Therefore

$$T\colon \begin{array}{l} X_1^{\infty} X_2^{p} \to L(\infty,\ \infty), \\ X_1^{1} X_2^{p} \to L(1/m,\ \infty). \end{array}$$

Interpolating,

$$T\colon\ X_1^{r,q} X_2^{p} \to L(r/m,\ q).$$

The theorem is proved.

Again, taking $r = q = p$ and $\eta$ = Poisson kernel, we get

$$\Big(\int\limits_0^\infty \Big(\sup_{t\leqslant u} \int\limits_{\mathbf{R}^{n-m}} |f(y_1 \ldots y_n,\ u)|^p\, dy_{m+1} \ldots dy_n\Big) t^{m-1}\, dt\Big)^{1/p} \leqslant C_p \|f\|_{L^p(\mathbf{R}^n)}.$$

We can strengthen the last conclusion somewhat by applying the following theorem (the notation is the same as in Theorem 3).

THEOREM 4. *Under the assumptions of Theorem 3, if $K(x_1)$ is also radial*

$$\Big(\int\limits_0^\infty t^{m-1}\big(Tf(t)\big)^p\, dt\Big)^{1/p}$$

$$\leqslant C_p \Big(\int\limits_0^\infty \varrho^{m-1} \int\limits_{\mathbf{R}^{n-m}} \Big(\int\limits_{\Sigma_{m-1}} |f(\varrho y_1',\ y_2)|\, d\sigma_{m-1}(y_1')\Big)^p\, dy_2\, d\varrho\Big)^{1/p}.$$

We omit the proof.

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# Weighted norm inequalities for parabolic fractional integrals


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**Abstract.** Norm inequalities are obtained for parabolic fractional integrals of distributions whose maximal functions belong to $L^p(\mathbf{R}_n,\ \omega(x)\,dx)$, where $0 < p < \infty$ and $\omega(x)$ is a weight satisfying an $A^p$-type condition and an anti-Hölder condition.


**§ 1. Introduction.** In this paper we obtain weighted norm inequalities for parabolic fractional integrals of distributions. The explicit definition of this fractional integrals and their existence are given in Theorem 2. The norm inequalities are stated in Theorem 5. The unweighted case has already been considered by A. P. Calderón and A. Torchinsky (see [1], [2] and [10]). For the classic case of harmonic functions and $p > 1$ weighted norm inequalities were obtained by B. Muckenhoupt and R. L. Wheeden in [9].

The basis of our method is a generalization of a result due to L. Carleson and extended by P. L. Duren (see [3] and [5]) and the result stated in Theorem 4. Theorem 5 is obtained from Theorem 4 by applying some techniques developed by L. I. Hedberg in [7] and G. V. Welland in [11] for the weighted case. A similar method but technically much simpler was already used in [8] in order to extend the results of B. Muckenhoupt and R. L. Wheeden in [9].

**§ 2. Definitions and notations.** We shall consider an $n\times n$ real matrix $P$, satisfying $(Px,\ x) \geqslant (x,\ x)$ for every $x \in \mathbf{R}_n$, where $(y,\ x)$ indicates the ordinary inner product in the $n$-dimensional Euclidean space $\mathbf{R}_n$. The transpose of $P$ with respect to this inner product will be denoted by $P^*$. $P$ defines the continuous group of transformations $\{t^P\}_{t>0}$, where $t^P = e^{\ln t \cdot P}$. For $x \in \mathbf{R}_n$, $x \neq 0$, the function $\varrho(x)$ is defined as the unique value of $t$ such that $|t^{-P}x| = 1$, where $|x|$ designates the norm of $x$ in $\mathbf{R}_n$, and $\varrho(0) = 0$. The function $\varrho(x)$ satisfies $\varrho(t^P x) = t\varrho(x)$ and $\varrho(x+y) \leqslant \varrho(x) + \varrho(y)$, thus it defines a translation invariant metric $\varrho(x-y)$. Likewise, since $(P^*x,\ x) = (x,\ Px) \geqslant (x,\ x)$, we can associate to $P^*$ a function $\varrho^*(x)$. We shall say that a function $\Omega(x)$ *is $\varrho$-homogeneous of degree $m$*

if for every $t > 0$ and $x \neq 0$ $\Omega(t^P x) = t^m \Omega(x)$. For more details about this metric $\varrho$ see [1].

Given a set $A \subset \boldsymbol{R}_n$, $\complement A$ represents the complement of $A$ in $\boldsymbol{R}_n$, and if $A$ is Lebesgue measurable, by $|A|$ we mean the Lebesgue measure of $A$. Let $B_\varrho(x; t) = \{y \in \boldsymbol{R}_n:\ \varrho(x-y) < r\}$, be the $\varrho$-ball with center $x$ and radius $r$. If $\gamma$ denotes the trace of the matrix $P$, it follows that $|B_\varrho(x; r)| = \pi^{n/2}\Gamma\left(\frac{n+2}{2}\right)^{-1} r^\gamma$. Let $\boldsymbol{R}^+_{n+1}$ be the upper-half space defined as $\boldsymbol{R}^+_{n+1} = \{(x, t):\ x \in \boldsymbol{R}_n,\ t > 0\}$. The $\varrho$-cone, with vertex at $x$ and amplitude $a$, will be the set $\Gamma^a_\varrho(x) = \{(y, t) \in \boldsymbol{R}^+_{n+1}:\ \varrho(x-y) < at\}$. Given a subset $A \subset \boldsymbol{R}_n$, $\Gamma^a_\varrho(A)$ indicates the union $\bigcup\{\Gamma^a_\varrho(x):\ x \in A\}$.

Let $S$ be the space of the infinetely differentiable, rapidly decreasing functions and let $S'$ be its strong dual, that is to say, the space of temperate distributions on $\boldsymbol{R}_n$. By $\varphi_t(x)$ we denote the function $t^{-\gamma}\varphi(t^{-P}x)$. For $\varphi \in S$ and $f \in S'$, $(\varphi_t * f)(x)$ will stand for $\langle f(y), \varphi_t(x-y)\rangle$. Given a function $\varphi \in S$ we define the norm

$$|||\varphi|||_k = \Big\{\int_{\boldsymbol{R}_n} (1+\varrho(x))^k \sum_{|m|\leqslant k} |\partial^m \varphi(x)|^2\, dx\Big\}^{1/2}$$

and denote by $A_k$ the class of all the functions $\varphi$ such that $|||\varphi|||_k \leqslant 1$. To a function $u(x, t)$ defined over $\boldsymbol{R}^+_{n+1}$ we associate the non-tangential maximal function

$$M_\varrho(a, u; x) = \sup_{(y,t)\in\Gamma^a_\varrho(x)} |u(y, t)|. \tag{2.1}$$

Given $f \in S'$ we consider

$$N_k(f; x, t) = \sup_{\varphi\in A_k} |(\varphi_t * f)(x)| \tag{2.2}$$

and the "grand maximal", of amplitude 1:

$$G_k(f, x) = M_\varrho\big(1, N_k(f; y, t), x\big). \tag{2.3}$$

Let $\omega(x)$ be a non-negative, measurable and locally integrable function on $\boldsymbol{R}_n$, the $\omega$-measure of the Lebesgue measurable subset $A$ of $\boldsymbol{R}_n$, is given by $m_\omega(A) = \int_A \omega(x)\,dx$. We say that any such a function $\omega(x)$ *belongs to the class* $A_p$, $1 < p < \infty$, if there exists a finite constant $C$ such that

$$\left(\frac{1}{|B_\varrho(x; r)|}\int_{B_\varrho(x;r)} \omega(x)\,dx\right)\left(\frac{1}{|B_\varrho(x; r)|}\int_{B_\varrho(x;r)} \omega(x)^{\frac{-1}{(p-1)}}\,dx\right)^{p-1} \leqslant C \tag{2.4}$$



for every $r > 0$ and $x \in \boldsymbol{R}_n$. By $A_\infty$ we mean the class which is the union of all the classes $A_p$ for $1 < p < \infty$. We define the class $(AH)_{1+\sigma}$, $\sigma > 0$, as the class of all the weights $\omega(x)$ belonging to $A_\infty$, such that

$$\left(\frac{1}{|B_\varrho(x; r)|}\int_{B_\varrho(x;r)} \omega(x)^{1+\sigma}\,dx\right)^{\frac{1}{(1+\sigma)}} \leqslant C\,\frac{1}{|B_\varrho(x; r)|}\int_{B_\varrho(x;r)} \omega(x)\,dx$$

holds for every $x \in \boldsymbol{R}_n$, $r > 0$ and $C$ a finite constant depending only on $\omega$.

We point out that it is a simple consequence of the definition of $A_\infty$, that for any $\omega \in A_\infty$, there exists a finite constant $C$ such that

$$m_\omega\big(B_\varrho(x; 2r)\big) \leqslant C m_\omega\big(B_\varrho(x; r)\big) \tag{2.5}$$

holds, for every $r > 0$ and $x \in \boldsymbol{R}_n$.

Finally, we wish to remark that for a measure $\nu$ satisfying (2.5), the $L^p$ norms, $0 < p < \infty$, with respect to $\nu$ of $G_k(f, x)$ and $G_h(f, x)$ are equivalent if $k$ and $h$ are large enough. The proof is the same as that given by C. Fefferman and E. M. Stein in [6] for $\nu$ equal to the Lebesgue measure.

## § 3. Results.

LEMMA 1. *Let $\nu \geqslant 0$ be a Borel measure on $\boldsymbol{R}_n$ such that $\nu(\boldsymbol{R}_n) = +\infty$ and*

$$\nu\big(B_\varrho(x; 2r)\big) \leqslant C\cdot\nu\big(B_\varrho(x; r)\big) \tag{3.1}$$

*holds for every $x \in \boldsymbol{R}_n$, $r > 0$, with $C$ a fixed finite constant. Let $A$ be an open subset of $\boldsymbol{R}_n$ with $\nu(A) < +\infty$. If $\{B_\varrho(\delta):\ \delta \in \Delta\}$ is a family of $\varrho$-balls satisfying*

$$\text{(i)}\quad B_\varrho(\delta) \subset A,\qquad \text{(ii)}\quad \bigcup\{B_\varrho(\delta):\ \delta\in\Delta\} = A, \tag{3.2}$$

*then there exists a sequence $\{B_\varrho(\delta_i):\ \delta_i \in \Delta\}$ such that*

$$B_\varrho(\delta_i) \cap B_\varrho(\delta_j) = \emptyset \quad \textit{if} \quad i \neq j \tag{3.3}$$

*and*

$$\bigcup B^*_\varrho(\delta_i) \supset A, \tag{3.4}$$

*where $B^*_\varrho$ is the ball with the same center as $B_\varrho$ and radius five times as large.*

For a proof of this lemma see [4].

In the following theorem we extend, to the parabolic case and more general measures, a result due to L. Carleson and L. P. Duren, see [3] and [5].

THEOREM 1. *Let $\nu$ be a Borel measure as in Lemma 1, and let $\mu$ be a Borel measure on $\boldsymbol{R}^+_{n+1}$ such that for every cylinder $Q(x; r) = B_\varrho(x; r)\times(0, r)$ $\boldsymbol{R}^+_{n+1}$ satisfies*

$$\mu\big(Q(x; r)\big) \leqslant C\cdot\big(\nu\big(B_\varrho(x; r)\big)\big)^\beta, \tag{3.5}$$

*where $\beta \geqslant 1$ and $C$ is a finite constant independent of $Q(x; r)$. Then, if $u(x, t)$ is a Borel measurable function on $\mathbf{R}_{n+1}^+$ such that $M_\varrho(1, u, x)$ belongs to $L^p(\mathbf{R}_n, d\nu)$, we have that*

$$\left(\iint_{\mathbf{R}_{n+1}^+} |u(x, t)|^{\beta p}\, d\mu\right)^{1/\beta p} \leqslant C\cdot\left(\int_{\mathbf{R}_n} M_\varrho(1, u, x)^p\, d\nu\right)^{1/p} \tag{3.6}$$

*holds, with $C$ a finite constant independent of $u(x, t)$.*

Proof. Let $\lambda > 0$ and consider the set

$$A = \{x: M_\varrho(1, u, x) > \lambda\}.$$

This set $A$ is open and, since $M_\varrho(1, u, x) \in L^p(\mathbf{R}_n, d\nu)$, we have $\nu(A) < +\infty$. For every point $x \in A$, we choose a positive number $r(x)$ which satisfies $B_\varrho(x; r(x)) \subset A$ and $B_\varrho^*(x; r(x)) \cap \complement A \neq \emptyset$. This number $r(x)$ exists because, by hypothesis, $A$ is open, $\nu(A) < +\infty$ and $\nu(\mathbf{R}_n) = +\infty$. The family $\{B_\varrho(x; r(x)): x \in A\}$ of open $\varrho$-balls satisfies the assumptions of Lemma 1, and therefore, there must be a sequence $B_i = B_\varrho(x_i; r(x_i))$ satisfying (3.3) and (3.4). Let $Q_i$ denote the cylinder $B_\varrho(x_i; 10\cdot r(x_i)) \times (0, 10\cdot r(x_i))$ and consider a point $(x, t) \in \complement\Gamma_\varrho^1(\complement A)$. For this $(x, t)$ we have that $\varrho(x-z) \geqslant t$ holds for every $z \in \complement A$. In particular, this shows that $x \in A$. Since by Lemma 1 the sequence $\{B_i^*\}$ covers $A$, there exists $i$ such that $x \in B_i$. From the definition of $r(x)$, we know that there is a point $z \in B_i^* \cap \complement A$. For this point $z$, we have

$$t \leqslant \varrho(x-z) \leqslant \varrho(x-x_i) + \varrho(x_i - z) < 10\cdot r(x_i)$$

thus,

$$(x, t) \in B_i^* \times (0, 10\cdot r(x_i)) \subset Q_i.$$

Therefore, we have proved that

$$\complement\Gamma_\varrho^1(\complement A) \subset \bigcup Q_i. \tag{3.7}$$

Now, let $D = \{(x, t): |u(x, t)| > \lambda\}$. We shall show that the set $D$ is contained in $\complement\Gamma_\varrho^1(\complement A)$. Take $(x, t) \notin \complement\Gamma_\varrho^1(\complement A)$, then $(x, t)$ belongs to $\Gamma_\varrho^1(z)$ for some $z \in \complement A$. By definition of $A$, this implies that $|u(x, t)| \leqslant M_\varrho(1, u, z) \leqslant \lambda$, which shows that $(x, t) \notin D$. Therefore, considering (3.7), we get

$$D \subset \complement\Gamma_\varrho^1(\complement A) \subset \bigcup Q_i$$

which implies

$$\mu(D) \leqslant \sum \mu(Q_i)$$

and by (3.5), we also have

$$\mu(D) \leqslant \sum \mu(Q_i) \leqslant C\cdot\sum \nu\big(B(x_i; 10\cdot r_i(x_i))\big)^\beta. \tag{3.8}$$



The assumption made that $\nu$ satisfies (3.1) and $\beta \geqslant 1$ applied to the term on the right-hand side of (3.8) gives

$$\mu(D) \leqslant C\cdot\left(\sum \nu(B_i)\right)^\beta$$

and since, by definition and (3.3), $\{B_i\}$ is a disjoint family and $B_i \subset A$, we get

$$\mu(D) \leqslant C\cdot(\nu(A))^\beta. \tag{3.9}$$

Next, we shall estimate the $\beta p$-norm of $u$ with respect to $\mu$. We have

$$\iint_{\mathbf{R}_{n+1}^+} |u(x, t)|^{\beta p}\, d\mu = \beta p \int_0^\infty \lambda^{\beta p - 1} \mu\big(\{(x, t): |u(x, t)| > \lambda\}\big)\, d\lambda.$$

By (3.9), this is bounded by

$$\begin{aligned} C\cdot\beta p \int_0^\infty \lambda^{\beta p-1} \nu\big(\{x: M_\varrho(1, u, x) > \lambda\}\big)^\beta\, d\lambda & \\ \leqslant C\cdot\sup\big(\nu(\{x: M_\varrho(1, u, x) > \lambda\})\cdot\lambda^p\big)^{\beta-1} & \times \\ \times\, \beta p \int_0^\infty \lambda^{p-1}\cdot\nu\big(\{x: M_\varrho(1, u, x) > \lambda\big)\, d\lambda. & \end{aligned} \tag{3.10}$$

But, since

$$\nu\big(\{x: M_\varrho(1, u, x) > \lambda\}\big)\cdot\lambda^p \leqslant \int_{\mathbf{R}_n} M_\varrho(1, u, x)^p\, d\nu$$

and

$$p\int_0^\infty \lambda^{p-1}\nu\big(\{x: M_\varrho(1, u, x) > \lambda\}\big)\, d\lambda = \int_{\mathbf{R}_n} M_\varrho(1, u, x)^p\, d\nu,$$

from (3.10), we get

$$\iint_{\mathbf{R}_{n+1}^+} |u(x, t)|^{\beta p}\, d\mu \leqslant C\cdot\left(\int_{\mathbf{R}_n} M_\varrho(1, u, x)^p\, d\nu\right)^\beta$$

which is equivalent to (3.6).

COROLLARY (to Theorem 1). *Let $\varphi \in S$ and $\omega \in (AH)_{1+\sigma}$. Assume that $f \in S'$ and satisfies that $M_\varrho(1, \varphi_t * f, x) \in L^p(\mathbf{R}_n, \omega(x)\,dx)$. Then, there exists a finite constant $C$, independent of $f$, such that*

$$|(\varphi_t * f)(x)| \leqslant C\big\{m_\omega(B_\varrho(x; 1))\big\}^{-(1/p)}\cdot t^{-\gamma\sigma/p(1+\sigma)}\cdot \|M_\varrho(1, \varphi_t * f, x)\|_{p,\omega} \tag{3.11}$$

*holds for every $t \geqslant 1$.*

Proof. Let $\mu$ be the Dirac $\delta$-measure at $(x, t)$ and $Q(r) = B_\varrho(y; r) \times (0, r)$. For this $\mu$ and $Q(r)$ we have $\mu(Q(r)) = 1$ or $0$ depending on whether $(x, t) \in Q(r)$ or not. By definition of $Q(r)$, if $(x, t) \in Q(r)$, then $\varrho(x-y) < r$ and $t < r$. This implies that $B_\varrho(x; t) \subset B_\varrho(y; 2r)$. Therefore,

$$m_\omega(B_\varrho(x; t)) \leqslant m_\omega(B_\varrho(y; 2r))$$

and by (2.5), we also have

$$m_\omega(B_\varrho(x; t)) \leqslant C\cdot m_\omega(B_\varrho(y; r));$$

thus,

$$\mu(Q(r)) \leqslant C\cdot m_\omega(B_\varrho(x; t))^{-1}\cdot m_\omega(B_\varrho(y; r))$$

which shows that the measure $\mu$ defined here, and the measure $d\nu(x) = \omega(x)dx$, satisfy the assumptions of Theorem 1 for $\beta = 1$. Then,

$$(3.12)\qquad |(\varphi_t * f)(x)|^p = \iint_{\boldsymbol{R}^+_{n+1}} |(\varphi_s * f)(y)|^p d\mu \leqslant C\cdot m_\omega(B_\varrho(x; t))^{-1}\cdot \int_{\boldsymbol{R}_n} M_\varrho(1, \varphi_s * f, y)^p \omega(y)\,dy.$$

Now, since $\omega(x) \in (AH)_{1+\sigma}$, we have

$$m_\omega(B_\varrho(x; t))^{-1} \leqslant C\cdot m_\omega(B_\varrho(x; 1))^{-1}\cdot t^{-\gamma\sigma/(1+\sigma)}$$

for $t \geqslant 1$. Therefore, from (3.12), we obtain

$$|(\varphi_t * f)(x)|^p < C\cdot m_\omega(B_\varrho(x; t))^{-1}\cdot t^{-\gamma\sigma/(1+\sigma)}\cdot \int_{\boldsymbol{R}_n} M_\varrho(1, \varphi_t * f, x)^p \omega(x)\,dx,$$

which is equivalent to (3.11).

Let $\varphi \in S$ and $f \in S'$. For $\eta > 0$ and $\alpha > 0$ we define the function $f_{\alpha,\eta}$ as

$$f_{\alpha,\eta}(x) = \int_\eta^{\eta^{-1}} (\varphi_s * f)(x)\cdot s^{\alpha\gamma}\,ds/s.$$

Of course, this definition depends on the $\varphi \in S$ chosen.

THEOREM 2. *Let $f \in S'$, $\varphi \in S$ and $0 < p < +\infty$. Assume that $\omega \in (AH)_{1+\sigma}$ and $M_\varrho(1, \varphi_t * f, x) \in L^p(\boldsymbol{R}_n, \omega(x)dx)$. Then, if $0 < \alpha < \sigma/p(1+\sigma)$, the limit of $f_{\alpha,\eta}$ for $\eta$ tending to zero, exists in $S'$ and the action on $S$ of the limit distribution $f_\alpha$ is given by*

$$(3.13)\qquad \langle f_\alpha, \psi\rangle = \int_0^\infty s^{\alpha\gamma}\Big(\int_{\boldsymbol{R}_n} (\varphi_s * f)(x)\psi(x)\,dx\Big)ds/s,$$

*for every $\psi \in S$.*

*Moreover, if $f \in S$, the Fourier transform of $f_\alpha$ coincides with the function*

$$\hat{f}(x)\cdot\varrho^*(x)^{-\alpha\gamma}\cdot\Omega(x),$$



*where $\Omega(x)$ is the $\varrho^*$-homogeneous function of degree zero given by*

$$(3.14)\qquad \Omega(x) = \varrho^*(x)^{\alpha\gamma}\int_0^\infty \varphi(s^{P^*}x)\cdot s^{\alpha\gamma}\,ds/s.$$

Proof. In order to prove (3.13), it is sufficient to prove that for every $\psi \in S$ the integral

$$(3.15)\qquad \int_0^\infty s^{\alpha\gamma}\Big|\int_{\boldsymbol{R}_n} (\varphi_s * f)(x)\psi(x)\,dx\Big|\,ds/s$$

is finite. It can be easily shown that

$$(3.16)\qquad \int_{\boldsymbol{R}_n} (\varphi_s * f)(x)\psi(x)\,dx = \int_{\boldsymbol{R}_n} \langle f(x), \psi(x+y)\rangle \varphi_s(y)\,dy.$$

Now, since $f \in S'$, there is a positive integer $N$ and a finite constant $C$ such that

$$|\langle f(x), \psi(x+y)\rangle| \leqslant C\cdot(\varrho(y)+1)^N,$$

therefore, the absolute value of (3.15) is smaller than or equal to a constant times

$$s^{-\gamma}\int_{\boldsymbol{R}_n} (\varrho(y)+1)^N |\varphi(s^{-P}y)|\,dy = \int_{\boldsymbol{R}_n} (s\varrho(y)+1)^N |\varphi(y)|\,dy.$$

If we assume $0 < s \leqslant 1$, the last integral is uniformly bounded in $s$ by

$$\int_{\boldsymbol{R}_n} (\varrho(y)+1)^N |\varphi(y)|\,dy < +\infty,$$

therefore,

$$\int_0^1 s^{\alpha\gamma}\Big|\int_{\boldsymbol{R}_n} (\varphi_s * f)(x)\psi(x)\,dx\Big|\,ds/s \leqslant C\cdot\int_0^1 s^{\alpha\gamma}\,ds/s < +\infty.$$

Let $s > 1$. By Corollary to Theorem 1, we have

$$\Big|\int_{\boldsymbol{R}_n} (\varphi_s * f)(x)\psi(x)\,dx\Big| \leqslant C\cdot s^{-\gamma\sigma/p(1+\sigma)}\cdot \int_{\boldsymbol{R}_n} m_\omega(B_\varrho(x; 1))^{-1/p}|\psi(x)|\,dx\,\|M_\varrho(1, \varphi_t * f, x)\|_{p,\omega}.$$

Simple computations show that, from (2.5), we can get a $\theta$ such that

$$m_\omega(B_\varrho(x; 1))^{-1/p} \leqslant C\cdot(\varrho(x)+1)^\theta$$

thus,

$$\Big|\int_{\boldsymbol{R}_n} (\varphi_s * f)(x)\psi(x)\,dx\Big| \leqslant C\cdot s^{-\gamma\sigma/p(1+\sigma)}.$$

Then, since by hypothesis, $\alpha < \sigma/p(1+\sigma)$, we get that

$$\int_1^\infty s^{\alpha\gamma}\Big|\int_{\boldsymbol{R}_n} (\varphi_s * f)(x)\psi(x)\,dx\Big|\,ds/s < +\infty.$$

Thus (3.15) is finite and (3.13) is proved.

If $f \in S$, $\psi \in S$, we have

$$\langle f_a, \hat{\psi}\rangle = \int_0^\infty s^{a\gamma}\Big(\int_{R_n}(\varphi_s * f)(x)\hat{\psi}(x)\,dx\Big)ds/s$$
$$= \int_0^\infty s^{a\gamma}\Big(\int_{R_n}\hat{f}(x)\cdot\hat{\varphi}(s^{P^*}x)\cdot\psi(x)\,dx\Big)ds/s$$
$$= \int \hat{f}(x)\cdot\varrho^*(x)^{-a\gamma}\Big(\varrho^*(x)^{a\gamma}\int_0^\infty \hat{\varphi}(s^{P^*}x)\cdot s^{a\gamma}\,ds/s\Big)\psi(x)\,dx.$$

Therefore,

$$\hat{f}_a(x) = \hat{f}(x)\cdot\varrho^*(x)^{-a\gamma}\cdot\Omega(x),$$

which ends the proof of the theorem.

We observe that the set of all the functions given by (3.14) for $\varphi \in S$, coincides with the set of all the $\varrho^*$-homogeneous functions of degree zero which are $C^\infty$ in $R_n - \{0\}$.

THEOREM 3. *Let $f \in S'$. There exists a constant $C_k$, independent of $f$, such that*

$$N_k(f, y, t) \leqslant C_k\Big(1+\frac{\varrho(y-z)}{t}\Big)^k N_k(f, z, s)$$

*for every $y, z$ and $|s-t| < t/2$.*

Proof. Let $\varphi \in A_k$ and consider $(z, s) \in R^+_{n+1}$ with $|t-s| < t/2$. If

$$\Phi(\xi) = (t^{-1}s)^\gamma\varphi\big(t^{-P}(y-z)+(t^{-1}s)^P\xi\big),$$

then,

$$(\varphi_t * f)(y) = (\Phi_s * f)(z).$$

Simple computations show that

$$|||\Phi|||_k \leqslant C_k\Big(1+\frac{\varrho(z-y)}{t}\Big)^k,$$

where $C_k$ is a finite constant independent of $f$. Therefore,

$$|(\varphi_t * f)(y)| \leqslant C\Big(1+\frac{\varrho(y-z)}{t}\Big)^k N_k(f, z, s),$$

or still,

$$N_k(f, y, t) \leqslant C\Big(1+\frac{\varrho(y-z)}{t}\Big)^k N_k(f, z, s),$$

which is the statement of the theorem.



LEMMA 2. *Let $f \in S'$ and $a > 0$. The function*

$$I^h_a(f, x) = \sup\{(t+s)^{a\gamma}|\varphi_t * \psi_s * f(y)|:\ \varrho(x-y) < t+s,\ \varphi \in A_h,\ \psi \in A_h,\ s > 0,\ t > 0\}$$

*satisfies, for $h$ big enough, the inequality*

$$I^h_a(f, x) \leqslant C\Big(\iint_{\Gamma^1_\varrho(x)} N_k(f, z, r)^q r^{aq\gamma-1}\,dz\,dr\Big)^{1/q},$$

*with $C$ a finite constant independent of $f$.*

Proof. We observe that the expression defining $I_a(f, x)$ is symmetric in $s$ and $t$, thus it is enough to consider only the supremum for, say, $s \geqslant t$. If $s \geqslant t$, we have

$$(t+s)^{a\gamma}|(\varphi_t * \psi_s * f)(y)| \leqslant 2^{a\gamma}s^{a\gamma}|(\varphi_t * \psi_s * f)(y)|.$$

Let $M > k+\gamma$. If $h$ is big enough, since $\varphi \in A_h$, we have

$$|\varphi(x)| \leqslant C\cdot(1+\varrho(x))^{-M},$$

with $C$ finite and independent of $\varphi \in A_h$. Therefore, if $\chi(r)$ is the characteristic function of $(0, 1)$, we get (see [1])

$$|\varphi(x)| \leqslant C\cdot\sum 2^{-iM}\chi(\varrho(x)/2^i).$$

Then,

$$|(\varphi_t * \psi_s * f)(y)| \leqslant \int |\varphi_t(y-z)|\cdot|(\psi_s * f)(z)|\,dz \tag{3.17}$$
$$\leqslant C\cdot t^{-\gamma}\sum 2^{-iM}\int_{R_n}\chi(\varrho(y-z)/2^i t)|\psi_s * f)(z)|\,dz.$$

Let $\xi$ and $r$ be such that $\varrho(y-\xi) < s$, $|s-r| < s/2$. By Theorem 3, we have

$$|(\psi_s * f)(z)| \leqslant C\Big(1+\frac{\varrho(z-\xi)}{s}\Big)^k N_k(f, \xi, r).$$

But since $\varrho(z-\xi) \leqslant \varrho(z-y)+\varrho(y-\xi) < 2^i t+s < 2^{i+1}s$, we get

$$|(\psi_s * f)(z)| \leqslant C 2^{k(i+2)}\cdot N_k(f, \xi, r).$$

Then, (3.17) implies

$$|(\varphi_t * \psi_s * f)(y)| \leqslant C\Big(\sum 2^{k(i+2)+\gamma i-iM}\Big)\cdot N_k(f, \xi, r).$$

Thus, since we choose $M$ such that $M > k+\gamma$, we have that the series on the right-hand side is convergent and we can write

$$|(\varphi_t * \psi_s * f)(y)| \leqslant C N_k(f, \xi, r) \tag{3.18}$$

for $\varrho(y-\xi) < s$, $|s-r| < s/2$.

Let $\varrho(x-y) < s+t < 2s$, then $\varrho(x-\xi) < 3s$ and (3.18) implies

$$s^{a\gamma}|(\varphi_t*\psi_s*f)(y)| \leqslant Cs^{a\gamma}\Big(s^{-(\gamma+1)}\iint\limits_{\substack{\varrho(x-\xi)<3s\\ |s-r|<s/2}} N_k(f,\xi,r)^q\,d\xi\,dr\Big)^{1/q}. \tag{3.19}$$

Still, applying Theorem 3 once more, we can see that the right-hand side of (3.19) is smaller than or equal to

$$Cs^{a\gamma}\Big(s^{-(\gamma+1)}\iint\limits_{\substack{\varrho(x-\xi)<2s/3\\ |s-r|<r/2}} N_k(f,\xi,r)^q\,d\xi\,dr\Big)^{1/q}$$

with a new constant $C$. Since $r/2 < s < 3r/2$, we can write all the powers of $s$ inside the integral, getting

$$C\Big(\iint\limits_{\substack{\varrho(x-\xi)<2s/3\\ |s-r|<r/2}} N_k(f,\xi,r)^q r^{a\gamma q-(\gamma+1)}\,d\xi\,dr\Big)^{1/q}.$$

Finally, observing that $\varrho(x-\xi) < 2s/3 < r$, we obtain that the last integral above is smaller than a constant times

$$\Big(\int\int\limits_{\Gamma_\varrho^1(x)} N_k(f,\xi,r)^q r^{a\gamma q-(\gamma+1)}\,d\xi\,dr\Big)^{1/q}.$$

This expression is independent of $s \geqslant t > 0$, $\varphi$ and $\psi \in A_h$, and $y$ such that $\varrho(x-y) < t+s$. Therefore, we get

$$I_a^h(f,x) \leqslant C\Big(\int\int\limits_{\Gamma_\varrho^1(x)} N_k(f,\xi,r)^q r^{a\gamma q-(\gamma+1)}\,d\xi\,dr\Big)^{1/q},$$

which proves the lemma.

THEOREM 4. *Let $\omega \in (AH)_{1+\sigma}$ and $f \in S'$ such that $G_k(f,x) \in L^p(\boldsymbol{R}_n, \omega(x)dx)$. Let $0 < a < 1/p\cdot\sigma/(1+\sigma)$ and $1/q = 1/p - a$. Then the function $I_a^h(f,x)$ satisfies*

$$\Big(\int I_a^h(f,x)^q\omega(x)^{q/p}\,dx\Big)^{1/q} \leqslant C\Big(\int G_k(f,x)^p\omega(x)\,dx\Big)^{1/p},$$

*with a constant $C$ independent of $f$.*

Proof. In Lemma 2 we have shown that

$$I_a^h(f,x) \leqslant C\Big(\int\int\limits_{\Gamma_\varrho^1(x)} N_k(f,\xi,r)^q r^{a\gamma q-(\gamma+1)}\,d\xi\,dr\Big)^{1/q}.$$

Then, taking the $q$-power, multiplying by $\omega(x)^{q/p}$ and integrating on $\boldsymbol{R}_n$, we get

$$\int\limits_{\boldsymbol{R}_n} I_a^h(f,x)^q\omega(x)^{q/p}\,dx \leqslant C\int\limits_{\boldsymbol{R}_n}\omega(x)^{q/p}\Big(\int\int\limits_{\Gamma_\varrho^1(x)}\{N_k(f,\xi,r)\}^q r^{a\gamma q-(\gamma+1)}\,d\xi\,dr\Big)dx$$
$$= \int\int\limits_{\boldsymbol{R}_{n+1}^+}\{N_k(f,\xi,r)\}^q r^{a\gamma q-(\gamma+1)}\Big(\int\limits_{B_\varrho(\xi,r)}\omega(x)^{q/p}\,dx\Big)d\xi\,dr.$$



Now, if we define the measure $\mu$ as

$$d\mu = r^{a\gamma q-(\gamma+1)}\Big(\int\limits_{B_\varrho(\xi,r)}\omega(x)^{q/p}\,dx\Big)d\xi\,dr, \tag{3.20}$$

we get that, for a cylinder $Q(z;s) = B_\varrho(z;s)\times(0,s)$, its $\mu$-measure is

$$\mu\big(Q(z;s)\big) = \int_0^s\int\limits_{B_\varrho(z;s)} r^{a\gamma q-(\gamma+1)}\Big(\int\limits_{B_\varrho(\xi;r)}\omega(x)^{q/p}\,dx\Big)d\xi\,dr \tag{3.21}$$
$$\leqslant Cs^{a\gamma q}\int\limits_{B_\varrho(z;2s)}\omega(x)^{q/p}\,dx.$$

But since $q/p = 1+qa < 1+q\sigma/p(1+\sigma)$, it follows $(q/p) < 1+\sigma$. Therefore by Hölder's inequality and since $\omega \in (AH)_{1+\sigma}$,

$$\Big(\frac{1}{|B_\varrho(z;2s)|}\int\limits_{B_\varrho(z;2s)}\omega(x)^{q/p}\,dx\Big)^{p/q} \leqslant \Big(\frac{1}{|B_\varrho(z;2s)|}\int\limits_{B_\varrho(z;2s)}\omega(x)^{1+\sigma}\,dx\Big)^{1/(1+\sigma)}$$
$$\leqslant C\cdot\frac{1}{|B_\varrho(z;2s)|}\int\limits_{B_\varrho(z;2s)}\omega\,dx.$$

Thus, applying this to (3.21) and recalling that $B_\varrho(z;2s) = C\cdot s^\gamma$, we get

$$\mu\big(Q(z;s)\big) \leqslant Cs^{a\gamma q}\int\limits_{B_\varrho(z;2s)}\omega(x)^{q/p}\,dx \leqslant Cs^{a\gamma q+\gamma-q\gamma/p}\Big(\int\limits_{B_\varrho(z;2s)}\omega(x)\,dx\Big)^{q/p}$$
$$= C\Big(\int\limits_{B_\varrho(z;2s)}\omega(x)\,dx\Big)^{q/p},$$

which by (2.5) is smaller than or equal to

$$C\Big(\int\limits_{B_\varrho(z;s)}\omega(x)\,dx\Big)^{q/p}.$$

This shows that the measures $d\mu$ defined in (3.20) and $\nu$, defined as $d\nu = \omega(x)dx$, satisfy the hypothesis of Theorem 1. Therefore, since $N_k(f,\xi,t)$ is a Borel measurable function on $\boldsymbol{R}_{n+1}^+$, and $M_\varrho(1, N_k(f,\xi,t), x) = G_k(f,x)$, we can apply Theorem 1 with $\beta = q/p$ to $u(x,t) = N_k(f,x,t)$ obtaining

$$\int\limits_{\boldsymbol{R}_n} I_a^h(f,x,t)^q\omega(x)^{q/p}\,dx \leqslant C\cdot\iint\limits_{\boldsymbol{R}_{n+1}^+} N_k(f,\xi,r)^q\,d\mu \leqslant C\cdot\int\limits_{\boldsymbol{R}_n} G_k(f,x)^p\omega(x)\,dx,$$

which proves the theorem.

THEOREM 5. *Let $\omega \in (AH)_{1+\sigma}$ and $p > 0$. Consider an $a$ such that $0 < a < \sigma/p(1+\sigma)$ and let $q$ satisfy $1/q = 1/p - a$. Let $f \in S'$ with $G_k(f,x) \in L^p(\boldsymbol{R}_n, \omega(x)dx)$, $k$ big enough. Then, if $f_a$ is the distribution defined in*

*Theorem 2 for a $\varphi \in A_h$, $h > 2(k+\gamma)$, we have that*

$$\Big(\int_{R_n} G_h(f_a, x)^q \omega(x)^{q/p} dx\Big)^{1/q} \leqslant C \cdot \Big(\int_{R_n} G_k(f, x)^p \omega(x) dx\Big)^{1/p}$$

*holds with a finite constant $C$ independent of $f$ and $\varphi \in A_h$.*

Proof. Take $\psi \in A_h$. By (3.13), we can write

$$(\psi_t * f_a)(y) = \int_0^\infty s^{a\gamma} (\psi_t * \varphi_s * f)(y) ds/s.$$

Then, for any $\delta > 0$, we have

$$|(\psi_t * f_a)(y)| \leqslant \int_0^\delta + \int_\delta^\infty s^{a\gamma} |(\psi_t * \varphi_s * f)(y)| ds/s.$$

Let $\eta$ be a positive number satisfying

$$0 < a - \eta < a + \eta < \frac{1}{p} \cdot \frac{\sigma}{1+\sigma}, \tag{3.22}$$

we get

$$|(\psi_t * f_a)(y)| \leqslant \sup_{s>0} s^{a\gamma - \eta\gamma} |(\psi_t * \varphi_s * f)(y)| \frac{\delta^{\eta\gamma}}{\eta\gamma} + \sup_{s>0} s^{a\gamma+\eta\gamma} |(\psi_t * \varphi_s * f)(y)| \frac{\delta^{-\eta\gamma}}{\eta\gamma},$$

which, by definition of $I_a^h(f, x)$, implies that, if $\varrho(x-y) < t$,

$$G_h(f_a, x) \leqslant I_{a-\eta}^h(f, x) \frac{\delta^{\eta\gamma}}{\eta\gamma} + I_{a+\eta}^h(f, x) \frac{\delta^{-\eta\gamma}}{\eta\gamma}.$$

Choosing $\delta = \{I_{a+\eta}^h(f, x)\}^{1/2\eta\gamma} \cdot \{I_{a-\eta}^h(f, x)\}^{-1/2\eta\gamma}$, we get,

$$G_h(f_a, x) \leqslant (\eta\gamma)^{-1} I_{a-\eta}^h(f, x)^{1/2} \cdot I_{a+\eta}^h(f, x)^{1/2}.$$

Then, taking the $q$-power, multiplying both sides by $\omega(x)^{q/p}$ and integrating, we obtain

$$\int G_h(f_a, x)^q \omega(x)^{q/p} dx \leqslant C \int I_{a-\eta}^h(f, x)^{q/2} \cdot I_{a+\eta}^h(f, x)^{q/2} \omega(x)^{q/p} dx. \tag{3.23}$$

Let $q_1$ and $q_2$ be such that

$$1/p - (a-\eta) = 1/q_1, \quad 1/p - (a+\eta) = 1/q_2,$$

then

$$q/q_1 = 1 + \eta q, \quad q/q_2 = 1 - \eta q.$$

Now, by (3.22), $0 < \eta < 1/p - a$. Then, $0 < q/q_1 < 1 + q(1/p - a) = 2$ and $0 = 1 - (1/p - a)q < 1 - \eta q < 1$. Then $2q_1/q > 1$, $2q_2/q > 1$ and $q/2q_1 + q/2q_1 = 1$. Therefore, by Hölder's inequality, from (3.23) we get

$$\int G_h(f_a, x)^q \omega(x)^{q/p} dx \leqslant C \Big(\int \{I_{a-\eta}^h(f, x)\}^{q_1} \omega(x)^{q_1/p} dx\Big)^{q/2q_1} \Big(\int \{I_{a+\eta}^h(f, x)\}^{q_2} \omega(x)^{q_2/p} dx\Big)^{q/2q_1}.$$



Hence, by Theorem 4, we obtain

$$\int_{R_n} G_h(f_a, x)^q \omega(x)^{q/p} dx \leqslant C \cdot \Big(\int_{R_n} G_k(f, x)^p \omega(x) dx\Big)^{q/p},$$

which proves the theorem.

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