(3), (4) give rise to the inequalities

$$\Big|1-\prod_{n=1}^{s-1}\cos\big(2\pi t(k)x_n\big)\Big|<2\pi k^{-1},$$

$$|\cos(2\pi p_r)-\cos\big(2\pi t(k)x_s\big)|<2\pi k^{-1},$$

$$\Big|1-\prod_{n=s+1}^{\infty}\cos\big(2\pi t(k)x_s\big)\Big|<2\pi k^{-1},$$

and these combine to give (6). This completes the proof of the first assertion of the theorem.

For the second part we proceed in a similar way but replace (4) by (4′),

$$\|x\|_1<\sum_{n=1}^{s}|x_n|+t^{-2}q^{-1}, \tag{4′}$$

and work with $\frac{1}{2}\big(1+\exp(2\pi i p_r)\big)$ in place of $\cos(2\pi p_r)$.

Remark. The statements of Corollaries 1, 2, 3 of Theorem 1 of [1] remain valid when the phrase "virtually all" is interpreted in the sense of "residual in l^2".

References

[1] G. Brown and W. Moran, *In general, Bernoulli convolutions have independent powers*, Studia Math. 47 (1973), pp. 141–152.

[2] C. Lin and S. Saeki, *Bernoulli convolutions in* LCA *groups*, to appear.

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Semi-stable probability measures on R^N

by

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Abstract. Let $\{\xi_n\}$ be a sequence of R^N-valued, independent and identically distributed random variables. Consider the sums

$$A_n(\xi_1+\ldots+\xi_{k_n})+c_n, \tag{0}$$

where A_n are non-singular linear operators in R^N, $c_n\in R^N$ and $k_n^{-1}k_{n+1}\to\gamma$. The limit law for sums of the form (0) is called semi-stable. The aim of this paper is to describe the class of all full semi-stable measures in R^N.

1. Introduction and notation. We begin with some notation. By M we denote the set of all Borel probability measures on the real Euclidean space R^N. We regard M as an Abelian topological semigroup with the convolution as a semigroup operation and the topology of weak convergence of measures.

We denote the convolution of two measures μ and ν by $\mu*\nu$. Throughout, the power μ^n is taken in the sense of the convolution. Moreover, by $\delta(x)$ we denote the probability measure concentrated at the point $x\in R^N$. The characteristic function (Fourier transform) $\hat{\mu}$ of a measure $\mu\in M$ is defined by the formula

$$\hat{\mu}(y)=\int_{R^N}\exp i(x,y)\mu(dx).$$

The group of all non-singular linear operators acting in R^N will be denoted by G.

For a Borel mapping $F: R^N\to R^N$ and a measure μ from M we denote by $F\mu$ the measure defined by the formula

$$F\mu(Z)=\mu(F^{-1}Z)$$

for any Borel subset Z of the space R^N. In particular, it is easy to verify that the mapping $(A,\mu)\to A\mu$ from $G\times M$ onto M is jointly continuous and the formulas

$$A(\mu*\nu)=A\mu*A\nu,\qquad \widehat{A\mu}(y)=\hat{\mu}(A^*y)$$

hold, here A^* denotes the adjoint operator. Given $\mu\in M$, we define $\bar{\mu}$ putting $\bar{\mu}(Z)=\mu(-Z)$, where $-Z=\{-y: y\in Z\}$.

For any $\mu \in M$ the measure $\mathring{\mu} = \mu * \overline{\mu}$ is called *symmetrization* of μ. A closed subset $S\mu$ of $\boldsymbol{R}^N$ is called the *support* of a measure μ if the complement of $S\mu$ has μ-measure zero and $\mu(U_x) > 0$ for any neighbourhood U_x of x, where x runs over $S\mu$. It is not difficult to see that for $\mu \in M$ the support $S\mu$ always exists and is unique. A measure μ from M is said to be *full* if its support $S\mu$ is not contained in any $(N$-$1)$-dimensional hyperplane of $\boldsymbol{R}^N$. In the excelent paper [3] M. Sharpe introduced and examined the notion of an operator-stable measure in $\boldsymbol{R}^N$. Namely, a probability distribution μ from M is called an *operator-stable measure* if it is a weak limit of measures of the form

$$\mu = \lim_{n\to\infty} A_n \nu^n * \delta(x_n),$$

where $\nu \in M$, $A_n \in G$, $x_n \in \boldsymbol{R}^N$. In [3] the class of all full operator-stable measures has been characterized. Recently, in [1], B. Kruglov considered the set of all limit distributions of the form

$$q = \lim c_n p^{k_n} * \delta(x_n),$$

where p runs over all probability measures on $\boldsymbol{R}^1$, $c_n > 0$, $x_n \in \boldsymbol{R}$ and the sequence $k_1 < k_2 < \ldots$ of positive integers is such that $k_n^{-1} k_{n+1} \to \gamma$ for some $1 \leqslant \gamma < \infty$. The author gave a description of that class of measures which is larger than the class of stable measures on $\boldsymbol{R}^1$.

Our purpose is to describe the class of all full measures in $\boldsymbol{R}^N$ which are the limit laws for sums of the form

$$A_n(\xi_1 + \ldots + \xi_{k_n}) + c_n,$$

where $\{\xi_n\}$ is a sequence of independent identically distributed random variables, $A_n \in G$, $c_n \in \boldsymbol{R}^N$ and $k_n^{-1} k_{n+1} \to \gamma$ for some $\gamma \geqslant 1$.

We can treat this problem regarding M as a metric semigroup. Then, we introduce the class of measures in question by the following

DEFINITION 1. A measure μ from M is said to be *semi-stable* if it is a weak limit of measures of the form

$$\mu = \lim_{n\to\infty} A_n \nu^{k_n} * \delta(b_n),$$

where $A_n \in G$, $\nu \in M$, $b_n \in \boldsymbol{R}^N$ and the sequence $k_1 < k_2 < \ldots$ of positive integers is such that $k_n^{-1} k_{n+1} \to \gamma$ for some $1 \leqslant \gamma < \infty$.

2. Characterization of full semi-stable measures in $\boldsymbol{R}^N$. In this section we shall prove the following theorem.

THEOREM. *A full probability measure μ from M is a semi-stable measure if and only if it is infinitely divisible and there exist a number $0 < c < 1$, a vector $b \in \boldsymbol{R}^N$ and an operator $B \in G$ such that the formula*

$$\mu^c = B\mu * \delta(b) \qquad (*)$$

holds. The spectrum of B is contained in the disc $\{|z|^2 \leqslant c\}$. Eigenvalues of B satisfying $|\lambda|^2 = c$ are simple, i.e. the elementary divisors of B associated with these eigenvalues are one-dimensional.

Furthermore, the measure μ can be decomposed into a product $\mu = \mu_1 * \mu_2$ of two measures μ_1 and μ_2, concentrated on B-invariant subspaces X_1 and X_2, respectively, and such that $\boldsymbol{R}^N = X_1 \oplus X_2$, μ_1 is a full semi-stable measure on X_1 of the Poisson-type (having no Gaussian component) and μ_2 is a full Gaussian measure on X_2. The spectrum of $B|X_1$ is then contained in the disc $\{|z|^2 < c\}$ and for the eigenvalues of $B|X_2$ the equality $|z|^2 = c$ holds.

The proof of our theorem will be preceded by several lemmas. In proofs of Lemmas 1 and 2 we use the technique developed in the fundamental work [4] by K. Urbanik.

LEMMA 1. *Let μ be a full measure for which the formula*

$$\mu = \lim_{n\to\infty} A_n \nu^{k_n} * \delta(b_n) \qquad (1)$$

holds, where $\gamma \in M$, $A_n \in G$, $b_n \in \boldsymbol{R}^N$ and $k_n^{-1} k_{n+1} \to \gamma < \infty$. Then $A_n \to \theta$.

Proof. Let us suppose the contrary and choose $z_k \in \boldsymbol{R}^N$ such that $\|z_k\| = 1$, $\|A_k{}^* z_k\| = \|A_k\|$. Without any loss of generality (passing to a subsequence, if necessary) we may assume that $\|A_k\| \to \delta > 0$, $y_k/\|y_k\| \to y$, where $y_k = A_k^* z_k$ and $z_k/\|A_k\| \to u$. By symmetrization of measures occurring in (1) we obtain

$$|\hat{\mu}(y)|^2 = \lim_{n\to\infty} |\hat{\nu}(A_n^* y)|^{2k_n}.$$

Hence $A_k \mathring{\nu}(y) \to 1 = \delta_0(y)$ almost uniformly on $\boldsymbol{R}^N$ and, consequently, $A_k \mathring{\nu}(tz_k/\|A_k\|) \to 1$ for $t \in \boldsymbol{R}$.

On the other hand, we have

$$\hat{\mathring{\nu}}\left(tA_k^* \frac{z_k}{\|A_k\|}\right) = \hat{\mathring{\nu}}\left(t\frac{y_k}{\|y_k\|}\right) \to \hat{\mathring{\nu}}(ty).$$

Thus, we get $\hat{\mathring{\nu}}(ty) = 1$ and, consequently, $\hat{\mathring{\nu}}^{k_n}(ty) = 1$ for $t \in \boldsymbol{R}$ and $y \in \boldsymbol{R}^N$.

Let us remark that $x_k = (A_k^*)^{-1} y \neq 0$ because $y \neq 0$.

Denote by $\overline{u}$ an arbitrary limit point of the sequence $x_k/\|x_k\|$. Then, since $A_n \mathring{\nu}^{k_n} \to \mathring{\mu}$, we obtain, passing to a subsequence, if necessary

$$\widehat{A_n \mathring{\nu}^{k_n}}\left(t\frac{x_n}{\|x_n\|}\right) \to \hat{\mathring{\nu}}(tu)$$

On the other hand,

$$\hat{\mathring{\nu}}^{k_n}(tA_n^*(A_n^*)^{-1}y) = \hat{\mathring{\nu}}^{k_n}(ty) \equiv 1.$$

Thus we obtain

$$\hat{\mathring{\mu}}(t\overline{u}) = 1 \quad \text{for} \quad t \in \boldsymbol{R} \text{ and } \|\overline{u}\| = 1.$$

By Proposition 1 in [3], μ is not a full measure what contradicts the assumption. The lemma is thus proved.

LEMMA 2. *If μ is a full measure for which the formula* (1) *holds, then the sequence of operators*

$$\{(A_k^*)^{-1}A_{k+1}^*\}\tag{2}$$

is precompact in G.

Moreover, if C is a limit point of the sequence (2), *then the formula*

$$\hat{\mu}(y) = [\hat{\mu}(Cy)]^\gamma e^{i(b,y)}\tag{3}$$

holds.

Proof. First, we shall show that the sequence (2) is precompact in the space of linear endomorphisms of $\boldsymbol{R}^N$.

Let us suppose the contrary, i.e. that the sequence of norms $\|(A_k^*)^{-1}A_{k+1}^*\|$ is unbounded. Let us choose vectors z_n in $\boldsymbol{R}^N$ such that $\|z_n\| = 1$ and $\|(A_k^*)^{-1}A_{k+1}^*z_k\| = \|(A_k^*)^{-1}A_{k+1}^*\|$. Taking a subsequence, if necessary, we may assume that

$$\|(A_k^*)^{-1}A_{k+1}^*\| \to \infty \quad \text{and} \quad y_k/\|y_k\| \to y, \quad \text{where} \quad y_k = (A_k^*)^{-1}A_{k+1}^*z_k.$$

Then for every $t \in \boldsymbol{R}$ we have, by (1),

$$\left|\hat{\nu}\left(tA_n^*\frac{y_n}{\|y_n\|}\right)\right|^{k_n} \to |\hat{\mu}(ty)|.$$

Simultaneously,

$$\left|\hat{\nu}\left(tA_n^*\frac{y_n}{\|y_n\|}\right)\right|^{k_n} = \left|\hat{\nu}\left(tA_{n+1}^*\frac{z_n}{\|y_n\|}\right)\right|^{k_{n+1}\frac{k_n}{k_{n+1}}} \to |\hat{\mu}(0)|^{1/\gamma} = 1$$

for every $t \in \boldsymbol{R}$. Hence it follows that the characteristic function of the symmetrized measure $\mathring{\mu}$ is equal to one on the subspace $\{tu\colon\ t \in \boldsymbol{R}\}$. Consequently (see [3], Proposition 1), μ is not a full measure, with contradicts the assumption. Now it is easy to see that the compactness of the sequence (2) in G follows immediately from (3). Indeed, since the support of $C\mu$ is contained in the image $C(\boldsymbol{R}^N)$, we infer that the operator C is non-singular. Thus it remains to prove (3).

Let C be a limit point of the sequence (2), say $C = \lim_{s\to\infty}(A_{n_s}^*)^{-1}A_{n_{s+1}}^*$. Then, by (1), we have

$$\begin{aligned}\hat{\mu}(y) &= \lim_{s\to\infty}\hat{\nu}(A_{n_{s+1}}^*y)^{k_{n_{s+1}}}e^{i(a_{n_s},y)}\\ &= \lim_{s\to\infty}\left\{\hat{\nu}\left(A_{n_s}^*(A_{n_s}^*)^{-1}A_{n_{s+1}}^*y\right)^{k_{n_s}}e^{i(a_{n_s},(A_{n_s}^*)^{-1}A_{n_{s+1}}^*y)}\right\}^{k_{n_{s+1}}/k_{n_s}}e^{i(b_s,y)}\end{aligned}$$

for some sequence $\{b_s\}$ of vectors in $\boldsymbol{R}^N$. It is almost evident that b_s converges weakly, say to a vector $b \in \boldsymbol{R}^N$.

Thus we get formula (3) and our lemma is proved.

DEFINITION 2. A Borel measure M on $\boldsymbol{R}^N$ is called a *Lévy–Khintchine spectral measure* (or briefly LK-*measure*) if M is a semi-finite measure in $\boldsymbol{R}^N$ concentrated on $\boldsymbol{R}^N - \{0\}$, finite on the complement of every neighbourhood of zero and such that

$$\int_{\|x\|<1}\|x\|^2M(dx) < \infty.\tag{4}$$

LEMMA 3. *Let M be a non-trivial LK-measure concentrated on the trajectory*

$$\tau_x = \{B^kx,\ k \in L\},$$

where $x \in \boldsymbol{R}^N$, $B \in G$, L — the set of all integers and let the formula

$$BM = cM\tag{5}$$

hold for some $0 < c < 1$. Denote by $\bar{\tau}_x$ a cyclic subspace generated by x. Then the spectrum of the reduced operator $B|\bar{\tau}_x$ is contained in the disc $\{|z|^2 < c\}$.

Proof. From (4) and (5) it follows that M is of the form

$$M(\{B^nx\}) = dc^{-n}, \quad d > 0,\ n \in L.\tag{6}$$

Indeed, we have $M(\{B^nx\}) = B^{-n}M\{x\} = c^{-n}M\{x\}$. In this case the condition (4) is equivalent to

$$\sum_{n=1}^{\infty}\|B^nx\|^2c^{-n} < \infty,\tag{7}$$

where $\|\cdot\|$ is an arbitrary norm in $\boldsymbol{R}^N$.

In the sequel the same letter β will denote a natural extension of $\beta|\bar{\tau}_x$ to a linear operator acting in the complex Euclidean space Z spanned over τ_x.

To define in (7) a suitable norm $\|\cdot\|$ we shall use the Jordan theorem about the canonical representation of linear operator acting in a complex Euclidean space. By this theorem there exist a basis $z_1, z_2, \ldots, z_n$ in Z, a system of integers $n_0, n_1, \ldots, n_k$, $0 = n_0 < n_1 < \ldots < n_k = n$ and a sequence of eigenvalues $\lambda_1, \ldots, \lambda_k$ (not necessarily different) indexed so that

$$\begin{aligned}Bz_i &= \lambda_jz_i + z_{i+1} \quad \text{for} \quad n_{j-1} < i < n_j,\\ Bz_{n_j} &= \lambda_jz_{n_j}.\end{aligned}\tag{8}$$

Let us establish a norm in Z putting

$$\|y\| = \sum_{s=1}^{n}|a_s| \quad \text{for} \quad y = \sum_{s=1}^{n}a_sz_s.\tag{9}$$

As the norm in (7) we accept the norm (9). From (8) and (9) it easily follows that each eigenvalue β of $B|Z$ satisfies

(10) $$\|B^n x\| \geqslant a|\beta|^n, \quad a > 0.$$

Comparing (10) with (7) we obtain $|\beta|^2 < c$, what ends the proof.

LEMMA 4. *If M is an LK-measure and the formula* (5) *holds for some $0 < c < 1$ and $B \in G$, then M is concentrated on a subspace $X \subset \boldsymbol{R}^N$ such that X is B-invariant and the spectrum of $B|X$ is contained in the open disc $\{|z|^2 < c\}$.*

Proof. From (5) it easily follows that the support of M is B-invariant. Thus it is a sum of trajectories τ_x.

Let us denote by $\Lambda(c,B)$ a set of LK-measures satisfying (5) and by V a real linear space of finite measures on $\boldsymbol{R}^N$ with the topology of weak convergence.

The set

$$\Omega(c,B) = \left\{N \in V\colon N(\boldsymbol{R}^N) \leqslant 1 \text{ and } N(dx) = \frac{\|x\|^2}{1+\|x\|^2} M(dx) \text{ where } M \in \Lambda(c, B)\right\}$$

is a compact convex subset of V. A support of every measure from Ω is a sum of some trajectories τ_x. Hence it follows that extreme points of Ω are measures concentrated on single trajectories τ_x. Thus the set of convex linear combinations of such measures is dense in Ω. Hence it follows that in $\Lambda(c,B)$ linear combinations of measures concentrated on single trajectories τ_x lie densely in $\Lambda(c,B)$.

Now, let

$$\boldsymbol{R}^N = X \oplus Y$$

be a decomposition of $\boldsymbol{R}^N$ into B-invariant subspaces such that

$$\text{spectrum } B|X \subset \{|z|^2 < c\},$$
$$\text{spectrum } B|Y \subset \{|z|^2 \geqslant c\}.$$

We can obtain such decomposition using the decomposition into cyclic subspaces of B.

Let $z \in \boldsymbol{R}^N$ and $z = x+y$ be a decomposition of z such that $x \in X$, $y \in Y$. The trajectory τ_z may be a support of a LK-measure if and only if $y = 0$ (comp. Lemma 3). Since linear combinations of such measures lie densely in $\Lambda(c,B)$, every spectral measure from $\Lambda(c,B)$ is concentrated on X, which completes the proof.

LEMMA 5. *If a measure μ is full and*

$$\mu = \lim_{n\to\infty} B_n \nu^{k_n} * \delta(b_n)$$

where $B_n \in G$, $b_n \in \boldsymbol{R}^N$, $k_n/k_{n+1} \to 1$, then μ is operator-stable in the sense of Sharpe.

Proof. Let us notice that by Lemma 1 the measure μ is infinitely divisible and thus its powers with any positive exponent exist. Let $a \in (0,1)$. Fix a sequence $l(n)$ of integers such that

$$\frac{k_{l(n)}}{k_n} \to a \quad \text{as} \quad n \to \infty.$$

Then

$$[\hat{\mu}(y)]^a = \lim_{n\to\infty} [\hat{\nu}(B_n^* y)]^{k_n \frac{k_{l(n)}}{k_n}} e^{ia(b_n,y)}$$
$$= \lim_{n\to\infty} \hat{\nu}(B_{l(n)}^* [(B_{l(n)}^*)^{-1} B_n^* y]^{k_{l(n)}} e^{i(b_{l(n)}(B_{l(n)}^*)^{-1}B_n^* y)} e^{i(a_n, y)})$$
$$= \lim_{n\to\infty} \widehat{C_n P_n}(y)\, \widehat{\delta(a_n)}(y),$$

where $C_n^* = (B_{l(n)}^*)^{-1} B_n^*$ and the sequence of the measures $P_n = B_{l(n)} \nu^{k_{l(n)}} * \delta(b_{l(n)})$ converges to a full measure μ.

The limit measure of the sequence $\mu^a = \lim C_n P_n * \delta(a_n)$ is also full.

By the compactness lemma of Sharpe ([3], p. 55) the sequence of operators $\{C_n\}$ is precompact in G, and the sequence of vectors $\{a_n\}$ is precompact in $\boldsymbol{R}^N$. Denoting by $C_{1/a}$ and a_a the limit points of these sequences for which

$$\mu^a = C_{1/a} \mu * \delta(a_a)$$

and putting $a = 1/n$, we obtain

$$\mu = C_n \mu^n * \delta(c_n), \quad c_n \in \boldsymbol{R}^N$$

which means operator-stability of μ in the sense of Sharpe.

Proof of Theorem. *Sufficiency*. Let μ be an infinitely divisible measure satisfying

(*) $$\mu^c = B\mu * \delta(b)$$

for some $0 < c < 1$, $B \in G$ and $b \in \boldsymbol{R}^N$. From (*) it easily follows that there exists a sequence of vectors $\{b_n\}$ of $\boldsymbol{R}^N$ such that

(11) $$\mu = (B^n \mu)^{\gamma^n} * \delta(b_n), \quad \text{where } \gamma = 1/c > 1,$$

holds. Putting $k_n = \text{Entier}(\gamma^n)$, we obtain

(12) $$\mu = \lim_{n\to\infty} B^n \mu^{k_n} * \delta(b_n),$$

where, obviously, $k_n \nearrow \infty$ and $k_n^{-1}k_{n+1} \to \gamma > 1$, which proves semi-stability of μ.

Necessity. Infinite divisibility of μ follows immediately from Lemma 1. The condition (*) in the case where $\gamma > 1$ follows from Lemma 2. If $\gamma = 1$, then μ is stable by Lemma 5 and Sharpe's formula holds:

$$\mu^t = t^A \mu * \delta(b_t), \quad t > 0.$$

Taking an arbitrary $0 < c < 1$ and putting $B = c^A$, $b_c = b$, we get (*) also in the case where $\gamma = 1$.

Let us write now the Lévy–Khintchine representation of $\hat{\mu}$

$$(13) \qquad \hat{\mu}(y) = \exp\Big[i(x_0, y) - \tfrac{1}{2}(Dy, y) + \int_{R^N - \{0\}} K(x, y) M(dx)\Big],$$

where $x_0 \in \boldsymbol{R}^N$, D is a symmetric non-negative linear operator in $\boldsymbol{R}^N$, M is a Lévy–Khintchine spectral measure and the kernel K is defined by

$$(14) \qquad K(x,y) = e^{i(x,y)} - 1 - \frac{i(x, y)}{1 + \|x\|^2}.$$

Writting (*) in terms of characteristic functions and taking finitenes of the following integral into account

$$(\bar{x}, y) = \int_{R^N} \frac{(Bx, y)\|x\|^2}{(1+\|x\|^2)(1+\|Bx\|)^2} M(dx),$$

we get, by uniqueness of the representation (13) of a infinitely divisible measure, the following conditions:

$$(15) \qquad BM = cM,$$

$$(16) \qquad BDB^* = cD.$$

Let $\boldsymbol{R}^N = X \oplus Y$ be a decomposition of $\boldsymbol{R}^N$ into a direct sum of B-invariant subspaces such that

$$\text{spectrum } B \mid X \subset \{|z|^2 < c\},$$
$$\text{spectrum } B \mid Y \subset \{|z| \geqslant c\}.$$

By virtue of Lemma 4 we have $M(Y) = 0$. In particular, if $X = \boldsymbol{R}^N$, then the measure μ is a full measure of the Poisson-type (without a Gaussian component). To simplify the notation we assume for a moment $Y = \boldsymbol{R}^N$. In this case μ reduces to a Gaussian measure with the characteristic function

$$(17) \qquad \hat{\mu}(y) = \exp[i(x_0, y) - \tfrac{1}{2}(Dy, y)],$$

where the operator D satisfies (16). This implies immediately that the spectrum of $B|Y$ lies in fact on the circle $\{|z|^2 = c\}$.

By $\tilde{X}$ (or $\tilde{A}$) we denote a natural linear extension of X (or A acting in X) to the complex case. Often the sign "$\sim$" will be omitted if it is clear what case we deal with. Talking about spectral properties of operators we always mean the properties of their natural complex extensions.

We shall show now that all the eigenvalues of $\tilde{B} \mid \tilde{Y}$ are simple. First, let us notice that the Gaussian measure defined by (17) is full and thus the operator $\tilde{D}|\tilde{Y}$ is non-singular in $\tilde{Y}$. From this it follows that the sesquilinear form

$$(18) \qquad \langle x, y\rangle_D = (Dx, y), \quad x, y \in \tilde{Y},$$

is an inner product in Y. For such a product we have, by (16),

$$(19) \qquad \langle B^*x, B^*y\rangle_D = c\langle x, y\rangle_D$$

and thus $B^{*\prime}B^* = cI$, where A' denotes the conjugation of A in the unitrary space $H = (\tilde{Y}, \langle\,,\cdot\,,\rangle)$. Thus $\tilde{B}$ is normal in H. This implies existence of a basis in $(\tilde{Y}, \langle\cdot,\cdot\rangle)$ such that for this basis B is of the diagonal form.

Summing up, there exists a decomposition of $\boldsymbol{R}^N$ into a direct sum $\boldsymbol{R}^N = X \oplus Y$ such that μ can be represented as a product $\mu = \mu_1 * \mu_2$, where μ_1 is a semi-stable full measure on X without a Gaussian component and μ_2 is a full Gaussian measure on Y. The spectrum of B is contained in the disc $\{|z|^2 \leqslant c\}$. Moreover, spectrum $B \mid X \subset \{|z|^2 < c\}$ and spectrum $B \mid Y \subset \{|z|^2 = c\}$. This ends the proof of necessity.

Remark. The pairs (c, B) which can occur in the formula (*), $\mu^c = B\mu * \delta(b)$, are characterized by the connection between the constant $0 < c < 1$ and spectral properties of B described in Theorem. More precisely, if (c,B) is such that $0 < c < 1$ and spectrum $B \subset \{|z|^2 \leqslant c\}$, where the elementary divisors of B corresponding to the eigenvalues lying on the circle $\{|z|^2 = c\}$ are one-dimensional, then there exists a semi-stable measure μ for which (*) holds.

Indeed, let a pair (c,B) have the properties mentioned above and let $\boldsymbol{R}^N = X \oplus Y$ be a decomposition of $\boldsymbol{R}^N$ identical with the one in the proof of Theorem. Moreover, let $X = \bigoplus_{j \in I} X_j$ be a decomposition of X into a sum of elementary subspaces cyclic with respect to B. Let us fix vectors $x_j \in X_j (j \in I)$ and put

$$(20) \qquad M(\{B^l x_k\}) = c^{-l} \quad \text{for} \quad k \in I,\ l = 0, \pm 1, \pm 2, \ldots$$

It is easily seen that M defined by (20) satisfies the condition $BM = cM$. We shall show that M is an LK-measure. To do this it suffices to show that for some norm $\|\cdot\|$ in X the inequality

$$\sum_{n=1}^{\infty} \|B^n x\|^2 c^{-n} < +\infty, \quad x \in X, \tag{21}$$

holds.

From the spectral properties of the operator $B|X$, in particular from the fact that the spectral radius of $B|X$ is smaller then $\sqrt{c}$, we infer that there exist constants γ and $\varrho < c$ such that

$$\|B^k z\|^2 < \gamma \varrho^k. \tag{22}$$

The inequality (22) implies (21).

Let us define a measure μ_1 putting

$$\hat{\mu}_1(y) = \exp \int_{R^N - \{0\}} K(x, y)\, M(dx),$$

where K and M are defined by (14) and (20), respectively. The measure μ_1 is, of course, semi-stable, without a Gaussian component and full on X.

Now we shall build a full Gaussian measure μ_2 concentrated on Y. Let us denote by $(y_1, \ldots, y_m)$ the basis in the real space Y, that is "diagonal" with respect to the operator B^*. More precisely, vectors of our basis satisfy either

(i) $B^* y_k = \lambda_k y_k$ for real λ_k, or

(ii) $B^* y_k = a y_k + b y_{k+1}$, $B^* y_{k+1} = -b y_k + a y_{k+1}$ for complex λ_k.

Here λ_k is an eigenvalue of B^* in Y; $\lambda_k = a + ib$, $a^2 + b^2 = c$.

We define a quadratic form (Dy, y) putting

$$(Dy, y) = \sum_{k=1}^{m} a_k^2 \quad \text{when} \quad y = \sum_{k=1}^{m} a_k y_k. \tag{23}$$

Let us verify the formula (16). In the case (i), putting $\psi(z) = (Dz, z)$, we have

$$\psi(B^* a y_k) = a^2 \lambda_k^2 = a^2 c = c\psi(a y_k).$$

In the case (ii), for $x = \alpha y_k + \beta y_{k+1}$, we have

$$\begin{aligned}\psi(B^* x) &= \psi\big(\alpha(a y_k + b y_{k+1}) + \beta(-b y_k + a y_{k+1})\big)\\ &= \alpha^2 a^2 + \alpha^2 b^2 + \beta^2 b^2 + \beta^2 a^2\\ &= c(\alpha^2 + \beta^2) = c\psi(\alpha y_k + \beta y_{k+1}) = c\psi(x).\end{aligned}$$

From these equalities the formula (16) follows easily.

Let us put now $\hat{\mu}_2(y) = \exp[-\frac{1}{2}(Dy, y)]$. Obviously, the measure μ_2 is full on Y. The measure $\mu = \mu_1 * \mu_2$ satisfies all the conditions of our remark.

References

[1] V. M. Kruglov, *On an extension of the class of stable distributions*, Teor. Ver. 17, 4 (1972), pp. 723–732 (in Russian).

[2] M. Loève, *Probability Theory*, 2nd edition, Van Nostrand, 1960.

[3] M. Sharpe, *Operator-stable probability distributions on vector groups*, Amer. Math. Soc. 136 (1969), pp. 51–65.

[4] K. Urbanik, *Levy's probability measures on Euclidean spaces*, Studia Math. 44 (1972), pp. 119–148.

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