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Abelian ergodic theorems for contraction semigroups

by

S. A. McGRATH (Annapolis, Md.)

Abstract. Let (X, Σ, μ) be a σ-finite measure space and $L_p(\mu) = L_p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, the usual Banach spaces. Let $\{T(t)\colon t \geqslant 0\}$ be a strongly continuous semigroup of $L_p(\mu)$ contractions for some $1 \leqslant p < \infty$. Let R_λ be the resolvent of $\{T(t)\}$. If $p > 1$ and $\{T(t)\}$ is a positive semigroup we show that $\lim_{\lambda\to\infty} \lambda R_\lambda f(x) = f(x)$ a.e. for $f \in L_p(\mu)$. In case $p = 1$, we show $\lambda R_\lambda f(x) \to f(x)$ a.e. for $f \in L_1(\mu)$ for an arbitrary semigroup of $L_1(\mu)$ contractions.

Introduction. Let (X, Σ, μ) be a σ-finite measure space and $L_p(\mu) = L_p(X, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$, the usual Banach space of complex-valued functions. Let $\{T(t)\colon t \geqslant 0\}$ be a strongly continuous semigroup of $L_p(\mu)$ contractions for some $1 \leqslant p < \infty$. This means that (i) $\|T(t)\|_p \leqslant 1$, $t \geqslant 0$; (ii) $T(s+t) = T(s)T(t)$, $s, t \geqslant 0$; (iii) $\|T(s)f - T(t)f\| \to 0$ as $s \to t$ for any $f \in L_p(\mu)$. To simplify the notation we assume $T(0) = I$; all results obtained hold with appropriate modification if $T(0) \neq I$.

For $\lambda > 0$, set

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} T(t) f(x)\, dt$$

for $f \in L_p(\mu)$. In case $p > 1$ we show that

$$(*) \qquad \lim_{\lambda\to\infty} \lambda R_\lambda f(x) = f(x) \text{ a.e.}$$

for $f \in L_p(\mu)$ and $\{T(t)\}$ a semigroup of *positive* $L_p(\mu)$ contractions. This means: $0 \leqslant f \in L_p(\mu) \Rightarrow T(t)f \geqslant 0$ for $t \geqslant 0$. If $p = 1$ we establish $(*)$ for an arbitrary strongly continuous semigroup of $L_1(\mu)$ contractions. This result extends a theorem in [2], p. 178. We remark that topological ergodic theorems for Abel means of operator semigroups have been studied in [5], [6], [10]. The question of pointwise convergence for Abel means has been considered in [2], [4], [9]. In [8] the author showed that for $p > 1$ and $\{T(t)\}$ a semigroup of positive $L_p(\mu)$ contractions

$$\|f^*\| \leqslant (p/p-1)\|f\|,$$

where $f^* = \sup_{\lambda>0} |\lambda R_\lambda f(x)|$. This estimate was obtained by use of a dilation theorem appearing in [1]. The author used this estimate to show that $\lim_{\lambda\to 0^+} \lambda R_\lambda f(x)$ exists and is finite a.e. for $f \in L_p(\mu)$.

Before proceeding further we will clarify the definition of $R_\lambda f(x)$. By Theorem III. 11.17 in [3], given $f \in L_p(\mu)$ there exists a scalar function $T(t)f(x)$, measurable with respect to the usual product measure on $[0, \infty)\times X$, such that (i) for a.e. t, $T(t)f(\cdot) = T(t)f$ and (ii) there exists a μ-null set $E(f)$, independent of λ, such that $x \notin E(f)$ implies $\int_0^\infty e^{-\lambda t} T(t)f(x)dt$, as a function of x, is in the equivalence class of $\int_0^\infty e^{-\lambda t} T(t)f dt$. The scalar representation $T(t)f(x)$ is uniquely determined up to a set of product measure zero. Defining $R_\lambda f(x) = \int_0^\infty e^{-\lambda t} T(t)f(x)\,dt$, we see that $R_\lambda f(x)$ is in the equivalence class of $R_\lambda f = \int_0^\infty e^{-\lambda t} T(t)f dt$ for every $\lambda > 0$. This justifies the definition of $R_\lambda f(x)$. We note that for $x \notin E(f)$, $R_\lambda f(x)$ is a continuous function of $\lambda > 0$.

Preliminary results.

1. LEMMA. *Let $\{T(t)\}$ be a strongly continuous semigroup of $L_p(\mu)$ contractions for some $1 \leqslant p < \infty$. Set $\mathscr{M} = \{\lambda R_\lambda f\colon 0 < \lambda < \infty,\ f \in L_p(\mu)\}$. Then $\mathscr{M}$ is dense in $L_p(\mu)$ and $\lim_{\lambda\to\infty} \lambda R_\lambda f(x) = f(x)$ a.e. for any $f \in \mathscr{M}$.*

Proof. $\mathscr{M}$ is dense in $L_p(\mu)$ since $\lambda R_\lambda f \to f$ in norm ([5], p. 321). We now show that (*) holds for functions in $\mathscr{M}$.

By the resolvent equation, we have

$$\lambda R_\lambda \eta R_\eta f = \frac{\lambda\eta}{\eta-\lambda}(R_\lambda f - R_\eta f).$$

Since $|R_\lambda f(x)| < \infty$ for a.e. x, we have $R_\lambda f(x) \to 0$ a.e. as $\lambda\to\infty$ by the Lebesgue dominated convergence theorem ([3], III. 6.16). Thus $\dfrac{\lambda\eta}{\eta-\lambda} R_\lambda f(x) \to 0$ as $\lambda\to\infty$. So for fixed $\eta > 0$, we have $\lim_{\lambda\to\infty} \lambda R_\lambda \eta R_\eta f(x) = \eta R_\eta f(x)$ a.e. for any $f \in L_p(\mu)$. ■

The following lemma is used in proving (*) when $p = 1$. Before stating it we introduce some notation. By Theorem 1 in [7] given a semigroup $\{T(t)\}$ of $L_1(\mu)$ contractions there exists a semigroup $\{\tilde{T}(t)\}$ of positive $L_1(\mu)$ contractions such that $\tilde{T}(t)|f| \geqslant |T(t)f|$ a.e. for any $f \in L_1(\mu)$ and $t \geqslant 0$. We set $S(t) = e^{-t}T(t)$, $t \geqslant 0$.

2. LEMMA. *Let $\{T(t)\}$ be a strongly continuous semigroup of $L_1(\mu)$ contractions. For fixed $0 < g \in L_1(\mu)$ set $h = \int_0^\infty \tilde{S}(t)g\,dt$ and define a measure m by $m(A) = \int_A h\,d\mu$, $A \in \Sigma$. Define $P(t)$ on $L_1(X, \Sigma, m)$ by*

$$P(t)f = [S(t)(fh)]/h.$$

Then $\{P(t)\}$ is a strongly continuous semigroup of $L_1(X, \Sigma, m)$ contractions such that $\|P(t)f\|_\infty \leqslant \|f\|_\infty$ for $f \in L_\infty(X, \Sigma, m)$.

Proof. We note that $h \in L_1(\mu)$ and $h > 0$ a.e. on X since $\tilde{S}(0) = I$ and $\{\tilde{S}(t)\}$ is positive and strongly continuous; hence $P(t)f(x)$ is finite a.e. Henceforth denote $L_p(X, \Sigma, m)$ by $L_p(m)$, $1 \leqslant p \leqslant \infty$. Clearly, $\{P(t)\}$ is a semigroup since $\{S(t)\}$ is. To see that $\|P(t)\|_1 \leqslant 1$ pick $f \in L_1(m)$. Then

$$\int |P(t)f|\,dm = \int |S(t)(fh)|\,d\mu \leqslant \int |fh|\,d\mu = \int |f|\,dm.$$

Since $\|P(r)f - P(t)f\| = \|S(r)(fh) - S(t)(fh)\|$ for $f \in L_1(m)$, we see that $\{P(t)\}$ is strongly continuous since $\{S(t)\}$ is. We have

$$\tilde{S}(t)h = \int_t^\infty \tilde{S}(r)g\,dr \leqslant \int_0^\infty \tilde{S}(r)g\,dr = h.$$

Hence $[\tilde{S}(t)(h)]/h \leqslant 1$ a.e. From the positivity of $\tilde{S}(t)$ it follows that $\|\tilde{S}(t)(fh)/h\|_\infty \leqslant \|f\|_\infty$ for $f \in L_\infty(m)$. Hence

$$\|P(t)f\|_\infty \leqslant \|\tilde{S}(t)(|f|h)/h\|_\infty \leqslant \|f\|_\infty. \ ■$$

Main results.

3. THEOREM. *Let $\{T(t)\colon t \geqslant 0\}$ be a strongly continuous semigroup of positive $L_p(\mu)$ contractions for some $1 < p < \infty$. Then $\lim_{\lambda\to\infty} \lambda R_\lambda f(x) = f(x)$ a.e. for every $f \in L_p(\mu)$.*

Proof. We have $\lim_{\lambda\to\infty} \lambda R_\lambda f(x) = f(x)$ for $f \in \mathscr{M}$ and $\mathscr{M}$ is dense in $L_p(\mu)$. Also $f^* < +\infty$ a.e. since $\|f^*\| \leqslant (p/p-1)\,\|f\|$ for any $f \in L_p(\mu)$ (see [8]). Employing Banach's convergence theorem ([3], IV. 11.3), we conclude that $\lim_{\lambda\to\infty} \lambda R_\lambda f(x)$ exists and is finite a.e. assuming $\lambda\to\infty$ through some countable subset of $(0, \infty)$, say the set of positive rationals. Since $\lambda R_\lambda f(x)$ depends continuously on λ for a.e. x, we have $\lim_{\lambda\to\infty} \lambda R_\lambda f(x)$ exists and is finite a.e. Since $\lambda R_\lambda f(x) \to f$ in norm, we have $\lim_{\lambda\to\infty} \lambda R_\lambda f(x) = f(x)$ a.e. ■

4. THEOREM. *Let $\{T(t)\colon t \geqslant 0\}$ be a strongly continuous semigroup of $L_1(\mu)$ contractions. Then $\lim_{\lambda\to\infty} \lambda R_\lambda f(x) = f(x)$ a.e. for $f \in L_1(\mu)$.*

Proof. By our Lemma 2 and Theorem 1 in [9], we have for $a > 0$ and $f \in L_1(m)$

$$m(E(a)) \leqslant (1/a) \int_{E(a)} |f|\,dm,$$

where $f^* = \sup_{\lambda>0} \left|\lambda \int_0^\infty e^{-\lambda t} P(t)f\,dt\right|$ and $E(a) = \{f^* > a\}$. Hence $f^* < \infty$ a.e. on X. Applying Banach's convergence theorem again we get

$$\lim_{\lambda\to\infty}\lambda\int_0^\infty e^{-\lambda t}P(t)f(x)\,dt = f(x)$$

a.e. for $f\in L_1(m)$. We note that $f\in L_1(\mu)$ implies $f/h\in L_1(m)$. Also

$$\int_0^\infty e^{-\lambda t}P(t)(f/h)\,dt = \Big\{\int_0^\infty e^{-\lambda t}S(t)f(x)\,dt\Big\}/h(x) \text{ a.e.}$$

and $\int_0^\infty e^{-\lambda t}S(t)f(x)\,dt = R_{\lambda+1}f(x)$. Thus

$$\lim_{\lambda\to 0}\lambda R_\lambda f(x) = \lim_{\lambda\to 0}(\lambda+1)\,R_{\lambda+1}f(x) = h(x)\lim_{\lambda\to 0}(\lambda+1)\int_0^\infty e^{-\lambda t}P(t)(f/h)\,dt$$
$$= h(x)\{f(x)/h(x)\} = f(x) \text{ a.e. } \blacksquare$$

Added in proof: Theorems 3 and 4 hold for pseudo-resolvents. The author has learned that an indirect proof of Theorem 4 for pseudo-resolvents was published in 1974 by C. Kipnis. The technique used in proving Theorem 4 may be adapted to obtain a direct proof of Kipnis' result.

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DEPARTMENT OF MATHEMATICS
U. S. NAVAL ACADEMY, ANNAPOLIS, MD. USA

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Corrigendum and addendum to the paper "In general, Bernoulli convolutions have independent powers"

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by

GAVIN BROWN and WILLIAM MORAN (Liverpool)

Abstract. In this paper we point out an error in our earlier paper with this title and prove that with a slight modification of the definitions the results remain true. Explicitly, we show that for virtually all (in the sense of Baire category) sequences $(x_n)\in l^2$ the infinite convolution

$$\nu(x) = \mathop{\ast}_{n=1}^{\infty}\tfrac{1}{2}(\delta(-x_n)+\delta(x_n))$$

has the property that the $\sigma(L^\infty(\nu), L^1(\nu))$ closure of $\{e^{int}: n\in\mathbf{Z}\}$ contains all constants in $[-1,1]$.

1. Corrigendum. We are indebted to Professor S. Saeki for pointing out to us that Remark 4 on p. 142 of [1] is false. In addition, we have subsequently found an error in the proof of the main theorem of [1]. The error arises in the final paragraph of the proof of Lemma 4 because the sets $M_i^{-1}U_i$ are not necessarily open in the relative topology of B. Nevertheless the main theorem of [1] remains true as stated and an appropriate variant of Remark 4 is obtained when, for example, the set F is replaced by the set F' defined by

$$F' = \Big\{(b_n):\ \sum_{n=1}^{\infty} b_n \leqslant \xi,\ b_n \geqslant 0\ (n = 1, 2, 3, \ldots)\Big\},$$

where ξ is any irrational number in $[0,1]$.

Since generalizations of the theorems stated in [1] will appear with full proofs in the forthcoming paper of Lin and Saeki [2], we refrain from giving the details of the corrections needed in our original arguments. Instead we wish to state and prove a variant of the main theorem of [1] which admits a simple direct proof and which yields a more natural interpretation of the title result of that paper.

2. Addendum. For any sequence $(x_n)_{n=1}^\infty$ of real numbers consider the (formal) Bernoulli convolution

(1) $$\nu(x) = \mathop{\ast}_{n=1}^{\infty}\tfrac{1}{2}\big(\delta(-x_n)+\delta(x_n)\big),$$