(3), (4) give rise to the inequalities
\[
1 - \sum_{k=1}^{\infty} \cos (\pi t(\xi_k) x_k) < 2\pi k^{-1},
\]
\[
[\cos (\pi t(\xi_k) x_k) - \cos (\pi t(0) x_k)] < 2\pi k^{-1},
\]
\[
1 - \sum_{k=1}^{\infty} \cos (\pi t(0) x_k) < 2\pi k^{-1},
\]
and these combine to give (6). This completes the proof of the first assertion of the theorem.

For the second part we proceed in a similar way but replace (4) by (4'),
\[
(4')
\]
\[
|\phi|_1 < \sum_{n=1}^{\infty} |\xi_n| + t^{-1} n^{-1},
\]
and work with \(1 + \exp (2\pi p t)\) in place of \(\cos (2\pi p t)\).

Remark. The statements of Corollaries 1, 2, 3 of Theorem 1 of [1] remain valid when the phrase “virtually all” is interpreted in the sense of “residual in \(\mathbb{P}^2\).”

References


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Semi-stable probability measures on \(\mathbb{R}^N\)

by

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Abstract. Let \(\{\xi_j\}\) be a sequence of \(\mathbb{R}^N\)-valued, independent and identically distributed random variables. Consider the sums
\[
\sum_{j=1}^{\infty} \xi_j, \quad \xi_0, \quad \xi_j, \quad \mathbb{E}^{-1} \xi_{n+1} \to \gamma.
\]
where \(\mathcal{A}_n\) are non-singular linear operators in \(\mathbb{R}^N\), \(\mathcal{A}_n\) is \(\mathbb{R}^N\) and \(\mathbb{E}^{-1} \xi_{n+1} \to \gamma\). The limit law for sums of the form (0) is called semi-stable. The aim of this paper is to describe the class of all full semi-stable measures in \(\mathbb{R}^N\).

1. Introduction and notation. We begin with some notation. By \(\mathcal{M}\) we denote the set of all Borel probability measures on the real Euclidean space \(\mathbb{R}^N\). We regard \(\mathcal{M}\) as an Abelian topological semigroup with the convolution as a semigroup operation and the topology of weak convergence of measures.

We denote the convolution of two measures \(\mu\) and \(\nu\) by \(\mu * \nu\). Throughout, the power \(\mu^n\) is taken in the sense of the convolution. Moreover, by \(\delta(x)\) we denote the probability measure concentrated at the point \(x \in \mathbb{R}^N\). The characteristic function (Fourier transform) \(\hat{\mu}\) of a measure \(\mu \in \mathcal{M}\) is defined by the formula
\[
\hat{\mu}(y) = \int_{\mathbb{R}^N} \exp \{i(x,y)\} d\mu(x).
\]
The group of all non-singular linear operators acting in \(\mathbb{R}^N\) will be denoted by \(\mathcal{G}\).

For a Borel mapping \(F: \mathbb{R}^N \to \mathbb{R}^N\) and a measure \(\mu\) from \(\mathcal{M}\) we denote by \(F\mu\) the measure defined by the formula
\[
F\mu(Z) = \mu(F^{-1}Z)
\]
for any Borel subset \(Z\) of the space \(\mathbb{R}^N\). In particular, it is easy to verify that the mapping \((\mathcal{A}_n, \mu) \to \mathcal{A}_n\mu\) from \(G \times \mathcal{M}\) onto \(\mathcal{M}\) is jointly continuous and the formulas
\[
A_{\mu}(\xi) = A_{\mu}\xi, \quad A_{\mu}(y) = A_{\mu}(A_{\gamma}y)
\]
hold, here \(A_{\mu}\) denotes the adjoint operator. Given \(\mu \in \mathcal{M}\), we define \(\hat{\mu}\) putting \(\hat{\mu}(Z) = \mu(-Z)\), where \(-Z = (-y : y \in Z)\).
For any \( \mu \in \mathcal{M} \) the measure \( \hat{\mu} = \mu * \delta \) is called symmetrization of \( \mu \). A closed subset \( S_0 \) of \( \mathbb{R}^N \) is called the support of a measure \( \mu \) if the complement of \( S_0 \) has \( \mu \)-measure zero and \( \mu(U_\varepsilon) > 0 \) for any neighbourhood \( U_\varepsilon \) of \( x \), where \( x \) runs over \( S_0 \). It is not difficult to see that for \( \mu \in \mathcal{M} \) the support \( S_0 \) always exists and is unique. A measure \( \mu \) from \( \mathcal{M} \) is said to be full if its support \( S_\mu \) is not contained in any \((N-1)\)-dimensional hyperplane of \( \mathbb{R}^N \). In the excellent paper [3] M. Sharpe introduced and examined the notion of an operator-stable measure in \( \mathbb{R}^N \). Namely, a probability distribution \( \mu \) from \( \mathcal{M} \) is called an operator-stable measure if it is a weak limit of measures of the form

\[
\mu = \lim_{n \to \infty} A_n \varphi^{\psi_n} \delta(\xi_n),
\]

where \( \varphi \in \mathcal{M} \), \( A_n, \varphi, \xi_n, \psi_n \in \mathbb{R}^N \). In [3] the class of all operator-stable measures has been characterized. Recently, in [1], B. Kraglov considered the set of all limit distributions of the form

\[
\varphi = \lim_{n \to \infty} \varphi_n \psi_n \varphi_n = \lim_{n \to \infty} \varphi_n \psi_n \varphi_n
\]

where \( p \) runs over all probability measures on \( \mathbb{R}^N \), \( \varphi_n > 0 \), \( \varphi_n \in \mathbb{R}^N \) and the sequence \( \varphi_n \psi_n \varphi_n = \lim \varphi_n \psi_n \varphi_n \) is such that \( \varphi_n \psi_n \varphi_n \) for some \( 0 \leq \gamma < \infty \). The author gave a description of that class of measures which is larger than the class of stable measures on \( \mathbb{R}^N \).

Our purpose is to describe the class of all full measures in \( \mathbb{R}^N \) which are the limit laws for sums of the form

\[
\varphi = \lim_{n \to \infty} A_n \varphi_n \delta(\xi_n),
\]

where \( \{\xi_n\} \) is a sequence of independent identically distributed random variables, \( A_n, \varphi_n \in \mathbb{R}^N \) and \( \varphi_n \psi_n \varphi_n \) for some \( 0 \leq \gamma < \infty \).

We can treat this problem regarding \( \mathcal{M} \) as a metric semigroup. Then, we introduce the class of measures in question by the following definition.

**Definition 1.** A measure \( \mu \) from \( \mathcal{M} \) is said to be semi-stable if it is a weak limit of measures of the form

\[
\mu = \lim_{n \to \infty} A_n \varphi^{\psi_n} \delta(\xi_n),
\]

where \( A_n, \varphi, \xi_n, \psi_n \in \mathcal{M} \), \( b_n \in \mathbb{R}^N \) and the sequence \( k_1 < k_2 \) of positive integers is such that \( k_1 \varphi_n \psi_n \varphi_n \) for some \( 0 \leq \gamma < \infty \).

**2. Characterization of full semi-stable measures in \( \mathbb{R}^N \).** In this section we shall prove the following theorem.

**Theorem.** A full probability measure \( \mu \) from \( \mathcal{M} \) is a semi-stable measure if and only if it is infinitely divisible and there exist a number \( 0 < c < 1 \), a vector \( b \in \mathbb{R}^N \) and an operator \( \mathcal{B} \in \mathcal{G} \) such that the formula

\[
\mu^c = B \mu \delta(b)
\]

holds. The spectrum of \( B \) is contained in the disc \( |z|^2 < c \). Eigenvalues of \( B \) satisfying \( |\lambda|^2 = c \) are simple, i.e. the elementary divisors of \( B \) associated to these eigenvalues are one-dimensional.

Furthermore, the measure \( \mu \) can be decomposed into a product \( \mu = \mu_1 \mu_2 \) of two measures \( \mu_1 \) and \( \mu_2 \), concentrated on \( \mathbb{B}\)-invariant subspaces \( X_1 \) and \( X_2 \), respectively, and such that \( \mathbb{B} \mathbb{X}_1 \otimes \mathbb{X}_2 \) is a full semi-stable measure on \( X_1 \) of the Poisson-type (having no Gaussian component) and \( \mu_2 \) is a full Gaussian measure on \( X_2 \). The spectrum of \( \mathbb{B} \mathbb{X}_1 \) is then contained in the disc \( |z|^2 < c \) and for the eigenvalues of \( \mathbb{B} \mathbb{X}_1 \) the equality \( |\lambda|^2 = c \) holds.

The proof of our theorem will be preceded by several lemmas. In proofs of Lemmas 1 and 3 we use the technique developed in the fundamental work [4] by K. Urbanik.

**Lemma 1.** Let \( \mu \) be a full measure for which the formula

\[
(1) \quad \mu = \lim_{n \to \infty} A_n \varphi^{\psi_n} \delta(\xi_n),
\]

holds, where \( \varphi \in \mathcal{M} \), \( A_n, \varphi, \xi_n \in \mathbb{R}^N \) and \( \varphi_n \psi_n \varphi_n \) for some \( 0 \leq \gamma < \infty \). Then \( \xi_n \to \theta \).

**Proof.** Let us suppose the contrary and choose \( \varphi_n \in \mathbb{R}^N \) such that \( |\varphi_n| = 1 \), \( |\varphi_n| = \|A_n\| \psi_n \). Without any loss of generality (passing to a subsequence, if necessary) we may assume that \( |A_n| \to \delta > 0 \), \( \varphi_n(\|A_n\|) = y \), where \( y_n = \varphi_n(\|A_n\|) \) and \( \varphi_n(\|A_n\|) = u \). By symmetrization of measures occurring in (1) we obtain

\[
|\varphi(y)|^2 = \lim_{n \to \infty} \varphi_n(y) = \mu_0(\varphi),
\]

Hence \( \varphi(y) \to 1 = \delta(\varphi) \) almost uniformly on \( \mathbb{R}^N \) and, consequently, \( \mu_0(\varphi) = 1 \) for \( \varphi \in \mathcal{M} \).

On the other hand, we have

\[
\varphi_n(\|A_n\|) = \varphi(\|A_n\|) \to \delta(\varphi).
\]

Thus, we get \( \varphi_n(y) = 1 \) and, consequently, \( \varphi_n(\|A_n\|) = 1 \) for \( \varphi \in \mathcal{M} \) and \( y \in \mathbb{R}^N \).

Let us remark that \( \varphi_n(\|A_n\|) \neq 0 \) because \( \varphi_n(\|A_n\|) = 0 \).

Denote by \( \mathbb{B} \) an arbitrary limit point of the sequence \( \varphi_n(\|A_n\|) \). Then, since \( \varphi_n(\|A_n\|) \to \delta(\varphi) \), we obtain, passing to a subsequence, if necessary

\[
\varphi_n(\|A_n\|) \to \delta(\varphi).
\]

On the other hand,

\[
\varphi_n(\|A_n\|) = \varphi_n(\|A_n\|) \to \delta(\varphi).
\]

Thus we obtain

\[
\mu_0(\mathbb{B}) = 1 \quad \text{for} \quad \varphi \in \mathcal{M} \quad \text{and} \quad \|B\| = 1.
\]
By Proposition 1 in [3], $\mu$ is not a full measure what contradicts the assumption. The lemma is thus proved.

**Lemma 2.** If $\mu$ is a full measure for which the formula (1) holds, then the sequence of operators

$$\left(\left[A_{k}^{*}\right]^{-1} A_{k+1}^{*}\right)$$

is precompact in $G$.

Moreover, if $G$ is a limit point of the sequence (2), then the formula

$$\tilde{\mu}(y) = \left(\tilde{\mu}(Cy)\right)^{y} e^{0,0}$$

holds.

**Proof.** First, we shall show that the sequence (2) is precompact in the space of linear endomorphisms of $\mathbb{R}^{N}$.

Let us suppose the contrary, i.e., that the sequence of norms $\|\left[A_{k}^{*}\right]^{-1} A_{k+1}^{*}\|_{G}$ is unbounded. Let $\varepsilon_{n}$ in $\mathbb{R}^{N}$ such that $\|\varepsilon_{n}\|_{G} = 1$ and $\|\left[A_{k}^{*}\right]^{-1} A_{k+1}^{*} C_{n}\|_{G} = \|\left[A_{k}^{*}\right]^{-1} A_{k+1}^{*}\|_{G}$. Taking a subsequence, if necessary, we may assume that

$$\|\left[A_{k}^{*}\right]^{-1} A_{k+1}^{*}\|_{G} \to \infty$$

and $y_{k} \rightharpoonup y$, where $y_{k} = \left(\left[A_{k}^{*}\right]^{-1} A_{k+1}^{*}\right) C_{n}$. Then for every $t \in \mathbb{R}$ we have, by (1),

$$\left|\tilde{\nu}\left(tA_{k}^{*} y_{k}\right)\right|_{G}^{y_{k}} \to |\tilde{\mu}(y)|_{G}.$$  

Simultaneously,

$$\left|\nu\left(tA_{k}^{*} y_{k}\right)\right|_{G}^{y_{k}} \to \left|\nu\left(tA_{k+1}^{*} y_{k}\right)\right|_{G}^{y_{k}} \to |\tilde{\mu}(0)|^{y} = 1$$

for every $t \in \mathbb{R}$. Hence it follows that the characteristic function of the symmetric measure $\tilde{\mu}$ is equal to one on the subspace $\{y: t \in \mathbb{R}\}$. Consequently (see [3], Proposition 1), $\mu$ is not a full measure, with contradicts the assumption. Now it is easy to see that the compactness of the sequence (2) in $G$ follows immediately from (3). Indeed, since the support of $C\mu$ is contained in the image $\mathcal{O}(\mathbb{R}^{N})$, we infer that the operator $C$ is non-singular. Thus it remains to prove (3).

Let $G$ be a limit point of the sequence (2), say $G = \lim_{m \to \infty} \left(A_{m}^{*}\right)^{-1} A_{m+1}^{*}$. Then, by (1), we have

$$\tilde{\mu}(y) = \lim_{m \to \infty} \left(\tilde{\mu}(A_{m}^{*} y)\right)^{y} e^{0,0} = \lim_{m \to \infty} \left(\tilde{\nu}\left(A_{m}^{*} A_{m+1}^{*}\right) y\right)^{y} e^{0,0}$$

for some sequence $\{b_{n}\}$ of vectors in $\mathbb{R}^{N}$. It is almost evident that $b_{n}$ converges weakly, say to a vector $b \in \mathbb{R}^{N}$.

Thus we get formula (3) and our lemma is proved.

**Definition 2.** A Borel measure $\mu$ on $\mathbb{R}^{N}$ is called a Lévy–Khinchine spectral measure (or briefly LK-measure) if $\mu$ is a semi-finite measure in $\mathbb{R}^{N}$ concentrated on $\mathbb{R}^{N} - \{0\}$, finite on the complement of every neighbourhood of zero and such that

$$\int_{|x| < c} |x|^2 \mu(dx) < \infty.$$  

**Lemma 3.** Let $\mu$ be a non-trivial LK-measure concentrated on the trajectory

$$\tau_{n} = \{A_{n} x, k \in \mathbb{N}\},$$

where $x \in \mathbb{R}^{N}$, $B \times G$, $\mathbb{N}$ — the set of all integers and let the formula

$$\mathbb{B}M = c \mathbb{M}$$

hold for some $0 < c < 1$. Denote by $\tau_{n}$ a cyclic subspace generated by $x$. Then the spectrum of the reduced operator $\mathbb{B}|\tau_{n}$ is contained in the disc $\{|z| < c\}$.

**Proof.** From (4) and (5) it follows that $\mu$ is of the form

$$M(Ba) = d e^{-\alpha}, \quad d > 0, \quad n e L.$$

Indeed, we have $M(Ba) = B^{-n}M(a) = e^{-n}M(a)$. In this case the condition (4) is equivalent to

$$\sum_{n=1}^{\infty} |Ba|^2 e^{-\alpha} < \infty,$$

where $||\cdot||$ is an arbitrary norm in $\mathbb{R}^{N}$.

In the sequel the same letter $\beta$ will denote a natural extension of $\beta|\tau_{n}$ to a linear operator acting in the complex Euclidean space $Z$ spanned over $\tau_{n}$.

To define in (7) a suitable norm $\|\cdot\|$ we shall use the Jordan theorem about the canonical representation of linear operator acting in a complex Euclidean space. By this theorem there exist a basis $e_{1}, e_{2}, \ldots, e_{n}$ in $Z$, a system of integers $n_{0}, n_{1}, \ldots, n_{m}, 0 = n_{0} < n_{1} < \ldots < n_{m} = n$ and a sequence of eigenvalues $\lambda_{1}, \ldots, \lambda_{m}$ (not necessarily different) indexed so that

$$B_{i} = \lambda_{i} e_{1} + e_{i+1}$$

for $n_{i} < i < n_{i+1}$,

$$B_{i} y_{i} = \lambda_{i} y_{i}.$$

Let us establish a norm in $Z$ putting

$$||y|| = \sum_{i=1}^{n} \alpha_{i}$$

for $y = \sum_{i=1}^{n} \alpha_{i} e_{i}$. 

3 — Studia Mathematicae 61.1
As the norm in (7) we accept the norm (9). From (8) and (9) it easily follows that each eigenvalue \( \beta \) of \( B|x| \) satisfies
\[
\|B|x\| \geq \alpha \beta^n, \quad \alpha > 0.
\]
Comparing (10) with (7) we obtain \( |\beta|^2 < c \), what ends the proof.

**Lemma 4.** If \( M \) is an LK-measure and the formula (5) holds for some \( 0 < c < 1 \) and \( B \in B \), then \( M \) is concentrated on a subspace \( X \subset \mathbb{R}^N \) such that \( X \) is \( B \)-invariant and the spectrum of \( B|x| \) is contained in the open disc \( \{ |z|^2 < c \} \).

**Proof.** From (5) it easily follows that the support of \( M \) is \( B \)-invariant. Thus it is a sum of trajectories \( \tau_x \).

Let us denote by \( A(\epsilon,B) \) a set of LK-measures satisfying (5) and by \( Y \) a real linear space of finite measures on \( \mathbb{R}^N \) with the topology of weak convergence.

The set
\[
\Omega(\epsilon,B) := \{ N \in Y : N(R^N) \leq 1 \text{ and } N(dx) = \frac{|z|^2}{1 + |z|^2} M(dx) \text{ where } M \in A(\epsilon,B) \}
\]
is a compact convex subset of \( Y \). A support of every measure from \( \Omega \) is a sum of some trajectories \( \tau_x \). Hence it follows that extreme points of \( \Omega \) are measures concentrated on single trajectories \( \tau_x \). Thus the set of convex linear combinations of such measures is dense in \( \Omega \). Hence it follows that in \( A(\epsilon,B) \) linear combinations of measures concentrated on single trajectories \( \tau_x \) lie densely in \( A(\epsilon,B) \).

Now, let
\[
\mathbb{R}^N = X \oplus Y
\]
be a decomposition of \( \mathbb{R}^N \) into \( B \)-invariant subspaces such that
\[
\text{spectrum } B|x| = \{ |z|^2 < c \}, \quad \text{spectrum } B|y| = \{ |z|^2 > c \}.
\]
We can obtain such decomposition using the decomposition into cyclic subspaces of \( B \).

Let \( x \in \mathbb{R}^N \) and \( s = x+y \) be a decomposition of \( s \) such that \( x \in X \), \( y \in Y \). The trajectory \( \tau_y \) may be a support of a LK-measure if and only if \( y = 0 \) (comp. Lemma 3). Since linear combinations of such measures lie densely in \( A(\epsilon,B) \), every spectral measure from \( A(\epsilon,B) \) is concentrated on \( X \), which completes the proof.

**Lemma 5.** If a measure \( \mu \) is full and
\[
\mu = \lim_{n \to \infty} B^{n} \mu_{a} \delta(b_{n}),
\]
where \( B_{a} \in G, b_{a} \in \mathbb{R}^N \), \( b_{n} \rightharpoonup b \), then \( \mu \) is operator-stable in the sense of Sharpe.

**Proof.** Let us notice that by Lemma 1 the measure \( \mu \) is infinitely divisible and thus its powers with any positive exponent exist. Let \( a \in (0,1) \).

Fix a sequence \( f(a) \) of integers such that
\[
\frac{k_{n}}{b_{n}} \to a \quad \text{as} \quad n \to \infty.
\]
Then
\[
\begin{align*}
[\hat{\mu}(y)]^n &= \lim_{n \to \infty} \hat{\mu}^{(n)}(B_{a}^{n} y) \frac{k_{n}}{b_{n}} e^{i\phi(b_{n})} \\
&= \lim_{n \to \infty} \hat{\mu}^{(n)}(B_{a}^{n} y) \frac{k_{n}}{b_{n}} e^{i\phi(b_{n})} e^{i\phi(B_{a}^{n} y - b_{n})} \\
&= \lim_{n \to \infty} \hat{C}_{\mu}^{(n)}(y) \delta(b_{n}),
\end{align*}
\]
where \( C_{\mu}^{(n)} = (B_{a}^{n})^{-1} B_{a}^{n} \) and the sequence of the measures \( P_{n} = P_{b_{n}}^{B_{a}^{n}} \delta(b_{n}) \) converges to a full measure \( \mu \).

The limit measure of the sequence \( \mu^{*} = \lim C_{\mu} P_{n} \delta(b_{n}) \) is also full.

By the compactness lemma of Sharpe ([3], p. 55) the sequence of operators \( \{ G_{n} \} \) is precompact in \( G \), and the sequence of vectors \( \{ a_{n} \} \) is precompact in \( \mathbb{R}^N \). Denoting by \( C_{\mu} \) and \( a_{n} \), the limit points of these sequences for which
\[
\mu^{*} = C_{\mu} \delta(a_{n})
\]
and putting \( a = 1/n \), we obtain
\[
\mu = C_{\mu} \delta(a_{n}), \quad a \in \mathbb{R}^N
\]
which means operator-stability of \( \mu \) in the sense of Sharpe.

**Proof of Theorem. Sufficiency.** Let \( \mu \) be an infinitely divisible measure satisfying
\[
(\ast) \quad \mu^{\gamma} = B \mu \delta(b)
\]
for some \( 0 < \gamma < 1 \), \( B \in G \) and \( b \in \mathbb{R}^N \). From (\ast) it easily follows that there exists a sequence of vectors \( \{ b_{n} \} \) of \( \mathbb{R}^N \) such that
\[
(11) \quad \mu = (B^{\gamma} \mu)^{\gamma} \delta(b_{n}), \quad \text{where} \quad \gamma = 1/c > 1,
\]
holds. Putting \( k_{n} = \text{Entier} (\gamma^{n}) \), we obtain
\[
(12) \quad \mu = \lim_{n \to \infty} B^{k_{n}} \mu_{a} \delta(b_{n}),
\]
where, obviously, $h_n \neq \infty$ and $h^{-1}_n h_{n+1} \to \gamma > 1$, which proves semi-

stability of $\mu$.

**Necessity.** Infinite divisibility of $\mu$ immediately from Lemma 1.

The condition $(\ast)$ in the case where $\gamma > 1$ follows from Lemma 2. If $\gamma = 1$, then $\mu$ is stable by Lemma 5 and Sharpe's formula holds:

$$
\mu^t = e^{\mu \star \delta(t)}, \quad t > 0.
$$

Taking an arbitrary $0 < c < 1$ and putting $B = c^d$, $h_n = h$, we get $(\ast)$
also in the case where $\gamma = 1$.

Let us write now the Lévy-Khintchine representation of $\mu$

$$
\hat{\mu}(y) = \exp \left( i\langle x, y \rangle - \frac{1}{2} \langle Dy, y \rangle + \int \int K(x, y) M(dx) \right),
$$

where $x \in \mathbb{R}^n$, $D$ is a symmetric non-negative linear operator in $\mathbb{R}^n$, $M$ is a Lévy-Khintchine spectral measure and the kernel $K$ is defined by

$$
K(x, y) = e^{i\langle x, y \rangle} - 1 - \frac{1}{1 + i\|y\|}.
$$

Writing $(\ast)$ in terms of characteristic functions and taking finiteness of the following integral into account

$$
\hat{\mathbb{E}}(\hat{\mu}) = \int \frac{(\hat{B}x, y) \|y\|^2}{(1 + i\|y\|)^2} M(dx),
$$

we get, by uniqueness of the representation (13) of an infinitely divisible measure, the following conditions:

$$
BM = cM, \quad BDB^* = cD.
$$

Let $\mathbb{R}^n = \mathbb{X} \oplus \mathbb{Y}$ be a decomposition of $\mathbb{R}^n$ into a direct sum of $B$-invariant subspaces such that

- spectrum $B | \mathbb{X} \subset \{\|x\|^2 < c\}$,
- spectrum $B | \mathbb{Y} \subset \{\|y\|^2 > \gamma\}$.

By virtue of Lemma 4 we have $M(\mathbb{Y}) = 0$. In particular, if $\mathbb{X} = \mathbb{R}^n$, then the measure $\mu$ is a full measure of the Poisson-type (without a Gaussian component). To simplify the notation we assume for a moment $\mathbb{Y} = \mathbb{R}^n$. In this case $\mu$ reduces to a Gaussian measure with the characteristic function

$$
\hat{\mu}(y) = \exp \left( i\langle x, y \rangle - \frac{1}{2} \langle Dy, y \rangle \right),
$$

where the operator $D$ satisfies (16). This implies immediately that the spectrum of $B \mathbb{Y}$ lies in fact on the circle $\{\|y\|^2 = c\}$.

By $\mathbb{X}$ (or $\mathbb{A}$) we denote a natural linear extension of $\mathbb{X}$ (or $\mathbb{A}$ acting in $\mathbb{X}$) to the complex case. Often the sign """" will be omitted if it is clear what case we deal with. Talking about spectral properties of operators we always mean the properties of their natural complex extensions.

We shall now show that all the eigenvalues of $B | \mathbb{Y}$ are simple. First, let us notice that the Gaussian measure defined by (17) is full and thus the operator $B | \mathbb{Y}$ is non-singular in $\mathbb{Y}$. From this it follows that the sesquilinear form

$$
\langle x, y \rangle_D = (Dx, y), \quad x, y \in \mathbb{Y},
$$

is an inner product in $\mathbb{Y}$. For such a product we have, by (16),

$$
\langle Bx, y \rangle_D = i\langle x, y \rangle_D
$$

and thus $B^* B^* = cI$, where $A^*$ denotes the conjugation of $A$ in the unitary space $\mathbb{H} = (\mathbb{Y}, \langle \cdot, \cdot \rangle)$. Thus $\hat{B}$ is normal in $\mathbb{H}$. This implies existence of a basis in $(\mathbb{Y}, \langle \cdot, \cdot \rangle)$ such that for this basis $\mathbb{H}$ is of the diagonal form.

Summing up, there exists a decomposition of $\mathbb{R}^n$ into a direct sum $\mathbb{R}^n = \mathbb{X} \oplus \mathbb{Y}$ such that $\mu$ can be represented as a product $\mu = \mu_1 \mu_2$, where $\mu_1$ is a semi-stable full measure on $\mathbb{X}$ without a Gaussian component and $\mu_2$ is a full Gaussian measure on $\mathbb{Y}$. The spectrum of $B$ is contained in the disc $\{\|y\|^2 < c\}$. Moreover, spectrum $B | \mathbb{X} \subset \{\|x\|^2 < c\}$ and spectrum $B | \mathbb{Y} \subset \{\|y\|^2 > \gamma\}$. This ends the proof of necessity.

Remark. The pairs $(\sigma, B)$ which can occur in the formula $(\ast)$, $\mu^* = B \mu \star \delta(\cdot)$, are characterized by the connection between the constant $0 < c < 1$ and spectral properties of $B$ described in Theorem. More precisely, if $(\sigma, B)$ is such that $0 < c < 1$ and spectrum $B | \mathbb{X} \subset \{\|x\|^2 < c\}$, then the elementary divisors of $B$ corresponding to the eigenvalues lying on the circle $\{\|y\|^2 = c\}$ are one-dimensional, then there exists a semi-stable measure $\mu$ for which $(\ast)$ holds.

Indeed, let a pair $(\sigma, B)$ have the properties mentioned above and let $\mathbb{R}^n = \mathbb{X} \oplus \mathbb{Y}$ be a decomposition of $\mathbb{R}^n$ identical with the one in the proof of Theorem. Moreover, let $\mathbb{X} = \mathbb{X} \oplus \mathbb{X}$ be a decomposition of $\mathbb{X}$ into a sum of elementary subspaces cyclic with respect to $B$. Let us fix vectors $x \in \mathbb{X}$ and put

$$
\mathbb{M}(\|x\|) = c^{-1/2} \quad \text{for} \quad \|x\| = 0, 1, \pm 1, \pm 2, \ldots
$$

It is easily seen that $\mathbb{M}$ defined by (20) satisfies the condition $BM = cM$. We shall show that $\mathbb{M}$ is an LK-measure. To do this it suffices to show that for some norm $\|\cdot\|$ in $\mathbb{X}$ the inequality...
holds.

From the spectral properties of the operator \( B \mid X \), in particular from the fact that the spectral radius of \( B \mid X \) is smaller then \( \gamma \), we infer that there exist constants \( \gamma \) and \( q < c \) such that

\[
\| B^k x \| \leq \gamma q^k.
\]

The inequality (22) implies (21).

Let us define a measure \( \mu_1 \) putting

\[
\mu_1(y) = \exp \int \mathcal{M}(x, y) \, M(dx),
\]

where \( \mathcal{K} \) and \( M \) are defined by (14) and (20), respectively. The measure \( \mu_1 \) is, of course, semi-stable, without a Gaussian component and full on \( X \).

Now we shall build a full Gaussian measure \( \mu_2 \) concentrated on \( Y \). Let us denote by \( (y_1, \ldots, y_m) \) the basis in the real space \( Y \), that is “diagonal” with respect to the operator \( B^* \). More precisely, vectors of our basis satisfy either

(i) \( B^* y_k = \lambda_k y_k \) for real \( \lambda_k \), or

(ii) \( B^* y_k = \alpha y_k + \beta y_{k+1} \), \( B^* y_{k+1} = -\beta y_k + \alpha y_k \) for complex \( \lambda_k \).

Here \( \lambda_k \) is an eigenvalue of \( B^* \) in \( Y \); \( \lambda_k = \alpha + \beta \), \( \alpha^2 + \beta^2 = c \).

We define a quadratic form \( (Dy, y) \) putting

\[
(y, y) = \sum_{k=1}^m \alpha_k \beta_k \quad \text{when} \quad y = \sum_{k=1}^m \alpha_k y_k.
\]

Let us verify the formula (16). In the case (i), putting \( y = (Dx, x) \), we have

\[
\psi(B^* \alpha y_k) = \alpha^2 \lambda_k = \alpha c = \psi(\alpha y_k).
\]

In the case (ii), for \( x = \alpha y_k + \beta y_{k+1} \), we have

\[
\psi(B^* x) = \psi(\alpha(x y_k + \beta y_{k+1}) + \beta(-y_k + \gamma y_{k+1})) = \alpha \alpha \alpha + \alpha \beta \beta + \beta \beta \beta + \beta \beta \beta = \psi(\alpha^2 + \beta^2) = \psi(\alpha y_k + \beta y_{k+1}) = \psi(x).
\]

From these equalities the formula (16) follows easily.

Let us put now \( \mu_1(y) = \exp \left\{ \frac{1}{2} (Dy, y) \right\} \). Obviously, the measure \( \mu_1 \) is full on \( Y \). The measure \( \mu = \mu_1 \cdot \mu_2 \) satisfies all the conditions of our remark.