1$. By computation we see

(4) $$\frac{1}{p} = \frac{1-\theta}{p_0'} + \frac{\theta}{p_1'} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0'} + \frac{\theta}{q_1'}.$$

Hence by (1), (3), (4), and Lemma 3.5,

$$\|T(f)\|_{L_q(\mu)} \leqslant KM_0^{1-s} M_1^{s} \|f\|_{H^p}$$

for all $f \in \mathscr{P}, j = 0, 1$, where K is as in the statement of the theorem. This concludes the proof.

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Harmonic analysis on the group of rigid motions of the Euclidean plane*

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Abstract. Aspects of Fourier analysis on $M(2)$ relevant to the study of L^p multiplier operators are developed. Relations between multiplier operators on $M(2)$ and $SO(3)$ or $SU(2)$ are studied. Applications are given to transplantation results for Bessel functions.

Introduction. The idea of considering the real line to be a limit of circles with increasingly large radii has long been used to relate Fourier analysis on the line, R, to Fourier analysis on the torus, T. In the study of multiplier operators, this idea leads to the following classical theorem: Let m be a continuous function on R. Suppose that for each $\lambda > 0$, there exists an operator M_λ acting continuously on $L^p(T)$, given by

$$M_\lambda f(x) = \sum_{n=-\infty}^{\infty} m\left(\frac{n}{\lambda}\right) a_n e^{inx},$$

where a_n is the nth Fourier coefficient of f. Assume that the operator norms $\|M_\lambda\|$ are uniformly bounded. Then m defines a bounded multiplier operator M on $L^p(R)$ ([3], p. 264).

We wish to generalize this result by replacing the torus, which may be identified with $SO(2)$, with the non-abelian group $SO(3)$, or with its universal covering group $SU(2)$, which is naturally identifiable with the unit sphere in two-dimensional complex space. By a limiting process analagous to the classical passage from the circle to the line, the group $SO(3)$ can be shown to tend to a non-compact non-abelian group: the group of rigid motions of the Euclidean plane, denoted by $M(2)$.

In this paper, we shall show how Fourier analysis on $M(2)$ is closely

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