that is, $(x_n \otimes y_n)$ is clearly a (weak) $\sigma(E\otimes F, E'\otimes F')$-Cauchy sequence. On the other hand, by hypothesis (cf. also Section 2), it follows that $A\otimes B$ is ε-bounded, so that, by Lemma 3.2, $(x_n\otimes y_n)$ is a $\sigma(G, G')$-Cauchy sequence in $A\otimes B$, that is, $A\otimes B$ is conditionally (weakly) $\sigma(G, G')$-compact.

(2) *implies* (1). Let $(z_n)_{n\in N}$ be a sequence in A and let $y\in B$ with $y\neq 0$. Then there exists by hypothesis a subsequence (x_n) of (z_n) such that $(x_n\otimes y)$ is a (weakly) $\sigma(G, G')$-Cauchy sequence. Now if $x'\in E'$ and $y'\in F'$ with $|\langle y, y'\rangle| = 1$, then for every $n, m\in N$ obviously follows

$$|\langle x_n\otimes y - x_m\otimes y, x'\otimes y'\rangle| = |\langle (x_n - x_m)\otimes y, x'\otimes y'\rangle| = |\langle x_n - x_m, x'\rangle|\cdot|\langle y, y'\rangle| = |\langle x_n - x_m, x'\rangle|.$$

Thus, (x_n) is clearly a weak Cauchy sequence in A and hence A is conditionally weakly compact and the proof is completed.

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MATHEMATICAL INSTITUTE
UNIVERSITY OF ATHENS, GREECE

Received January 28, 1975 (941)

STUDIA MATHEMATICA, T. LXI. (1977)

The algebra of finitely additive measures on a partially ordered semigroup

by

J. S. PYM (Sheffield) and H. L. VASUDEVA (Chandigarh)

Abstract. The algebra of all finitely additive measures on a discrete semigroup which is a product of totally ordered sets provided with the multiplication max is studied. It is found that all proper maximal left ideals are the kernels of complex homomorphisms, and that the quotient of the algebra by its radical is isomorphic with the usual measure algebra on the almost periodic compactification of the original semigroup.

The algebra of all finitely additive measures on a discrete semigroup is in general very difficult to study. This is in part because any other algebra of measures (finite or countably additive) on the same semigroup provided with any topology can be obtained as a quotient of this one, and so we are in a sense asking to study all these algebras at once. Success therefore depends on severely restricting the class of semigroups under consideration. This policy was followed in [5] where we treated the case of a totally ordered semigroup (that is, a totally ordered set given the multiplication max). In the present paper, we offer similar results for finite direct products of such semigroups.

In Section 2, we show that every maximal left ideal of the algebra (which is, of course, non-commutative) is the kernel of a complex-valued homomorphism, and thus is two-sided. The exact form of the complex homomorphisms is in fact given in Theorem 2.9. The quotient of the algebra by its radical is therefore commutative; it turns out to be the algebra of countably additive measures on a certain compact semigroup. This semigroup is the almost periodic compactification of the original semigroup (and in fact coincides with the weakly almost periodic compactification, which we found in [5]). Moreover, it is a finite product of compact totally ordered semigroups.

Our justifications for presenting these results are first, that the appearance of the almost periodic (rather than the weakly almost periodic) compactification should be recorded. Secondly, the proofs we gave in [5] do not extend to the present case. Moreover, although the greater generality gives an appearance of greater complexity, the methods of this paper are basically simpler than those of [5]. Thirdly, there is good reason to believe that if any infinite product of totally ordered semigroups is considered, the structure of the algebra of finitely additive measures is very much more complicated. There are suggestions of this in the works of Baartz [1] and Newman [3], and we have further evidence which we hope to publish elsewhere. Incidentally, the method described in the last section of [5] for obtaining results about measure algebras on locally compact semigroups from results about finitely additive measure algebras can be used here to recover (with new proofs) the relevant part of Baartz's work; as the idea is described in [5], we have not included the details here.

1. Preliminaries. Let $T_1, \ldots, T_k$ be totally ordered sets each with the semigroup structure obtained by defining $xy = \max\{x, y\}$ and each with a minimal element and a maximal element. Provide $S = T_1 \times \ldots \times T_k$ with the product structure; then S is a lattice, has the multiplication max, and has a minimal and a maximal element. We shall denote the *minimal* (resp. *maximal*) *element* of any partially ordered set by 0 (resp. ω); this will cause no real confusion. We can regard each T_i $(1 \leqslant i \leqslant k)$ as a subsemigroup of S by identifying it with its image under the map $x \mapsto (0, \ldots, 0, x, 0, \ldots, 0)$ (where the x appears in the ith place). In this way, S is seen to be the direct sum of $T_1, \ldots, T_k$.

A *prime subsemigroup* of a semigroup is a non-empty subsemigroup whose complement is an ideal, and so 0 belongs to every prime subsemigroup. A *prime ideal* is the complement of a prime subsemigroup. A *segment* U_i in the totally ordered semigroup T_i is a subset with the properties

(1) $0 \in U_i$

and

(2) if $x \in U_i$ and $y \leqslant x$, then $y \in U_i$.

It is easy to see that the prime subsemigroups of T_i are just the segments.

PROPOSITION 1.1 ([1]). (1) *U is a prime subsemigroup iff there are segments U_i in T_i $(1 \leqslant i \leqslant k)$ such that*

$$U = U_1 \times \ldots \times U_k \; (= U_1 \cdot U_2 \cdots U_k).$$

(2) *I is a prime ideal in S iff there are prime ideals I_i in T_i $(1 \leqslant i \leqslant k)$ such that*

$$I = \bigcup_{i=1}^{k} T_1 \times \ldots \times T_{i-1} \times I_i \times T_{i+1} \times \ldots \times T_k.$$

Proof. (1) Obviously if U is of the prescribed form it is a prime subsemigroup. Conversely if U is a prime subsemigroup, $U \cap T_i = U_i$ is a prime subsemigroup of T_i and so a segment. Thus, $U \supseteq U_1 \cdot U_2 \cdots U_k = U_1 \times \ldots \times U_k$. On the other hand, if $x = (x_1, \ldots, x_k) \notin U_1 \times \ldots \times U_k$, then for some i, $x_i \notin U_i$, i.e. $y = (0, \ldots, 0, x_i, 0, \ldots, 0) \notin U$; since $S \setminus U$ is an ideal, $x = xy \notin U$.

(2) follows immediately from (1).

Let U be a prime subsemigroup of S. In the notation of Proposition 1.1, if $I = S \setminus U$, we write $\tilde{U} = I_1 \times \ldots \times I_k$, and call it the *ideal opposite* U. Notice that $\tilde{U}$ may be empty even though $U \neq S$.

We shall need a notation for certain subsets of S. We put

$$[0, x] = \{y: y \leqslant x\}; \quad [x, \omega] = \{y: y \geqslant x\};$$

and, if $x = (x_1, \ldots, x_k)$

$$[0, x[\; = \{y = (y_1, \ldots, y_k): 0 \leqslant y_i < x_i \; (1 \leqslant i \leqslant k)\};$$
$$]x, \omega] = \{y = (y_1, \ldots, y_k): x_i < y_i \leqslant \omega \; (1 \leqslant i \leqslant k)\}.$$

Thus, if $U = [0, x]$, $\tilde{U} = \,]x, \omega]$; and if $U = [0, x[$, $\tilde{U} = [x, \omega]$.

We shall denote by $\mathscr{B} = \mathscr{B}(S)$ the space of all complex-valued bounded functions on S made into a Banach space with its usual (supremum) norm. Its dual space, $\mathscr{F} = \mathscr{F}(S)$, is the Banach space of all finitely additive measures on the discrete space S. The semigroup structure of S gives rise to an algebra multiplication $*$ in $\mathscr{F}$:

$$\mu * \nu(f) = \int \cdot \int f(xy)\, d\nu(y) \cdot d\mu(x) \qquad (f \in \mathscr{B}),$$

where it is convenient to use the integral notation, though in general the Fubini theorem does not hold (i.e. $*$ is not commutative). This multiplication will be called *convolution*.

There is a natural embedding $x \mapsto \delta_x$ of S in $\mathscr{F}$, where $\delta_x(f) = f(x)$ for $f \in \mathscr{B}$. This map is an injection and preserves the semigroup structure of S. It is also easily checked that $\delta_x * \mu = \mu * \delta_x$ for $x \in S$, $\mu \in \mathscr{F}$. Thus, $\mathscr{F}$, with its dual norm, becomes a Banach algebra with identity δ_0.

A set E is said to *carry* $\mu \in \mathscr{F}$ iff $\mu(f) = 0$ whenever f vanishes on E. This is the same as saying that $\mu(\chi_E f) = \mu(f)$ for all $f \in \mathscr{B}$, where χ_E is the characteristic function of E. For any $E \subseteq S$, $\mu \in \mathscr{F}$, $f \in \mathscr{B}$, we write $\mu_E(f) = \mu(\chi_E f) = \int_E f\, d\mu$, and we call μ_E the *restriction* of μ to E.

PROPOSITION 1.2. *If μ (resp. ν) is carried by E (resp. F), then $\mu * \nu$ is carried by EF.*

Proof. This is easy to see since, for $f \in \mathscr{B}$,

$$\mu * \nu(f) = \int_E \cdot \int_F f(xy)\, d\nu(y) \cdot d\mu(x).$$

A special role in our work is played by certain measures which are located at the vertex of the prime subsemigroup U, i.e. at the corner opposite 0. An *inner vertex measure* of U is a member of the set

$$\mathscr{I}(U) = \{\mu \in \mathscr{F}\colon \text{ for each } x \in U,\ \mu \text{ is carried by } U \cap]x, \omega]\}.$$

Thus, if $\mu \in \mathscr{I}(U)$, the restriction of μ to $U \cap]x, \omega]$ for $x \in U$ is independent of x and is always equal to μ. The intersection of the carriers of μ is empty (so μ is entirely without support in the sense of [4]). If U has a maximal element, say $U = [0, y]$, then any measure in $\mathscr{I}(U)$ is carried by $[0, y] \cap \cap]y, \omega] = \emptyset$, and so $\mathscr{I}(U) = \{0\}$. If U_i is a segment in the totally ordered semigroup T_i, and if U_i has no maximal element, then $U_i \cap]x, \omega] \neq \emptyset$ for each $x \in U_i$ and $\mathscr{I}(U_i) \neq \{0\}$ (any linear functional dominated by the sublinear $f \mapsto \limsup_{x \in U_i} |f(x)|$ is in $\mathscr{I}(U_i)$). Moreover, if $\mathscr{I}(U) \neq \{0\}$ we can find $\mu \in \mathscr{I}(U)$ with $\mu(1) \neq 0$; for $\mu \in \mathscr{I}(U)$ iff $|\mu| \in \mathscr{I}(U)$, and $|\mu|(1) = 0$ iff $\mu = 0$.

An *outer vertex measure* is an element of

$$\mathscr{O}(U) = \{\mu \in \mathscr{F}\colon \text{ for each } x \in \tilde{U},\ \mu \text{ is carried by } \tilde{U} \cap [0, x[\,\}.$$

Similar remarks apply to this collection of measures. Notice that $\mathscr{I}(U) = \mathscr{O}(U) = \{0\}$ is possible; for example, if $U = [0, x]$ and $\tilde{U} = [y, \omega]$ (which, in the totally ordered case, means that y is the immediate successor of x).

We shall need to calculate some convolutions later, and it may be helpful if we give some typical examples here.

Proposition 1.3. *Let U be a prime subsemigroup.*

(1) *If μ is carried by U and, for each $x \in U$, ν is carried by $[x, \omega]$, then $\mu * \nu = \mu(1).\nu$.*

(2) *If, for each $x \in \tilde{U}$, μ is carried by $[0, x]$ and ν is carried by $\tilde{U}$, then $\nu * \mu = \mu(1)\nu$.*

Proof (1) We have, for $f \in \mathscr{B}$,

$$\mu * \nu(f) = \int_U \cdot \int_{[x, \omega]} f(st)\, d\nu(t) \cdot d\mu(s),$$

where x is any element of U. In calculating the inner integral, s is fixed, and is taken from the range of the outer integral, i.e. from U. We may therefore take $x = s$, and then since $x \leqslant t$, we have $st = t$. The inner integral is therefore simply $\nu(f)$, and the result follows.

(2) is proved in a similar way.

In particular, the result of Proposition 1.3 (1) holds if $\mu, \nu \in \mathscr{I}(U)$, and it can be seen that if $\mathscr{I}(U)$ is large enough, $\mathscr{F}$ cannot be commutative.

2. Maximal left ideals. We fix a proper maximal left ideal $\mathscr{L}$ in $\mathscr{F}$. The *primitive ideal* $\mathscr{K}$ associated with $\mathscr{L}$ is defined by

$$\mathscr{K} = \{\mu\colon \mu * \mathscr{F} \subseteq \mathscr{L}\}.$$

Notice that as $\mathscr{F}$ has an identity, $\mathscr{K} \subseteq \mathscr{L}$, and also that $\mathscr{K}$ is a two-sided ideal. We shall need the following elementary result about primitive ideals, whose proof we include because it is short (see [6]).

Proposition 2.1. *Let $\mathscr{L}, \mathscr{K}$ be as above and let $\mathscr{I}, \mathscr{J}$ be left ideals in $\mathscr{F}$.*

(1) *If $\mathscr{I}\mathscr{J} \subseteq \mathscr{K}$, then either $\mathscr{I} \subseteq \mathscr{K}$ or $\mathscr{J} \subseteq \mathscr{K}$.*

(2) *If $\mathscr{I}\mathscr{J} \subseteq \mathscr{L}$, then either $\mathscr{I} \subseteq \mathscr{K}$ or $\mathscr{J} \subseteq \mathscr{L}$.*

Proof. (1) Suppose $\mathscr{J} \nsubseteq \mathscr{K}$, so that $\mathscr{J}\mathscr{F} \nsubseteq \mathscr{L}$. The linear span $[\mathscr{J}\mathscr{F}]$ of $\mathscr{J}\mathscr{F}$ is a left ideal, not contained in $\mathscr{L}$, so that $[\mathscr{J}\mathscr{F}] + \mathscr{L} = \mathscr{F}$ since $\mathscr{L}$ is maximal. Hence $\mathscr{I}\mathscr{F} = \mathscr{I}[\mathscr{J}\mathscr{F}] + \mathscr{I}\mathscr{L} \subseteq \mathscr{K} + \mathscr{L} \subseteq \mathscr{L}$. Thus $\mathscr{I} \subseteq \mathscr{K}$.

(2) If $\mathscr{J} \nsubseteq \mathscr{L}$, then $\mathscr{J} + \mathscr{L} = \mathscr{F}$ and the argument can proceed as before.

We now begin to discover the structure of $\mathscr{L}$.

Proposition 2.2. *$\{x \in S\colon \delta_x \in \mathscr{K}\} = \{x \in S\colon \delta_x \in \mathscr{L}\} = K$ (say) is a prime ideal in S.*

Proof. Observe first from the definition of $\mathscr{K}$ that if $\mu * \mathscr{F} = \mathscr{F} * \mu$, then $\mu \in \mathscr{K}$ iff $\mu \in \mathscr{L}$. Since δ_x is in the centre of $\mathscr{F}$ for each x, it follows that the two sets we are considering coincide. Obviously K is a two-sided ideal in S.

Now let $x, y \in S$ and suppose $xy \in K$. Then

$$\mathscr{F} * \delta_x * \mathscr{F} * \delta_y = \mathscr{F} * \mathscr{F} * \delta_x * \delta_y \subseteq \mathscr{F} * \delta_{xy} \subseteq \mathscr{K}.$$

Thus, by Proposition 2.1 (1), either $\mathscr{F} * \delta_x \subseteq \mathscr{K}$ or $\mathscr{F} * \delta_y \subseteq \mathscr{K}$, i.e. either $x \in K$ or $y \in K$. It follows that K is prime.

We see from this proposition that $G = S \setminus K$ is a prime subsemigroup, and we may use Proposition 1.1 to conclude that $G = U_1 \times \ldots \times U_k$ where each U_i is a segment in T_i. We now partition the index set $A = \{1, \ldots, k\}$ by writing

$$A_1 = \{i\colon \mathscr{I}(U_i) \neq \{0\} \text{ and } \mathscr{I}(U_i) \subseteq K\};$$
$$A_2 = \{i\colon \mathscr{O}(U_i) \neq \{0\} \text{ and } \mathscr{O}(U_i) \subseteq K\} \setminus A_1;$$
$$A_3 = A \setminus (A_1 \cup A_2).$$

In the definition of A_1, A_2, A_3, by $\mathscr{I}(U_i)$ (resp. $\mathscr{O}(U_i)$) we mean the inner (resp. outer) measures of U_i when it is regarded as a prime subsemigroup of T_i. We also put

$$S_a = \prod_{i \in A_a} T_i, \quad G_a = \prod_{i \in A_a} U_i \quad (1 \leqslant a \leqslant 3)$$

and identify S with $S_1 \times S_2 \times S_3$, G with $G_1 \times G_2 \times G_3$. We shall have occasion to refer to, for example, $\mathscr{I}(G_a)$, regarding G_a as embedded in S_a.

In order to avoid consideration of many special cases in what follows, we shall restrict our attention to one situation only.

We shall assume that $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, $A_3 \neq \emptyset$ *and that* $\mathscr{I}(U_i) \neq \{0\}$ *for* $i \in A_2$, $\mathscr{O}(U_i) \neq \{0\}$ *for* $i \in A_3$.

This restriction applies except to the statement of our main Theorem 2.9, in which the general situation is described. (To deal with the other cases, if $\mathscr{I}(U_i) = \{0\}$ for some $i \in A_2$, then $U_i = [0, x_i]$ for some $x_i \in T_i$ (see the remarks before Proposition 1.3), and we replace $\varrho \in \mathscr{I}(U_i)$ in the proofs below by δ_{x_i}. If $\mathscr{O}(U_i) = \{0\}$ for some $i \in A_3$, we replace $\sigma \in \mathscr{O}(U_i)$ by $\varrho \in \mathscr{I}(U_i)$ or, if this is not possible, by δ_{x_i} as before. Proofs will need some alteration.)

We now continue under our general assumption. We require some more convolution formulas.

LEMMA 2.3. (1) *Let* $\varrho \in \mathscr{I}(G_a)$ *and let* μ *be arbitrary. Then* $\varrho * \mu * \varrho = \varrho(1)\mu * \varrho$.

(2) *Let* $\sigma \in \mathscr{O}(G_a)$ *and let* μ *be arbitrary. Then* $\sigma * \mu * \sigma = \sigma(1)\sigma * \mu$.

Proof. We shall prove only (1). We take $a = 1$ for convenience, and we regard S as $S_1 \times (S_2 \times S_3)$. Since, for each $x \in G_1$, ϱ is carried by $]x, \omega] \times \{0\}$, $\mu * \varrho$ is carried by

$$\big(S_1 \times (S_2 \times S_3)\big) \cdot (]x, \omega] \times \{0\}) =]x, \omega] \times (S_2 \times S_3),$$

using Proposition 1.2. Now, for any f,

$$\varrho * \mu * \varrho(f) = \int_{G_1 \times \{0\}} \cdot \int_{]s_1, \omega] \times (S_2 \times S_3)} f(s_1 t_1, s_2 t_2)\, d\mu * \varrho(t_1, t_2) \cdot d_\varrho(s_1, s_2),$$

where the argument of Proposition 1.3 has been used to write the range of the inner integral as $]s_1, \omega] \times (S_2 \times S_3)$. In this inner integral, $t_1 \geqslant s_1$, and because of the range of the outer integral, $s_2 = 0$. Hence $s_1 t_1 = t_1$, $s_2 t_2 = t_2$; the inner integral is $\mu * \varrho(f)$, and the result follows.

We can now prove a key lemma.

LEMMA 2.4. *Let* $x \in G_1$, $\varrho \in \mathscr{I}(G_2)$, $\sigma \in \mathscr{O}(G_3)$ *and let* $\varrho(1) \neq 0$, $\sigma(1) \neq 0$. *Then* $\delta_x * \sigma * \varrho \notin \mathscr{L}$.

Proof. We suppose $\delta_x * \sigma * \varrho \in \mathscr{L}$, and we show first that either $\delta_x \in \mathscr{K}$ or $\sigma \in \mathscr{K}$ or $\varrho \in \mathscr{K}$. Indeed, we have

$$(\mathscr{F} * \delta_x) * (\mathscr{F} * \sigma) * (\mathscr{F} * \varrho) = \mathscr{F} * \mathscr{F} * \sigma * \mathscr{F} * \delta_x * \sigma * \varrho \subseteq \mathscr{L}$$

using Lemma 2.3(2) and the fact that δ_x commutes with everything. By Proposition 2.1, either $\mathscr{F} * \delta_x \subseteq \mathscr{K}$ or $\mathscr{F} * \sigma \subseteq \mathscr{K}$ or $\mathscr{F} * \varrho \subseteq \mathscr{L}$; since $\mathscr{F}$ has an identity, this means either $\delta_x \in \mathscr{K}$ or $\sigma \in \mathscr{K}$ or $\varrho \in \mathscr{L}$.

Suppose (to obtain a contradiction) that $\varrho \in \mathscr{L}$ but $\varrho \notin \mathscr{K}$. Then $\mathscr{F} * \varrho * \mathscr{F} \nsubseteq \mathscr{L}$ by definition of $\mathscr{K}$ and hence the linear span $[\mathscr{F} * \varrho * \mathscr{F}]$ is a left ideal not contained in the maximal left ideal $\mathscr{L}$. Hence $\mathscr{F} = [\mathscr{F} * \varrho * \mathscr{F}] + \mathscr{L}$. As $\delta_0 \in \mathscr{F}$, there is $\varphi \in [\mathscr{F} * \varrho * \mathscr{F}]$ such that $\delta_0 - \varphi \in \mathscr{L}$. As $\mathscr{L}$ is a left ideal, $\varrho - \varrho(1)\varphi = \varrho * (\delta_0 - \varphi) \in \mathscr{L}$ (using Lemma 2.3(1)).

Since $\varrho \in \mathscr{L}$ and $\varrho(1) \neq 0$, we conclude that $\varphi \in \mathscr{L}$, and hence (as $\delta_0 - \varphi \in \mathscr{L}$) that $\delta_0 \in \mathscr{L}$, which is false.

We next show that $\delta_x \in \mathscr{K}$ is impossible. Indeed, this is equivalent to $x \in K$, which contradicts $x \in G_1 \subseteq G = S \setminus K$.

To obtain a contradiction from the assertion $\varrho \in \mathscr{K}$ is harder. First observe from Proposition 1.3 that since $\mathscr{K}$ is an ideal, $\varrho \in \mathscr{K}$ implies $\mathscr{I}(G_2) \subseteq \mathscr{K}$. Now suppose that $G_2 = U_{i_1} \times \ldots \times U_{i_r}$ (i.e. that $A_2 = \{i_1, \ldots, i_r\}$). Choose $\lambda_j \in \mathscr{I}(U_{i_j})$ $(1 \leqslant j \leqslant r)$ with $\lambda_j(1) \neq 0$. An easy argument about supports shows that $\lambda_1 * \ldots * \lambda_r \in \mathscr{I}(G_2)$. (It might be worth remarking at this point that this convolution product depends on the order in which the λ_j's are taken.) Now using Lemma 2.3(1),

$$(\mathscr{F} * \lambda_1) * \ldots * (\mathscr{F} * \lambda_r) = \mathscr{F} * (\lambda_1 * \ldots * \lambda_r) * \mathscr{F} * \lambda_2 * \ldots * \mathscr{F} * \lambda_r \subseteq \mathscr{K},$$

as $\mathscr{K}$ is an ideal. So by Proposition 2.1, $\mathscr{F} * \lambda_j \in \mathscr{K}$ for at least one j. But this implies $\lambda_j \in \mathscr{K}$ and hence that $\mathscr{I}(U_{i_j}) \subseteq \mathscr{K}$. This means $i_j \in A_1$, a contradiction since $i_j \in A_2$.

The assertion $\sigma \in \mathscr{K}$ leads to a contradiction in a similar way. This completes the proof of Lemma 2.4.

We shall need one more convolution formula.

LEMMA 2.5. *Let* G_1, G_2, G_3 *be as above. Fix* $x \in G_1$. *Suppose that, for every* $y \in \tilde{G}_3$, μ *is carried by* $[0, x] \times G_2 \times [0, y[$. *Let* $\varrho \in \mathscr{I}(G_2)$, $\sigma \in \mathscr{O}(G_3)$, *and let* ν *be arbitrary. Then*

$$\sigma * \mu * \nu * \delta_x * \varrho = \mu(1)\sigma * \nu * \delta_x * \varrho.$$

Proof. We first put $\tau = \nu * \delta_x * \varrho$. Since δ_x is carried by $[x, \omega] \times \{(0, 0)\}$ and for each $z \in G_2$, ϱ is carried by $\{0\} \times]z, \omega] \times \{0\}$, we see that τ is carried by $[x, \omega] \times]z, \omega] \times S_3$. Therefore, for any $f \in \mathscr{B}$,

$$\sigma * \mu * \tau(f) = \int_{\{(0,0)\} \times \tilde{G}_3} \cdot \int_{[0,x] \times G_2 \times [0, s_3[} \cdot \int_{[x,\omega] \times]t_2, \omega] \times S_3} f(s_1 t_1 u_1, s_2 t_2 u_2, s_3 t_3 u_3)\, d\tau(u_1, u_2, u_3) \times d\mu(t_1, t_2, t_3) \cdot d\sigma(s_1, s_2, s_3),$$

where the argument of Proposition 1.3 has been used to replace y in the range for the middle integral by s_3, and z in the range for the innermost integral by t_2. Considering ranges for the various variables shows us that $t_1 \leqslant x \leqslant u_1$, so that $s_1 t_1 u_1 = s_1 u_1$; that $t_2 \leqslant u_2$ so that $s_2 t_2 u_2 = s_2 u_2$; and that $t_3 \leqslant s_3$, so that $s_3 t_3 u_3 = s_3 u_3$. The t variables therefore disappear, and the integral therefore has the value $\mu(1)\sigma * \tau(f)$.

LEMMA 2.6. *Let* G_1, G_2, G_3 *be as above. Fix* $x \in G_1$. *Suppose that, for every* $y \in \tilde{G}_3$, μ *is carried by* $[0, x] \times G_2 \times [0, y[$. *Then* $\mu \in \mathscr{L}$ *iff* $\mu(1) = 0$.

Proof. Let $\varrho \in \mathscr{I}(G_2)$, $\sigma \in \mathscr{O}(G_3)$ with $\varrho(1) \neq 0$, $\sigma(1) \neq 0$, so that $\varrho, \sigma \notin \mathscr{K}$. Then $\delta_x * \sigma * \varrho \notin \mathscr{L}$, by 2.4, and $\sigma \notin \mathscr{L}$ (for $\sigma \in \mathscr{L}$ implies $\sigma(1)\sigma * \mathscr{F} = \sigma * \mathscr{F} * \sigma \in \mathscr{L}$

using Lemma 2.3 (2), whence $\sigma \in \mathscr{K}$). Therefore $\mathscr{F} * \delta_x * \sigma * \varrho + \mathscr{L}$ is a left ideal properly containing $\mathscr{L}$, and is thus $\mathscr{F}$ itself. Since $\delta_0 \in \mathscr{F}$, we can find $\nu \in \mathscr{F}$ so that $\delta_0 - \nu * \delta_x * \sigma * \varrho \in \mathscr{L}$. From Lemmas 2.3 and 2.5 we deduce that

(i) $$\sigma * \mu - \sigma(1)\mu(1)\sigma * \nu * \delta_x * \varrho = \sigma * \mu * (\delta_0 - \nu * \delta_x * \sigma * \varrho) \in \mathscr{L}$$

and also that

(ii) $$\sigma - \sigma(1)\sigma * \nu * \delta_x * \varrho = \sigma * (\delta_0 - \nu * \delta_x * \sigma * \varrho) \in \mathscr{L}.$$

Now suppose $\mu \in \mathscr{L}$. We see then from (i) that $\mu(1)\sigma * \nu * \delta_x * \varrho \in \mathscr{L}$. As $\sigma \notin \mathscr{L}$, (ii) shows that $\sigma * \nu * \delta_x * \varrho \notin \mathscr{L}$. We conclude that $\mu(1) = 0$.

Again, if $\mu(1) = 0$, (i) shows that $\sigma * \mu \in \mathscr{L}$. Then, again from Lemma 2.3

$$(\mathscr{F} * \sigma) * (\mathscr{F} * \mu) = \mathscr{F} * \sigma * \mathscr{F} * \sigma * \mu \subseteq \mathscr{L},$$

and so from Proposition 2.1, either $\sigma \in \mathscr{K} \subseteq \mathscr{L}$ or $\mu \in \mathscr{L}$. Since we know that $\sigma \notin \mathscr{L}$, our conclusion follows.

Our final result is similar to the one we have just proved, but requires the taking of a limit. We next establish its existence.

PROPOSITION 2.7. *Let G_1, G_2, G_3 be as above. Let $\mu \in \mathscr{F}$. For $x \in G_1$, $y \in \tilde{G}_3$ denote by μ_{xy} the restriction of μ to $[0, x] \times G_2 \times [0, y[$. Consider G_1 as a directed set ordered by $\leqslant$ and $\tilde{G}_3$ as directed by $\geqslant$. Then the three limits $\lim_x \lim_y \mu_{xy}$, $\lim_y \lim_x \mu_{xy}$, $\lim_{xy} \mu_{xy}$ exist in the sense of norm convergence and are equal.*

Proof. First, each measure is a linear combination of four positive measures and so we may assume $\mu \geqslant 0$. We begin by considering the convergence of $\mu_{xy}(f)$ for $f \in \mathscr{B}$, and again because each $f \in \mathscr{B}$ is a linear combination of positive functions we may assume $f \geqslant 0$. Observe then that if $x_1 \leqslant x_2$ we have $[0, x_1] \subseteq [0, x_2]$ whence $\mu_{x_1 y} \leqslant \mu_{x_2 y}$ for each y. Hence, for a fixed y, $(\mu_{xy}(f))$ is an increasing net; it is bounded above (by $\mu(f)$) and so $\lim_x \mu_{xy}(f)$ exists. Clearly, $\lim_x \mu_{xy}$ is a positive element of $\mathscr{F}$ and for each x, $\mu_{xy} \leqslant \lim_x \mu_{xy}$. Therefore

$$\|(\lim_x \mu_{xy}) - \mu_{xy}\| = \big((\lim_x \mu_{xy}) - \mu_{xy}\big)(1) \to 0,$$

and the convergence is in norm.

For similar reasons, (μ_{xy}) is decreasing in y, and the limit exists. We therefore have, for each x, y,

$$\lim_y \mu_{xy} \leqslant \mu_{xy} \leqslant \lim_x \mu_{xy}.$$

The left-hand member of this chain is increasing in x, and the right-hand member decreasing in y; therefore the iterated limits both exist in the norm and

$$\lim_x \lim_y \mu_{xy} \leqslant \lim_y \lim_x \mu_{xy}.$$

If we can show that these limits are equal, it will follow from the above inequalities that the double limit $\lim_{xy} \mu_{xy}$ also exists and coincides with them. To this end, notice that if $y_2 \leqslant y_1$, the restriction of $(\lim_x \mu_{xy_1}) - \mu_{xy_1}$ to $S_1 \times S_2 \times [0, y_2[$ is just $(\lim_x \mu_{xy_2}) - \mu_{xy_2}$. Therefore

$$0 \leqslant (\lim_x \mu_{xy_2}) - \mu_{xy_2} \leqslant (\lim_x \mu_{xy_1}) - \mu_{xy_1},$$

and hence

$$0 \leqslant (\lim_y \lim_x \mu_{xy}) - (\lim_y \mu_{xy}) \leqslant (\lim_x \mu_{xy_1}) - \mu_{xy_1}.$$

We now take limits over x to find

$$0 \leqslant (\lim_y \lim_x \mu_{xy}) - (\lim_x \lim_y \mu_{xy}) \leqslant 0,$$

and the proof is finished.

LEMMA 2.8. *For any $\mu \in \mathscr{F}$, $\mu - \lim_{xy} \mu_{xy} \in \mathscr{K}$.*

Proof. To begin, fix $y \in \tilde{G}_3$ and put $\nu_y = \mu - \lim_x \mu_{xy}$. We claim that for each $x \in G_1$, ν_y is carried by the complement of $[0, x] \times G_2 \times [0, y[$. Indeed, for each $x_1 \geqslant x$, the restriction of $\mu - \mu_{x_1 y}$ to $[0, x] \times G_2 \times [0, y[$ is zero, and so the same is true for $\mu - \lim_x \mu_{xy}$. Therefore, for each $x \in G_1$, ν_y is carried by a union of sets of the following three types:

$$]x_i, \omega] \times \prod_{j \neq i} T_j; \qquad \tilde{U}_i \times \prod_{j \neq i} T_j; \qquad [y_i, \omega] \times \prod_{j \neq i} T_j.$$

We will show $\nu_y \in \mathscr{K}$ by proving that measures carried by each of these sets are in $\mathscr{K}$.

We start with the easiest case. Let λ be carried by $[y_i, \omega] \times \prod_{j \neq i} T_j$. As $y \in \tilde{G}_3$, $y \in K$ and so $y_i \in K$ (remember that K is a prime ideal), or equivalently, $\delta_{y_i} \in \mathscr{K}$. Then $\lambda = \delta_{y_i} * \lambda \in \mathscr{K}$ using Proposition 1.3(1) and the fact that $\mathscr{K}$ is an ideal.

Next take λ carried by $\tilde{U}_i \times \prod_{j \neq i} T_j$ with $i \in A_2$. Take $\sigma \in \mathscr{O}(U_i)$ with $\sigma(1) \neq 0$; since $i \in A_2$, $\sigma \in \mathscr{K}$. Then $\sigma(1)\lambda = \lambda * \sigma \in \mathscr{K}$ using Proposition 1.3 (2) and that $\mathscr{K}$ is an ideal.

Finally if λ is carried by $]x_i, \omega] \times \prod_{j \neq i} T_j$ for every $x_i \in U_i$ (where $i \in A_1$), we take $\varrho \in \mathscr{I}(U_i)$ with $\varrho(1) \neq 0$, so that $\varrho \in \mathscr{K}$, and we find that $\varrho(1)\lambda = \varrho * \lambda \in \mathscr{K}$ using Proposition 1.3 (1) and that $\mathscr{K}$ is an ideal.

We have now shown that $\nu_y \in \mathscr{K}$. But $\mathscr{K}$ is closed [6], and so $\lim_y \nu_y \in \mathscr{K}$. This is the required result.

We now give our main theorem. We state the full result (though we shall only prove the case in which $A_1 \neq \emptyset$, $A_2 \neq \emptyset$ and $A_3 \neq \emptyset$). We need some more notation to do this. If $A_3 = \emptyset$ (resp. $A_1 = \emptyset$) we denote by μ_x (resp. μ_y) the restriction of μ to $[0, x] \times G_2$ for $x \in G_1$ (resp. $G_2 \times [0, y[$ for $y \in \tilde{G}_3$), or simply the restriction to $[0, x]$ (resp. $[0, y[$) if also $A_2 = \emptyset$.

THEOREM 2.9. *Each maximal left ideal of $\mathscr{F}$ is the kernel of a complex-valued homomorphism. With the above notations*

$$\mathscr{L} = \mathscr{K} = \{\mu\colon \lim_{xy} \mu_{xy}(1) = 0\} \quad \text{if } A_1 \neq \emptyset \text{ and } A_3 \neq \emptyset;$$

$$\mathscr{L} = \mathscr{K} = \{\mu\colon \lim_{x} \mu_{x}(1) = 0\} \quad \text{if } A_1 \neq \emptyset \text{ and } A_3 = \emptyset;$$

$$\mathscr{L} = \mathscr{K} = \{\mu\colon \lim_{y} \mu_{y}(1) = 0\} \quad \text{if } A_1 = \emptyset \text{ and } A_3 \neq \emptyset;$$

$$\mathscr{L} = \mathscr{K} = \{\mu\colon \mu_{G_2}(1) = 0\} \quad \text{if } A_1 = A_3 = \emptyset.$$

Proof. Let $\mu \in \mathscr{F}$. Consider the measure

$$(\lim_y \mu_{xy}) - (\lim_y \mu_{xy})(1) \cdot \delta_0.$$

For every $y \in \tilde{G}_3$ this is carried by $[0, x] \times G_2 \times [0, y[$. Moreover, its value at the function 1 is 0. By Lemma 2.6, it belongs to $\mathscr{L}$. Since $\mathscr{L}$ is closed, the limit over x also belongs to $\mathscr{L}$, i.e.

$$(\lim_{xy} \mu_{xy}) - (\lim_{xy} \mu_{xy})(1) \cdot \delta_0 \in \mathscr{L}.$$

Now, if $(\lim_{xy} \mu_{xy})(1) = 0$, we see that $\lim_{xy} \mu_{xy} \in \mathscr{L}$ and so, from Lemma 2.8, $\mu \in \mathscr{L}$. Conversely, if $\mu \in \mathscr{L}$, from Lemma 2.8 we see that $\lim_{xy} \mu_{xy} \in \mathscr{L}$, and since $\delta_0 \notin \mathscr{L}$, we conclude that $\lim_{xy} \mu_{xy}(1) = 0$. Hence $\mathscr{L}$ is the kernel of the map $\mu \mapsto \lim_{xy} \mu_{xy}(1)$.

We next show that $\mu \mapsto \mu_{xy}(1)$ is a homomorphism. Let χ be the characteristic function of the prime subsemigroup $[0, x] \times G_2 \times [0, y[$. Then χ is a complex homomorphism of S. Hence $\mu \mapsto \mu(\chi) = \mu_{xy}(1)$ is a complex homomorphism of $\mathscr{F}$. Then $\mu \mapsto \lim_{xy} \mu_{xy}(1)$, being a limit of homomorphisms, is itself a homomorphism.

As $\mathscr{L}$ is now seen to be two-sided, it coincides with $\mathscr{K}$.

3. The quotient of $\mathscr{F}$ by its radical.

THEOREM 3.1. *Let $\mathscr{R}$ be the radical of $\mathscr{F}$. Let aS denote the almost periodic compactification of S (when S has the discrete topology).*

(i) *aS is a direct product of compact, totally ordered, totally disconnected semigroups.*

(ii) *The algebra $\mathscr{M}(aS)$ of bounded regular Borel measures on aS is isomorphic with $\mathscr{F}/\mathscr{R}$.*

Proof. We show first that $aS = aT_1 \times \ldots \times aT_k$ (a fact which can be deduced from the very general considerations in [2]) and that each aT_i is totally ordered and totally disconnected.

Let T be any totally ordered semigroup with both maximal and minimal elements and with the discrete topology. Form a new set aT by adjoining a supremum (resp. infimum) for each segment (resp. complement of a segment) which has no maximal (resp. minimal) element; give aT its natural order. An alternative description of aT is this: for each segment U of T take two points s_U, t_U; if $U \subseteq V$, write $s_U \leqslant t_U \leqslant s_V \leqslant t_V$; identify t_U with s_V if and only if for some $x \in T$, $U = [0, x[$ and $V = [0, x]$; embed T in aT by mapping x to $s_{[0,x]}$. Every set in aT clearly has a supremum and an infimum, so aT is compact in the order topology. Moreover, if U is any segment in T, then $[0, s_U] = [0, t_U[$ and $]s_U, \omega] = [t_U, \omega]$ are both open and closed in aT, and as these sets form a sub-base for the open sets of aT, this space is totally disconnected.

To see that aT is in fact the almost periodic compactification of T, we prove it has the universal mapping property. Let $\varphi\colon T \to W$ be a homomorphism of T into a compact jointly continuous semigroup W. As W is jointly continuous, the closure $\overline{\varphi(T)}$ consists entirely of idempotents, and as T is totally ordered, $\overline{\varphi(T)}$ has a natural structure as a totally ordered semigroup. Since $\overline{\varphi(T)}$ is compact, every set in it has both a supremum and an infimum. We extend φ to a map $\bar{\varphi}\colon aT \to \overline{\varphi(T)}$ by defining

$$\bar{\varphi}(s_U) = \sup\{\varphi(x)\colon x \in T,\ x \leqslant s_U\},$$

$$\bar{\varphi}(t_U) = \inf\{\varphi(x)\colon x \in T,\ x \geqslant t_U\}.$$

It is easy to check that $\bar{\varphi}$ is well defined, continuous, and is the unique continuous extension of φ.

We now prove (i). The canonical map $S = T_1 \times \ldots \times T_k \to aT_1 \times \ldots \times aT_k$ extends by the universal mapping property to a continuous map $aS \to aT_1 \times \ldots \times aT_k$. Since $T_1 \times \ldots \times T_k$ is dense in $aT_1 \times \ldots \times aT_k$ and aS is compact, this mapping is surjective. To find an inverse map to it, first consider the composition (for $1 \leqslant i \leqslant k$)

$$T_i \to T_1 \times \ldots \times T_k = S \to T_i$$

consisting of a canonical injection into the direct sum followed by the projection from the direct product. It is the identity, and so its extension

$$aT_i \to aS \to aT_i$$

must also be the identity. Thus aT_i may be considered to be a subset of aS for each i. Since multiplication in aS is jointly continuous, the map $aT_1 \times \ldots \times aT_k \to aS$ defined by $(x_1, \ldots, x_k) \to x_1 \ldots x_k$ (the product in aS) is continuous. Its image is compact and contains the image of $T_1 \times \ldots \times T_k = S$ which is dense, and so the image is the whole of aS. But now the composite map

$$aT_1 \times \ldots \times aT_k \to aS \to aT_1 \times \ldots \times aT_k$$

can be seen to be the identity on $T_1 \times \ldots \times T_k$; and since both maps are surjections, they must in fact be isomorphisms.

We now proceed to (ii). Let $U_i \subseteq T_i$ $(1 \leqslant i \leqslant k)$ be a segment. Let $\mathscr{U}$ be the closed subspace of $\mathscr{B}$ spanned by all the characteristic functions $\chi = \chi_{U_1 \times \ldots \times U_k}$. We map $\mathscr{U}$ into the space $\mathscr{C}(aS)$ of all continuous functions on aS by sending χ to the characteristic function $\overline{\chi}$ of $[0, s_{U_1}] \times \ldots \times [0, s_{U_k}]$ (by the proof that aT is totally disconnected, χ is the characteristic function of a set which is both open and closed, and so is continuous), and we extend to $\mathscr{U}$ by linearity and continuity. Moreover, the collection of all $\overline{\chi}$'s separates the points of aS, and so we may apply the Stone–Weierstrass theorem to see that the map $\mathscr{U} \to \mathscr{C}(aS)$ is an isomorphism.

It is well known that $\mathscr{M}(aS)$ is an algebra. The kernel of $\overline{\chi}$ is a prime ideal of aS, so that $\mu \to \mu(\overline{\chi})$ is a complex homomorphism of $\mathscr{M}(aS)$. Again since the linear span of the $\overline{\chi}$'s is dense in $\mathscr{C}(aS)$, we see that if $\mu(\overline{\chi}) = 0$ for all $\overline{\chi}$, then $\mu = 0$. Hence $\mathscr{M}(aS)$ is semisimple.

Next, the map $\mathscr{C}(aS) \to \mathscr{U} \subseteq \mathscr{B}$ is a isometric embedding; hence the adjoint $\psi\colon \mathscr{F} \to \mathscr{M}(aS)$ is a quotient map. We complete the proof by showing $\operatorname{Ker}\psi = \mathscr{R}$. Now if $\mu \in \operatorname{Ker}\psi$, then $\psi(\mu)(\overline{\chi}) = 0$ for each $\overline{\chi}$ of the above form; thus, $\mu(\chi) = 0$, or $\mu(U_1 \times \ldots \times U_k) = 0$ for all segments $U_1, \ldots, U_k$. In particular, with the notation of Section 2, if $x \in G_1$, $y \in \tilde{G}_3$,

$$\mu([0, x] \times G_2 \times [0, y[) = \mu_{xy}(1) = 0,$$

and so also $\lim_{xy} \mu_{xy}(1) = 0$. By Theorem 2.9, this means that $\mu \in \mathscr{R}$. On the other hand, as $\mathscr{M}(aS)$ is semisimple, $\mathscr{R} \subseteq \operatorname{Ker}\psi$, and the conclusion is achieved.

The theorem corresponding to Theorem 3.1 which we gave in [5], asserted that when S was totally ordered (i.e. $k = 1$) $\mathscr{F}/\mathscr{R}$ was isomorphic to the measure algebra on the weakly almost periodic compactification of S; in the totally ordered case, then, these two compactifications coincide. In fact, this is also true in our present, more general, situation. We hope to publish the proof elsewhere.

Acknowledgement. The authors would like to thank the Science Research Council for the grant which supported this research.

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DEPARTMENT OF PURE MATHEMATICS,
THE UNIVERSITY, SHEFFIELD, ENGLAND
and
DEPARTMENT OF MATHEMATICS
PUNJAB, UNIVERSITY, CHANDIGARH, INDIA

Received February 3, 1975 (944)