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Weighted norm inequalities relating the g_λ^* and the area functions

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Abstract. The purpose of this paper is to obtain weak-type and strong-type weighted estimates relating the parabolic g_λ^* and area functions.

§ 1. Notations and definitions. The n-dimensional euclidean space will be denoted by $\boldsymbol{R}_n$. The scalar product of two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of $\boldsymbol{R}_n$ is the number $(x\cdot y) = \sum_1^n x_i y_i$ and the norm of x is defined as $|x| = (x\cdot x)^{1/2} = \{\sum_1^n x_i^2\}^{1/2}$. By $\boldsymbol{R}_{n+1}^+$ we mean the upper half-space, that is, the set $\{(x,t)\colon x\in \boldsymbol{R}_n, t>0\}$. Let P be an $n\times n$ matrix satisfying $(Px\cdot x)\geqslant (x\cdot x)$ for every x in $\boldsymbol{R}_n$. For $t>0$ we define t^P as $t^P = e^{\ln t\cdot P}$. The set $\{t^P\colon t>0\}$ is a group under the operation multiplication of matrices. Let $x\in \boldsymbol{R}_n$ be given and define $h(t) = (t^P x\cdot t^P x)$; this function is strictly increasing if $x\neq 0$, tends to infinity for t tending to infinity and tends to zero for t tending to zero. Then, for $x\neq 0$ there exists a unique $t = \varrho(x)$ such that $|t^{-P}x| = 1$. It can be shown that the limit of $\varrho(x)$ for x tending to zero is equal to zero and therefore, it turns out that the function $\varrho(x)$ defined as above for $x\neq 0$ and zero for $x = 0$ is a continuous function on $\boldsymbol{R}_n$ which satisfies: $\varrho(x+y)\leqslant \varrho(x)+\varrho(y)$ and $\varrho(s^P x) = s\varrho(x)$. Thus, the function $d(x,y) = \varrho(x-y)$ is a translation invariant metric on $\boldsymbol{R}_n$. The ϱ-ball with center at x and radius $r>0$ is the set $B_\varrho(x;r) = \{y\colon \varrho(x-y)<r\}$. If by $m(S)$ we denote the Lebesgue measure of a measurable set S, then $m(B_\varrho(x;r))$ is equal to $\pi^{n/2}\Gamma((n+2)/2)^{-1}r^\gamma$; where γ denotes the trace of the matrix P. The ϱ-cone $\Gamma_a(x)$ with vertex at x and amplitud $a>0$ will be the subset of $\boldsymbol{R}_{n+1}^+$ given by $\Gamma_a(x) = \{(y,t)\colon \varrho(x-y)<at\}$. If D is a subset of $\boldsymbol{R}_n$, $\Gamma_a(D)$ will stand for the union $\bigcup\{\Gamma_a(x)\colon x\in D\}$. Let $\omega(x)$ be a non-negative, measurable and locally integrable function on $\boldsymbol{R}_n$. If $f(x)$ is measurable

function, we define its q-norm with respect to the weight $\omega(x)$ as $\|f\|_{q,\omega} = \{\int_{R_n} |f(x)|^q \omega(x)\,dx\}^{1/q}$. By $m_\omega(D)$ we denote the ω-measure of a Lebesgue measurable set D defined as $m_\omega(D) = \int_D \omega(x)\,dx$.

Let $u(x, t)$ be a solution of the parabolic differential equation

$$\frac{\partial u}{\partial t} = t^{-1}(t^{P^*}\partial \cdot t^{P^*}\partial)u,$$

where $\partial = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ and $t > 0$. For such a function $u(x, t)$ we define its area function $S(a, x)$ of amplitude $a > 0$ as

$$S^2(a, x) = \iint_{R^+_{n+1}} |t^{P^*}\partial u(y, t)|^2 \chi\big(\varrho(x-y)/at\big)(at)^{-\gamma} \frac{dy\,dt}{t},$$

where $\chi(s)$ stands for the characteristic function of the interval $(0, 1)$ and γ = trace of P.

We also define a $g^*_\lambda(x)$ function associated to $u(x, t)$ as

$$\{g^*_\lambda(x)\}^2 = \iint_{R^+_{n+1}} |t^{P^*}\partial u(y, t)|^2 \{1+\varrho(t^{-P}(x-y))\}^{-\lambda\gamma} t^{-\gamma} \frac{dy\,dt}{t},$$

where $\lambda > 1$ and γ has the same meaning as above.

Observe that for $P = I$ (identity) t^{P^*} is equal to $t \cdot I$ and the parabolic differential equation becomes

$$(\partial u/\partial t) = t \cdot \sum_1^n (\partial^2 u/\partial x_i^2).$$

Furthermore, the solutions of this equation are the functions $u(x, t) = v(x, t^2/2)$, where $v(x, t)$ is any solution of the standard n-dimensional heat equation

$$(\partial v/\partial t) = \sum_1^n (\partial^2 v/\partial x_i^2).$$

Finally, we shall say that a non-negative, measurable and locally integrable function $\omega(x)$ belongs to the class A^P_ϱ, $\infty > p > 1$, if there exists a finite constant c such that

$$\left\{\frac{1}{m(B_\varrho)}\int_{B_\varrho} \omega(x)\,dx\right\} \cdot \left\{\frac{1}{m(B_\varrho)}\int_{B_\varrho} \omega(x)^{-1/p-1}\,dx\right\}^{p-1} \leqslant c$$

for every ϱ-ball B_ϱ. If $p = 1$, this expression should be interpreted as

$$\sup_{y \in B_\varrho}\left\{\frac{1}{m(B_\varrho)}\int_{B_\varrho} \omega(x)\,dx\right\} \leqslant c\,\omega(y).$$

§ 2. Statements of the results. The results obtained in this paper show that even in the case where weighted norm inequalities are considered, the theory of the g^*_λ function can be subordinate to that of the area function. For the case $\omega(x) \equiv 1$, that is, when the norms are taken relatively to the Lebesgue measure, this result is due to A. P. Calderón and A. Torchinsky (see [1]) whose method we borrow in order to deal with the weighted case. In the classical case of harmonic functions, weighted norm inequalities and weak type results for the area function and the g^*_λ function were obtained by R. Gundy, B. Muckenhoupt, C. Segovia and R. L. Wheeden (see [2], [4] and [5]).

The main results in this paper are stated in the following theorems:

THEOREM 1. *Let $\omega(x)$ belong to A^p_ϱ, $\infty > p \geqslant 1$, and $0 < q < 2$. Then, $\|S(a, x)\|_{q,\omega}$ is finite for every amplitude $a \geqslant 1$ if $\|S(1, x)\|_{q,\omega}$ is finite. Moreover, there is a finite constant c, not depending on $u(x, t)$, such that*

$$\|S(a, x)\|^q_{q,\omega} \leqslant c \cdot a^{(p-q/2)\gamma} \|S(1, x)\|^q_{q,\omega}$$

holds for every $a \geqslant 1$.

THEOREM 2. *Let $\omega(x)$ belong to A^p_ϱ, $\infty > p \geqslant 1$ and $0 < q < 2$. Then, if $\lambda > 2p/q$, there exists a finite constant c, not depending on $u(x, t)$, such that*

$$\|g^*_\lambda\|^q_{q,\omega} \leqslant c \cdot \|S(1, x)\|^q_{q,\omega}$$

holds.

THEOREM 3. *Let $\omega(x)$ belong to A^1_ϱ, $\lambda = 2/q$ and $0 < q < 2$. Then there exists a finite constant c, not depending on $u(x, t)$, such that*

$$m_\omega\big(\{x\colon\ g^*_\lambda(x) > t\}\big) \leqslant c \cdot t^{-q} \|S(1, x)\|^q_{q,\omega}$$

holds for every $t > 0$.

§ 3. The proofs. First, we shall state in Propositions 1 and 2 some known results about weights belonging to A^p_ϱ.

PROPOSITION 1. *If $\omega(x)$ belongs to A^p_ϱ, $\infty > p \geqslant 1$, there exists a finite constant c such that*

$$\left\{\frac{m(E)}{m(B_\varrho)}\right\}^p \leqslant c \cdot \frac{m_\omega(E)}{m_\omega(B_\varrho)}$$

holds for every ball B_ϱ and every Lebesgue measurable subset E of B_ϱ.

PROPOSITION 2. *Let $\omega(x)$ belong to A^p_ϱ, $\infty > p \geqslant 1$, and let $M(f, x)$ be the Hardy maximal function of a Lebesgue measurable function $f(x)$ defined as*

$$M(f, x) = \sup_{x \in B_\varrho}\left\{\frac{1}{m(B_\varrho)}\int_{B_\varrho} |f(y)|\,dy\right\}.$$

Then there exists a finite constant c such that,

$$m_\omega\{x:\ M(f,x)>s\}\leqslant cs^{-p}\|f(x)\|_{p,\omega}^p$$

holds for every $s>0$. Of course, the constant c does not depend on f.

The proofs of these propositions will not be given here (see [3]).

The following lemma will supply the geometric background needed in the sequel.

LEMMA 1. *Let A be an open subset of $\boldsymbol{R}_n$ and χ_A its characteristic function. If for $a\geqslant 1$ we define U as the set*

$$U=\{x:\ M(\chi_A,x)>(4a)^{-\gamma}\},$$

then we have

(i) *$\Gamma_a(\complement U)$ is contained in $\Gamma_1(\complement A)$.*

(ii) *If $(z,t)\in\Gamma_a(\complement U)$, then $m\big(B_\varrho(z;t)\big)\leqslant 2m\big(B_\varrho(z;t)\cap\complement A\big)$.*

Proof. The lemma is obviously true if $\Gamma_a(\complement U)=\emptyset$. Therefore, we shall assume that $\Gamma_a(\complement U)$ is not the empty set, which implies that $A\neq\boldsymbol{R}_n$. Let us see (i). If $(z,t)\in\Gamma_a(\complement U)$, then either $z\in\complement A$ or $z\in A$. In the first case it is apparent that $(z,t)\in\Gamma_1(\complement A)$. If we are in the second case, i.e. $z\in A$, let us call δ the distance from z to the closed and non-empty set $\complement A$. This number δ is positive and finite, and $B_\varrho(z;\delta)$ is contained in A. The assumption that (z,t) belongs to $\Gamma_a(\complement U)$ implies that there is $y\in\complement U$ with $\varrho(z-y)<at$. Thus, writing $r=\delta+\varrho(z-y)$, we get

$$B_\varrho(z;\delta)\subset B_\varrho(y;r)$$

and also

$$B_\varrho(z;\delta)\subset B_\varrho(z;\delta)\cap A\subset B_\varrho(y;r)\cap A.$$

This, together with the definition of U, implies that

$$m\big(B_\varrho(z;\delta)\big)\leqslant m\big(B_\varrho(y;r)\cap A\big)\leqslant(4a)^{-\gamma}m\big(B_\varrho(y;r)\big)$$

since $y\in\complement U$. From these inequalities and the fact that the Lebesgue measure of a ϱ-ball is equal to a fixed constant times the γ power of its radius, we get

$$\delta\leqslant r/(4a).$$

Recalling that $r=\delta+\varrho(z-y)$ and $\varrho(z-y)<at$, we obtain

$$\delta\leqslant\frac{\delta+\varrho(z-y)}{4a}<\frac{\delta+at}{4a}$$

and since $a\geqslant 1$, it follows that $\delta<t$. Then, by the very definition of δ, there exists an $x\in\complement A$ satisfying $\varrho(x-z)<t$, which means that $(z,t)\in\Gamma_1(\complement A)$. This proves (i).

Next, we prove (ii). If $(z,t)\in\Gamma_a(\complement U)$, there is $y\in\complement U$ such that $\varrho(z-y)<at$. Then $B_\varrho(z;t)\subset B_\varrho\big(y;(1+a)t\big)$ and since $y\in\complement U$, we get

$$m\big(B_\varrho(z;t)\cap A\big)\leqslant m\big(B_\varrho\big(y;(1+a)t\big)\cap A\big)\leqslant(4a)^{-\gamma}m\big(B_\varrho\big(y;(1+a)t\big)\big)$$

and therefore

$$m\big(B_\varrho(z;t)\cap A\big)\leqslant\left(\frac{1+a}{4a}\right)^{\gamma}m\big(B_\varrho(y;t)\big)=\left(\frac{1+a}{4a}\right)^{\gamma}m\big(B_\varrho(z;t)\big).$$

Now, observing that $(1+a)/4a<1/2$, from

$$m\big(B_\varrho(z;t)\big)=m\big(B_\varrho(z;t)\cap A\big)+m\big(B_\varrho(z;t)\cap\complement A\big)$$

we obtain

$$(1-2^{-\gamma})\,m\big(B_\varrho(z;t)\big)\leqslant m\big(B_\varrho(z;t)\cap\complement A\big)$$

which implies (ii).

In the next lemma we establish an inequality which is the base of the method used in this paper.

LEMMA 2. *Let $\omega(x)$ be a weight belonging to A_ϱ^p, $\infty>p\geqslant 1$, and let A be an open set in $\boldsymbol{R}_n$. If U is the set associated to A as in Lemma 1, then there exists a finite constant c, which does not depend on $u(x,t)$, such that*

$$a^{\gamma(1-p)}\int_{\complement U}S^2(a,x)\,\omega(x)\,dx\leqslant C\cdot\int_{\complement A}S^2(1,x)\,\omega(x)\,dx$$

holds.

Proof. From the definition of the function $S(a,x)$ and by a change in the order of integration, we get

$$a^{\gamma(1-p)}\int_{\complement U}S^2(a,x)\,\omega(x)\,dx=a^{\gamma(1-p)}a^{-\gamma}\int_{\complement U}\left\{\iint_{\Gamma_a(x)}|t^{P^*}\partial u(y,t)|^2t^{-\gamma}\frac{dy\,dt}{t}\right\}\omega(x)\,dx$$

$$=a^{-\gamma p}\iint_{\Gamma_a(\complement U)}|t^{P^*}\partial u(y,t)|^2t^{-\gamma}\left\{\int_{B_\varrho(y;at)\cap\complement U}\omega(x)\,dx\right\}\frac{dy\,dt}{t}\tag{3.0}$$

$$\leqslant a^{-\gamma p}\iint_{\Gamma_a(\complement U)}|t^{P^*}\partial u(y,t)|^2m_\omega\big(B_\varrho(y;at)\cap\complement U\big)t^{-\gamma}\frac{dy\,dt}{t}.$$

Now, if we apply Proposition 1 to the sets $E=B_\varrho(y;t)$ and $B=B_\varrho(y;at)$, we get

$$m_\omega\big(B_\varrho(y;at)\big)\leqslant ca^{\gamma p}m_\omega\big(B_\varrho(y;t)\big).\tag{3.1}$$

Applying Proposition 1 once again, this time to $E=B_\varrho(y;t)\cap\complement A$ and $B=B_\varrho(y;t)$, we get

$$m_\omega\big(B_\varrho(y;t)\big)\leqslant c\left\{\frac{m\big(B_\varrho(y;t)\big)}{m\big(B_\varrho(y;t)\cap\complement A\big)}\right\}^p\cdot m_\omega\big(B_\varrho(y;t)\cap\complement A\big).\tag{3.2}$$

Therefore, (3.1) and (3.2) plus part (ii) of Lemma 1 imply

$$m_\omega\big(B_\varrho(y;\,at)\big) \leqslant c a^{\gamma p} m_\omega\big(B_\varrho(y;\,t)\cap \complement A\big).$$

From this inequality it follows that the last integral in (3.0) is smaller than or equal to

$$C\cdot \iint\limits_{\Gamma_a(\complement U)} |t^{P^*}\partial u(y,t)|^2\Big\{\int\limits_{B_\varrho(y;t)\cap\complement A}\omega(x)\,dx\Big\}t^{-\gamma}\frac{dy\,dt}{t}$$

$$= C\cdot \iint\limits_{\Gamma_a(\complement U)} |t^{P^*}\partial u(y,t)|^2 m_\omega\big(B_\varrho(y;\,t)\cap\complement A\big)t^{-\gamma}\frac{dy\,dt}{t}.$$

Finally, since by part (i) of Lemma 1 we know that $\Gamma_a(\complement U)\subset\Gamma_1(\complement A)$, we have that the last integral above is smaller than or equal to

$$C\cdot \iint\limits_{\Gamma_1(\complement A)} |t^{P^*}\partial u(y,t)|^2\Big\{\int\limits_{B_\varrho(y;t)\cap\complement A}\omega(x)\,dx\Big\}t^{-\gamma}\frac{dy\,dt}{t} = C\cdot\int\limits_{\complement A} S^2(1,x)\,\omega(x)\,dx$$

which proves the lemma.

Proof of Theorem 1. Let $A = \{x\colon S(1,x) > sa^{\gamma/2}\}$. It is easy to see that the set A is open. Let U be the set associated to A as in Lemma 1. Then

$$m_\omega\big(\{x\colon S(a,x)>s\}\big)\leqslant m_\omega(U)+m_\omega\big(\complement U\cap\{x\colon S(a,x)>s\}\big).$$

Our immediate task will be finding estimates for the terms in the second member above. By Tchebyshev's inequality and Lemma 2, we have

$$m_\omega\big(\complement U\cap\{x\colon S(a,x)>s\}\big)\leqslant s^{-2}\int\limits_{\complement U} S^2(a,x)\,\omega(x)\,dx$$

$$\leqslant C\cdot a^{(p-1)\gamma}s^{-2}\int\limits_{\complement A} S(1,x)\,\omega(x)\,dx.$$

Now, since $\complement A = \{x\colon S(1,x)\leqslant a^{\gamma/2}s\}$, the last integral is bounded by

$$2\int\limits_0^{a^{\gamma/2}s} t\cdot m_\omega\big(\{x\colon S(1,x)>t\}\big)\,dt. \tag{3.3}$$

On the other hand, by Proposition 2, we have

$$m_\omega(U) = m_\omega\big(\{x\colon M(\chi_A,x)>(4a)^{-\gamma}\}\big)\leqslant C\cdot(4a)^{\gamma p}m_\omega(A). \tag{3.4}$$

Therefore, from (3.3) and (3.4), we conclude that

$$m_\omega\big(\{x\colon S(a,x)>s\}\big)\leqslant C\cdot a^{\gamma p}m_\omega\big(\{x\colon S(1,x)>a^{\gamma/2}s\}\big)+$$

$$+C\cdot a^{(p-1)\gamma}s^{-2}\int\limits_0^{a^{\gamma/2}s} t\cdot m_\omega\big(\{x\colon S(1,x)>t\}\big)\,dt.$$

This estimate of the measure of the set $\{x\colon S(a,x)>s\}$ allows us to compute the q-norm of $S(a,x)$ as follows:

$$\|S(a,x)\|^q_{q,\omega} = q\cdot\int\limits_0^\infty s^{q-1}m_\omega\big(\{x\colon S(a,x)>s\}\big)\,ds$$

$$\leqslant C\cdot a^{\gamma p}\int\limits_0^\infty s^{q-1}m_\omega\big(\{x\colon S(1,x)>a^{\gamma/2}s\}\big)\,ds+$$

$$+C\cdot a^{(p-1)\gamma}\int\limits_0^\infty s^{q-1}s^{-2}\int\limits_0^{a^{\gamma/2}s} t\cdot m_\omega\big(\{x\colon S(1,x)>t\}\big)\,dt\,ds.$$

By a change of variables, the first term of the second member becomes

$$C\cdot a^{\gamma(p-(q/2))}\int\limits_0^\infty t^{q-1}\cdot m_\omega\big(\{x\colon S(1,x)>t\}\big)\,dt = C\cdot a^{\gamma(p-(q/2))}\|S(1,x)\|^q_{q,\omega}.$$

As for the second term, a change of the order of integration gives

$$C\cdot a^{(p-1)\gamma}\int\limits_0^\infty t\cdot m_\omega\big(\{x\colon S(1,x)>t\}\big)\cdot\Big(\int\limits_{a^{-\gamma/2}t}^\infty s^{q-3}\,ds\Big)\,dt$$

$$= C\cdot a^{(p-1)\gamma}\cdot a^{-(q-2)\gamma/2}\int\limits_0^\infty t^{q-1}\cdot m_\omega\big(\{x\colon S(1,x)>t\}\big)\,dt$$

$$= C\cdot a^{\gamma(p-(q/2))}\cdot\|S(1,x)\|^q_{q,\omega}.$$

Thus, we obtain the inequality

$$\|S(a,x)\|^q_{q,\omega}\leqslant C\cdot a^{\gamma(p-(q/2))}\cdot\|S(1,x)\|^q_{q,\omega}$$

which was claimed in the statement of Theorem 1.

LEMMA 3. *Let $\chi(t)$ denote the characteristic function of the interval $0\leqslant t<1$ and $0<\mu<\infty$. Then*

$$2^{-\mu}(1-2^{-\mu})^{-1}\cdot(1+s)^{-\mu}\leqslant\sum_{-\infty}^{\infty}\chi(s\cdot 2^{-k})\cdot 2^{-\mu k}\leqslant 2^{\mu}(1-2^{-\mu})^{-1}\cdot(1+s)^{-\mu}$$

holds for every $s\geqslant 0$.

Proof. For a given $s\geqslant 0$. let h be the least non-negative integer such that $s<2^h$. Then, $\chi(s\cdot 2^{-k})$ will be different from zero (and therefore, equal to one) if and only if $k\geqslant h$. Thus, we have

$$\sum_{k=0}^\infty\chi(s\cdot 2^{-k})\cdot 2^{-\mu k} = \sum_{k=h}^\infty 2^{-\mu k} = 2^{-\mu h}(1-2^{-\mu})^{-1}. \tag{3.5}$$

In order to estimate the value of this sum, we consider first the case $h>0$. Then, by the definition of h we have $2^{h-1}\leqslant s<2^h$, which implies that

$$2^{h-1}<2^{h-1}+1\leqslant s+1<2^h+1<2^{h+1}$$

or

(3.6) $$2^{h-1} < s+1 < 2^{h+1}.$$

If $h = 0$, that is to say, when $0 \leqslant s < 1$, we can see directly that (3.6) is still valid. Then, from (3.6) we get that

$$(1+s)^{-\mu}\cdot 2^{-\mu} < 2^{-\mu h} < 2^{\mu}(1+s)^{-\mu}$$

holds for every $s \geqslant 0$. This, together with (3.5), gives the estimate claimed in the lemma.

Proof of Theorem 2. Let us apply Lemma 3 to $s = \varrho(x-y)/t$ and $\mu = \lambda\gamma$. Then,

$$\left\{1+\frac{\varrho(x-y)}{t}\right\}^{-\lambda\gamma} \leqslant C\cdot\sum_{k=0}^{\infty}\chi\big(\varrho(x-y)2^{-k}t^{-1}\big)\cdot 2^{-\lambda\gamma k}$$

therefore,

$$\begin{aligned}(g_\lambda^*(x))^2 &= \iint_{R_{n+1}^+}\left\{1+\frac{\varrho(x-y)}{t}\right\}^{-\lambda\gamma}|t^{P^*}\partial u(y,t)|^2\cdot t^{-\gamma}\frac{dy\,dt}{t}\\ &\leqslant C\cdot\sum_{k=0}^{\infty}2^{-\lambda\gamma k}\iint_{R_{n+1}^+}\chi\big(\varrho(x-y)2^{-k}t^{-1}\big)|t^{P^*}\partial u(y,t)|^2 t^{-\gamma}\frac{dy\,dt}{t}\\ &= C\cdot\sum_{k=0}^{\infty}2^{-(\lambda-1)\gamma k}\cdot S^2(2^k, x).\end{aligned}$$

Now, since $q/2 < 1$, we can write

(3.7)
$$\begin{aligned}\|g_\lambda^*(x)\|_{q,\omega}^q &= \int\{(g_\lambda^*(x))^2\}^{q/2}\omega(x)\,dx \leqslant C\cdot\int\Big(\sum_{k=0}^{\infty}2^{-(\lambda-1)\gamma k}\cdot S^2(2^k,x)\Big)^{q/2}\omega(x)\,dx\\ &\leqslant C\cdot\sum_{k=0}^{\infty}2^{-(\lambda-1)\gamma kq/2}\cdot\|S(2^k,x)\|_{q,\omega}^q\end{aligned}$$

but, from Theorem 1, we already know that

$$\|S(2^k,x)\|_{q,\omega}^q \leqslant C\cdot 2^{\gamma k(p-(q/2))}\|S(1,x)\|_{q,\omega}^q,$$

therefore the series in (3.7) is less than or equal to

$$C\cdot\Big(\sum_{k=0}^{\infty}2^{(p-(\lambda q/2))\gamma k}\Big)\cdot\|S(1,x)\|_{q,\omega}^q,$$

where the series inside the parentheses is geometric and converges since $(\lambda q/2)-p$ is greater than zero. This proves the first part of the theorem.

Proof of Theorem 3. Let E, A_k and U_k be the sets defined as

$$\begin{aligned}E &= \{x\colon M(S^q(1,\cdot),x) > s^q\cdot 4^{-\gamma}\},\\ A_k &= \{x\colon S(1,x) > 2^{k\gamma/q}\cdot s\},\\ U_k &= \{x\colon M(\chi_{A_k},x) > 2^{-\gamma k}\cdot 4^{-\gamma}\},\end{aligned}$$

where $k = 0, 1, 2, \ldots$ We observe that the sets U_k defined here are related to the sets A_k as in Lemma 1. Let us see that the sets U_k are contained in the set E for every k. If x belongs to U_k, there is a ball $B_\varrho(x;r)$ such that

$$m\big(B_\varrho(x;r)\cap A_k\big) \geqslant m\big(B_\varrho(x;r)\big)\cdot 2^{-\gamma k}\cdot 4^{-\gamma}.$$

Therefore,

$$\begin{aligned}\int_{B_\varrho(x;r)}S^q(1,y)\,dy &\geqslant \int_{B_\varrho(x;r)\cap A_k}S^q(1,y)\,dy \geqslant s^q\cdot 2^{\gamma k}\cdot m\big(B_\varrho(x;r)\cap A_k\big)\\ &\geqslant m\big(B_\varrho(x;r)\big)\cdot s^q\cdot 4^{-\gamma}\end{aligned}$$

which shows that $x \in E$.

Now, for $s > 0$, we have

(3.8) $$s^2 m_\omega\big(\{x\colon g_\lambda^*(x) > s\}\cap \complement E\big) \leqslant \int_{\complement E}(g_\lambda^*(x))^2\omega(x)\,dx.$$

Applying Lemma 3 with $\mu = \gamma\lambda$ and proceeding as in the proof of Theorem 2, we get

$$\int_{\complement E}(g_\lambda^*(x))^2\omega(x)\,dx \leqslant \sum_{k=0}^{\infty}2^{-(\lambda-1)\gamma k}\int_{\complement E}S^2(2^k,x)\,\omega(x)\,dx.$$

Since, as we have shown above, $U_k \subset E$, we have

$$\int_{\complement E}S^2(2^k,x)\,\omega(x)\,dx \leqslant \int_{\complement U_k}S^2(2^k,x)\,\omega(x)\,dx.$$

Also, by Lemma 2, the last integral is majorized by

$$C\cdot\int_{\complement A_k}S^2(1,x)\,\tilde{\omega}(x)\,dx.$$

Collecting these results, we have

(3.9) $$\int_{\complement E}(g_\lambda^*(x))^2\omega(x)\,dx \leqslant C\cdot\sum_{k=0}^{\infty}2^{-(\lambda-1)\gamma k}\int_{\complement A_k}S^2(1,x)\,\omega(x)\,dx.$$

Interchanging the order of summation and integration in the second member, it becomes

(3.10) $$C\cdot\int_{R_n}S^2(1,x)\cdot\Big(\sum_{k=0}^{\infty}2^{-(\lambda-1)\gamma k}\chi_{\complement A_k}(x)\Big)\omega(x)\,dx.$$

In order to estimate the series inside the integral, let h be the least non-negative integer such that $x \in \complement A_k$. Then the sum of the series is equal to

$$\sum_{k=h}^{\infty} 2^{-(\lambda-1)\gamma k} = 2^{-(\lambda-1)\gamma h}(1-2^{-(\lambda-1)\gamma})^{-1}.$$

We have, by the definition of h, that $S(1, x) \leqslant 2^{h\gamma/q} \cdot s$, therefore,

$$2^{-(\lambda-1)h\gamma} \leqslant \big(s^{-1} \cdot S(1, x)\big)^{-(\lambda-1)q}.$$

Thus, the sum of the series is smaller than or equal to

$$\big(s^{-1} \cdot S(1, x)\big)^{-(\lambda-1)q} \cdot (1-2^{-(\lambda-1)\gamma})^{-1}.$$

So, we can majorize (3.10) by

$$C \cdot s^{(\lambda-1)q} \int_{R_n} S^2(1, x) \cdot S(1, x)^{-(\lambda-1)q} \omega(x)\, dx$$

and taking into account that $(\lambda-1)q = \big((2/q)-1\big)q = 2-q$, the integral above can be written as

$$C \cdot s^{2-q} \int_{R_n} S^q(1, x)\, \omega(x)\, dx.$$

This expression majorizes the second member of (3.8) and therefore, from (3.8) we get

$$m_\omega\big(\{x:\ g_\lambda^*(x) > s\} \cap \complement E\big) \leqslant C \cdot s^{-q} \cdot \int_{R_n} S^q(1, x)\, \omega(x)\, dx.$$

Finally, consider the inequality

$$m_\omega\big(\{x:\ g_\lambda^*(x) > s\}\big) \leqslant m_\omega\big(\{x:\ g_\lambda^*(x) > s\} \cap \complement E\big) + m_\omega(E).$$

By Proposition 2, we know that

$$m_\omega(E) \leqslant C \cdot s^{-q} \int_{R_n} S^q(1, x)\, \omega(x)\, dx.$$

Then, this and the estimation we obtained above for the first term on the right-hand side of (3.11) imply

$$m_\omega\big(\{x:\ g_\lambda^*(x) > s\}\big) \leqslant C \cdot s^{-q} \int_{R_n} S^q(1, x)\, \omega(x)\, dx$$

which is the statement of the theorem.

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