STUDIA MATHEMATICA, T. LXI. (1977)

Spectral radius characterizations of commutativity in Banach algebras

by

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Abstract. We prove that if the spectral radius is subadditive or submultiplicative on a complex Banach algebra, then the algebra is commutative modulo the radical.

1. Introduction. It follows from I. M. Gelfand's theory of commutative (complete) normed algebras that the spectrum of elements in such algebras possesses some nice algebraic and continuity properties. In view of the general importance of the notion of spectrum, it should be quite interesting and desirable to examine, conversely, to a what extent these nice properties can affect the commutativity, the essence of the Gelfand theory.

As regards purely algebraic properties, namely the subadditivity and/or the submultiplicativity of the spectral radius, we shall show here that the effect is absolute: each one of these two properties itself implies the commutativity of the algebra (modulo the radical, of course). Accordingly, these two properties of the spectral radius are, in fact, equivalent with each other, a phenomenon which seems to be rather surprising and interesting by itself.

With respect to continuity properties of the spectrum, the problem remains open; nevertheless we feel that the results and ideas of the present paper can be used also in studying various continuity properties of spectra, including further characterizations of commutativity. We discuss briefly some conjectures at the end of this paper.

Concerning our present purpose, significant steps have been made by C. Le Page [8], and R.A. Hirschfeld and W. Żelazko [6]. But the main characterizations obtained there, involving the norm, are not purely algebraic (let us recall the familiar experience that a linear algebra structure can be endowed, in general, with different, i.e. non-equivalent, complete algebra norms). Moreover, they are not quite satisfactory since the algebra in question is *a priori* assumed to be function (uniform), while the general commutative algebra, of course, need not be function. Nevertheless, on p. 198 in [6] it is stated implicitly that both the properties together, subadditivity and submultiplicativity (supposed on a general Banach algebra), imply the commutativity of the algebra modulo its radical. Our main contribution here consists just in showing that each one of these two conditions alone makes the same effect as well. Clearly, the Hirschfeld–Żelazko characterization of function algebras appears then immediately, in a more natural way, from this general result.

As to the submultiplicativity, the result has been recently announced without proof also by B. Aupetit [1], but we have had no occasion to compare his proof during the preparation of the present paper. In our proof, we show at first that the submultiplicativity implies the subadditivity, and then from the latter we derive the commutativity. However, let us mention at this point that it is possible to overleap from submultiplicativity to commutativity directly and even more easily, though at the cost of losing the equivalence with subadditivity. For if the spectral radius is submultiplicative, it is then immediate that its kernel coincides with the radical, while, assuming subadditivity only, this step requires a proof that represents, indeed, the deepest part of the result. Therefore we cannot be sure about the way taken by B. Aupetit. As to the subadditivity, some attempts have been made by Gh. Mocanu [9] and K. Srinivasacharyulu [13], under certain superfluous hypotheses like entire lack of quasi-nilpotents or even completeness of the spectral norm.

Let us quote one more remark. Although the spectral radius on a radical algebra, being zero constant, possesses all the best qualities, no commutativity properties in the algebra, however, can be recognized. Therefore any spectral characterization of commutativity, in general, should be expected modulo the radical, as confirmed also by the present paper.

Besides methods developed in the papers cited above, also a recent idea due to V. Pták and the author [11] plays an important role in the proofs below.

2. The main result. We consider an arbitrary Banach algebra A over the complex field. The spectrum of an element x in A will be denoted by $\sigma(x)$, and the spectral radius by $|x|_\sigma$. (If the algebra A does not have a unit element, write A_1 for its usual unitization [2], p. 15; in this case, by the spectrum of an element in A we mean its spectrum in A_1.) The spectral radius (since the spectrum) is completely determined by the purely algebraic structure of A; on the other hand, it is one of the highly fascinating features of the Banach algebras theory (cf. W. Rudin [12], p. 237) that the spectral radius can be, at the same time, expressed by means of the norm as follows

$$|x|_\sigma = \lim |x^n|^{1/n}.$$

However, the reader should be warned that this formula will be used only in the proof of Lemma 5.

We say that the spectral radius is *subadditive on A* if there is a (positive) constant $\varkappa$ such that the inequality

$$(1)\qquad |x+y|_\sigma \leqslant \varkappa(|x|_\sigma + |y|_\sigma)$$

holds for all x, y in A. Similarly, the spectral radius is said to be *submultiplicative on A* if

$$(2)\qquad |xy|_\sigma \leqslant \gamma |x|_\sigma |y|_\sigma$$

for all x, y in A, with some (positive) constant γ. Clearly (if the algebra is not radical), the constants must be $\geqslant 1$.

The (Jacobson) radical of A will be abbreviated to $\operatorname{rad} A$. The unit element will be denoted by 1, and often left out in expressions like $\lambda - x$.

The main result of the present paper can now be summarized in the following

THEOREM. *Let A be a complex Banach algebra. Then the following three conditions are equivalent:*

1° *the spectral radius is subadditive on A;*

2° *the spectral radius is submultiplicative on A;*

3° *the algebra $A/\operatorname{rad} A$ is commutative.*

Observe that if A has a unit, then it is almost trivial and well known that the spectrum of a class in the Banach algebra $A/\operatorname{rad} A$ is precisely the same as the spectrum of any of its members in A. Hence the same is true also for the spectral radii so that, by Gelfand's theory, condition 3° implies both 1° and 2°.

If A does not have a unit, let us recall that $\operatorname{rad} A = A \cap \operatorname{rad} A_1$ (cf [2], p. 126). Therefore if $A/\operatorname{rad} A$ is commutative, then so is $A_1/\operatorname{rad} A_1$, and the preceding argument can be repeated to show that the spectral radius is subadditive and submultiplicative on A_1, hence on A as well.

Thus, to complete the proof of the theorem, it will be sufficient to verify that 2° ⇒ 1° and 1° ⇒ 3°, for algebras with unit or not. And this is accomplished in what follows.

Before embarking upon that proof, however, one could now feel that the constants $\varkappa$ and γ used in the definitions of subadditivity and submultiplicativity were, in fact, illusory. This is indeed the case since, by virtue of the theorem just now stated, both the properties of the spectral radius occur either simultaneously with $\varkappa = \gamma = 1$ (if $A/\operatorname{rad} A$ is commutative) or never with any constants (if $A/\operatorname{rad} A$ is non-commutative). But the introduction of the constants was necessitated by the method of proof, namely by our endeavour to pass simply from algebras without unit to algebras with unit adjoint; otherwise they play no essential role. Nevertheless, even the fact that, for example, a (formal) subadditivity with some positive constant $\varkappa$ entails the subadditivity with $\varkappa = 1$ seems to be by no means trivial, and may be of independent interest. Also, this more general respect will be useful in applications.

Finally, we are able to pass to the proof of Theorem. It is divided into five lemmas.

3. The proofs. In the first two lemmas, let A be a Banach algebra without unit.

Lemma 1. *If the spectral radius is subadditive on A, then it is subadditive also on A_1.*

Proof. Suppose (1) on A with $\varkappa \geqslant 1$. Take a, b in A. Then, whatever complex numbers α, β might be, we have

$$\begin{aligned}|(a+\alpha)+(b+\beta)|_\sigma &= |(a+b)+(\alpha+\beta)|_\sigma \\ &\leqslant |a+b|_\sigma + |\alpha+\beta| \leqslant \varkappa(|a|_\sigma+|b|_\sigma)+|\alpha|+|\beta| \\ &\leqslant \varkappa(|a|_\sigma+|\alpha|)+\varkappa(|b|_\sigma+|\beta|) \leqslant 3\varkappa(|a+\alpha|_\sigma+|b+\beta|_\sigma),\end{aligned}$$

where the last inequality is a consequence of the fact that

$$|a|_\sigma+|\alpha| \leqslant 3\,|a+\alpha|_\sigma,$$

cf. [2], p. 77. To see this, observe that $\sigma(a)$ contains zero, so that $\alpha \in \sigma(a+\alpha)$, hence $|\alpha| \leqslant |a+\alpha|_\sigma$. Furthermore, $|a|_\sigma = |a+\alpha-\alpha|_\sigma \leqslant |a+\alpha|_\sigma + |\alpha| \leqslant 2\,|a+\alpha|_\sigma$. Consequently, $|a|_\sigma+|\alpha| \leqslant 3\,|a+\alpha|_\sigma$ as desired. We have thus obtained subadditivity on A_1 with constant $3\varkappa$.

Notation. If λ is a complex number, and S a non-empty compact subset of the complex plane, we denote by $d(\lambda, S)$ the distance of the point λ from the set S, i.e.

$$d(\lambda, S) = \inf\{|\lambda - s|: s \in S\}.$$

We employ this notion in the proofs of Lemmas 2 and 3.

Lemma 2. *If the spectral radius is submultiplicative on A, then it is subadditive on A_1.*

Proof. Suppose (2) on A with $\gamma \geqslant 1$. Fix a, b in A. Take a complex number λ such that

(3) $$d(\lambda, \sigma(a)) > \gamma^{1/2}\,|a|_\sigma$$

and, at the same time,

(4) $$d(\lambda, \sigma(b)) > \gamma^{1/2}\,|b|_\sigma.$$

Then $\lambda - a$ has an inverse in A_1; write $(\lambda-a)^{-1} = c+\mu$ with c in A, μ complex. From the equality $(\lambda-a)(\mu+c) = 1$, one has $\mu = 1/\lambda$, hence

$$c = (\lambda-a)^{-1} - 1/\lambda.$$

Also, observe that

(5) $$|\lambda| \geqslant d(\lambda, \sigma(b)) > \gamma^{1/2}\,|b|_\sigma \geqslant |b|_\sigma,$$

since $\sigma(b)$ contains zero. Now we have

$$\begin{aligned}\lambda-(a+b) &= (\lambda-a)-b = (\lambda-a)[1-(\lambda-a)^{-1}b] \\ &= (\lambda-a)(1-b/\lambda-cb) = (\lambda-a)[1-cb(1-b/\lambda)^{-1}](1-b/\lambda);\end{aligned}$$

here the element $1-b/\lambda$ is invertible since, by (5), the spectral radius of b/λ is strictly less than one.

Since A is an ideal in A_1, we have, by assumption, that

$$|cb(1-b/\lambda)^{-1}|_\sigma \leqslant \gamma\,|c|_\sigma\,|b(1-b/\lambda)^{-1}|_\sigma.$$

But now

$$\sigma(c) = \left\{\frac{1}{\lambda-\alpha} - \frac{1}{\lambda}:\ \alpha \in \sigma(a)\right\} = \left\{\frac{\alpha}{(\lambda-\alpha)\lambda}:\ \alpha \in \sigma(a)\right\},$$

so that

(6) $$|c|_\sigma \leqslant \frac{|a|_\sigma}{|\lambda|\,d(\lambda, \sigma(a))} < \frac{1}{|\lambda|}\gamma^{-1/2}.$$

Similarly, the spectrum of the element $b(1-b/\lambda)^{-1}$ is the set

$$\left\{\beta\left(1-\frac{\beta}{\lambda}\right)^{-1}:\ \beta \in \sigma(b)\right\} = \left\{\frac{\lambda\beta}{\lambda-\beta}:\ \beta \in \sigma(b)\right\},$$

so that

(7) $$|b(1-b/\lambda)^{-1}|_\sigma \leqslant |\lambda|\,|b|_\sigma/d(\lambda, \sigma(b)) < |\lambda|\,\gamma^{-1/2}.$$

Multiplying the estimates (6) and (7), we obtain

$$|cb(1-b/\lambda)^{-1}|_\sigma < 1,$$

and so the element $\lambda-(a+b)$, being represented as a product of three invertible elements, is invertible as well.

Thus, we have proved that if λ satisfies both (3) and (4), then it cannot lie in $\sigma(a+b)$. It follows, however, that (imagine a picture!)

$$|a+b|_\sigma \leqslant (1+\gamma^{1/2})\max(|a|_\sigma, |b|_\sigma),$$

and all the more

$$|a+b|_\sigma \leqslant (1+\gamma^{1/2})(|a|_\sigma+|b|_\sigma),$$

for all a, b in A. By Lemma 1, the spectral radius is now subadditive on A_1 with $\varkappa = 3(1+\gamma^{1/2})$.

In view of the (already mentioned) fact that $\operatorname{rad} A = A \cap \operatorname{rad} A_1$, we have reduced the problem completely to algebras with unit. Therefore suppose from now on that A is a Banach algebra with unit. In this case, it is possible to give a slightly simpler proof to the following

Lemma 3. *If the spectral radius is submultiplicative on A, then it is also subadditive on A.*

Proof. Fix a, b in A, and take a λ such that

$$d(\lambda, \sigma(a)) > \gamma |b|_\sigma.$$

Then

$$\lambda-(a+b) = (\lambda-a)-b = (\lambda-a)[1-(\lambda-a)^{-1}b],$$

where

$$|(\lambda-a)^{-1}b|_\sigma \leqslant \gamma |(\lambda-a)^{-1}|_\sigma |b|_\sigma = \gamma \frac{|b|_\sigma}{d((\lambda, \sigma(a))} < 1,$$

so that this λ does not belong to $\sigma(a+b)$. Consequently,

$$|a+b|_\sigma \leqslant |a|_\sigma + \gamma |b|_\sigma \leqslant \gamma(|a|_\sigma + |b|_\sigma),$$

which says that the spectral radius is subadditive on A, this time even with the same constant $\varkappa = \gamma$.

Thus we are now in a position when the implication $2^\circ \Rightarrow 1^\circ$ is proved in general, and it remains to show that $1^\circ \Rightarrow 3^\circ$ in the case of algebras with unit. The following key lemma is analogous to Proposition 6 in C. Le Page [8].

LEMMA 4. *Let A be a Banach algebra with unit. Let c be a fixed regular element in A. Suppose that $|cxc^{-1}-x|_\sigma = 0$ for all x in A. Then the elements $cxc^{-1}-x$ lie in* rad$A$ *for all x in A.*

Proof. We have to show that the elements $cxc^{-1}-x$ lie in all primitive ideals of A. Recall that a primitive ideal P is defined as

$$P = L: A = \{a \in A: aA \subset L\},$$

where L is a maximal left ideal in A. Denote by $a \to a^\sim$ the canonical mapping of A onto $A-L$, the Banach space of cosets modulo L. Then the corresponding left regular representation T: $a \to T(a)$ of the algebra A into the Banach algebra of bounded linear operators on $A-L$, defined by the formula

$$T(a)w^\sim = (aw)^\sim, \quad w \in A,$$

is known to be strictly irreducible, and its kernel coincides just with P; see [2], p. 123. Also, $T(1)$ is the identity operator on $A-L$ (this is trivial). Since, clearly,

$$\sigma(T(a)) \subset \sigma(a)$$

for all a in A, we see that all the operators

$$T(c)T(x)T(c^{-1})-T(x) = T(cxc^{-1}-x)$$

are quasi-nilpotent.

We wish to show that the operator $T(c^{-1})$ is scalar. If this were not so, then there is a vector $v^\sim$ in $A-L$ such that $u^\sim = T(c^{-1})v^\sim$ is linearly independent of $v^\sim$. The representation T being strictly irreducible, there is an operator $T(y)$, for a suitable y in A, such that

$$T(y)u^\sim = u^\sim, \quad T(y)v^\sim = 0.$$

Then we have

$$\begin{aligned}[T(c)T(y)T(c^{-1})-T(y)]v^\sim &= T(c)T(y)u^\sim = T(c)u^\sim \\ &= T(c)T(c^{-1})v^\sim \\ &= T(1)v^\sim = v^\sim \neq 0,\end{aligned}$$

so that the operator $T(cyc^{-1}-y)$ is not quasi-nilpotent.

This contradiction reveals that $T(c^{-1})$ is indeed a scalar multiple of the identity on $A-L$, hence it commutes with anything. Thus $T(cxc^{-1}-x)$ is always the zero operator on $A-L$, which means that the elements under consideration lie in the kernel of T, i.e. in P.

Since the radical is the intersection of the primitive ideals (cf. [2], p. 124), we are done.

Now it remains to prove the last

LEMMA 5. *Suppose that the spectral radius is subadditive on A. Then the algebra $A/$*rad$A$ *is commutative.*

Proof. Take a, b fixed elements in A. We have to show that $ab-ba$ lies in rad$A$. Define a function f from the complex plane into the Banach algebra A as follows

$$f(\lambda) = e^{\lambda a}be^{-\lambda a},$$

for all λ complex.

I would like to thank E. Kirchberg who pointed out me a gap in the first version of this proof. An elementary proof proceeds as follows. For each $\lambda \neq 0$ put

$$g(\lambda) = (f(\lambda)-b)/\lambda$$

and let $g(0) = ab-ba$. Then g is an entire function with values in A. Let λ be fixed. From the Cauchy integral formula for g^n we obtain the estimate

$$(8) \qquad |g(\lambda)|_\sigma \leqslant |g^n(\lambda)|^{1/n} \leqslant \max_{\mu \in \Gamma} |g^n(\mu)|^{1/n},$$

where Γ is a circle centred at λ.

Let $\varepsilon > 0$ be given. From the subadditivity we get

$$|g(\mu)|_\sigma \leqslant 2\varkappa |b|_\sigma / |\mu|$$

for all $\mu \neq 0$. Thus we can choose Γ (centred at λ) so large that

$$|g(\mu)|_\sigma < \varepsilon \quad \text{for all } \mu \in \Gamma.$$

Letting now $n = 2^k$, we can find (by a simple topological argument like in the Dini theorem) an index $k = m$ such that $|g^n(\mu)|^{1/n} < \varepsilon$ for all $\mu \in \Gamma$, with $n = 2^m$. Then (8) yields $|g(\lambda)|_\sigma < \varepsilon$, so it must be $|g(\lambda)|_\sigma = 0$.

This conclusion being true for all b in A, we infer, from Lemma 4, that

$$e^{\lambda a} b e^{-\lambda a} - b \in \operatorname{rad} A.$$

Multiplying this inclusion from the right by $e^{\lambda a}$, we obtain

$$e^{\lambda a} b - b e^{\lambda a} \in \operatorname{rad} A$$

for all λ complex. Accordingly, in the Banach algebra $A/\operatorname{rad} A$ we have

$$e^{\lambda[a]}[b] = [b]e^{\lambda[a]}$$

for all λ complex, where $[x]$ denotes the class of $x \in A$ modulo $\operatorname{rad} A$. Expanding both sides (in $A/\operatorname{rad} A$) into powers of λ, comparison of the coefficients of λ yields

$$[a][b] = [b][a],$$

or

$$ab - ba \in \operatorname{rad} A$$

as desired. This completes the proof of Theorem.

4. Two applications. We now exhibit two (previously known) commutativity criteria, having effect, however, only on a narrower class of function (or uniform) algebras. The first one is due to R. A. Hirschfeld and W. Żelazko [6].

COROLLARY 1. *Let A be a (complex) Banach algebra such that $|x| \leqslant \omega |x|_\sigma$ for all x in A, where ω is a (positive) constant. Then A is commutative.*

Proof. From assumption we have $\operatorname{rad} A = 0$, so it suffices to verify that the spectral radius is, for example, subadditive:

$$|x+y|_\sigma \leqslant |x+y| \leqslant |x| + |y| \leqslant \omega(|x|_\sigma + |y|_\sigma).$$

Alternatively, one could verify also submultiplicativity in the same easy way.

For the second application, let us recall the notion of numerical range. That is defined, for a in A, as the set

$$V(a) = \{\varphi(a)\colon \varphi(1) = |\varphi| = 1,\ \varphi \text{ linear functional on } A\},$$

where the unit is to be added if A does not have it (i.e. consider A_1 in place of A in the above formula). Then the numerical radius is naturally

$$|a|_\nu = \sup\{|\lambda|\colon \lambda \in V(a)\}.$$

The following observation (in the case of algebras with unit) is due to F. F. Bonsall and J. Duncan [3], p. 41; see also K. Srinivasacharyulu [13].

COROLLARY 2. *Let A be a (complex) Banach algebra such that $|x|_\nu \leqslant \omega |x|_\sigma$ for all x in A, where ω is a (positive) constant. Then A is commutative.*

Proof. We have $\operatorname{rad} A = 0$ since $|x|_\nu = 0$ implies $x = 0$, see [2], p. 56. Further, always $|x|_\sigma \leqslant |x|_\nu$, see [2], p. 53. So we have

$$|x+y|_\sigma \leqslant |x+y|_\nu \leqslant |x|_\nu + |y|_\nu \leqslant \omega(|x|_\sigma + |y|_\sigma)$$

as expected. Note that, this time, submultiplicativity of the spectral radius is not obvious at a first glance.

In view of the well-known inequality $|x| \leqslant e|x|_\nu$ (see [2], p. 56), Corollary 2 is also an immediate consequence of Corollary 1. And conversely, if $|x| \leqslant \omega |x|_\sigma$, then $|x|_\nu \leqslant \omega |x|_\sigma$ since obviously $|x|_\nu \leqslant |x|$. Thus, the conditions stated in Corollaries 1 and 2 are, in fact, equivalent with each other, and each of them characterizes the class of function algebras.

5. Some open problems. It follows from the Gelfand theory that the spectrum (and the spectral radius as well) on a commutative Banach algebra is Lipschitzian in the sense that

(8) $$\operatorname{dist}(\sigma(a), \sigma(b)) \leqslant |a-b| \quad \text{for all } a, b.$$

Here dist stands, naturally, for the Hausdorff distance of spectra, i.e.

$$\operatorname{dist}(\sigma(a), \sigma(b)) = \max\{\sup_{\alpha\in\sigma(a)} d(\alpha, \sigma(b)),\ \sup_{\beta\in\sigma(b)} d(\beta, \sigma(a))\}.$$

It is therefore quite natural to ask whether, conversely, this necessary condition (or some its modification) ensures the commutativity of the algebra modulo the radical. In the light of the ideas presented in this paper, it seems to be very plausible that even a uniform continuity of the spectral radius might be sufficient. However, we can support this conjecture only by the case of finite-dimensional algebras, as easily seen from the classical Wedderburn structure theorem for such algebras (recall that the spectral radius on the full n by n matrix algebra is not uniformly continuous unless $n = 1$). It would be interesting to try the problem, say, for C^*-algebras.

On the other hand, let us mention that conditions like (8) can be satisfied on large non-commutative sets (not being algebras!). For example, we have proved in [11] that on the set of all normal operators on a Hilbert space just the inequality (8) holds true. This is one more reason for our question.

To indicate another kind of related questions, suppose that, in a Banach algebra A, the set $N = \{x \in A\colon |x|_\sigma = 0\}$ is a closed linear subspace.

It then follows, by a result of I. N. Herstein [5], p. 228, that N is a Lie ideal, i.e. $cx-xc \in N$ for all $x \in N$, $c \in A$. Moreover, N is also a subalgebra. To show this, take a, b arbitrary elements of N. Then by assumption we have $a+b \in N$, hence also $(a+b)^2 \in N$ and so, again by assumption, $ab+ba \in N$. But by Herstein's result also $ab-ba \in N$. Therefore, once more by assumption, we obtain $ab \in N$ as claimed. (Note that closedness of N plays a role in Herstein's result only.)

Conversely, let the set N of quasi-nilpotents be such that $a, b \in N$ implies $ab \in N$. It is then possible to show, by a similar method as we have used in the proofs of Lemmas 2 and 3 (see also V. Pták [10], p. 268) that N is a linear subspace (i.e. subalgebra) of A. Indeed, let a, b be arbitrary elements of N; it is enough to show that, for example, -1 does not belong to $\sigma(a+b)$, in other words, to prove that $1+a+b$ is regular. Since both $1+a$ and $1+b$ are regular, we can write

$$1+a+b = (1+a)(1+b)-ab = (1+a)(1-uv)(1+b),$$

where $u = (1+a)^{-1}a$ and $v = b(1+b)^{-1}$ have zero spectra. Since, by assumption, uv is quasi-nilpotent, the assertion follows.

These remarks lead to a question whether, under some of the above hypotheses, N must be an ideal of A, i.e. $N = \operatorname{rad} A$. If $\operatorname{codim} N = 1$, the conclusion follows immediately from the Gleason–Kahane–Żelazko theorem (see W. Rudin [12], p. 233). But is there any deeper relation?

An affirmative answer to our question follows easily also, for example, in C^*-algebras. Moreover, by a well-known theorem of I. Kaplansky (see J. Dixmier [4], p. 58), every C^*-algebra without nilpotents is commutative. Accordingly, for C^*-algebras we can formulate the following strengthening of our main theorem.

Proposition. *Let A be a C^*-algebra. Then the following three conditions are equivalent:*

1° *the sum of arbitrary two quasi-nilpotents is again a quasi-nilpotent;*

2° *the product of arbitrary two quasi-nilpotents is again a quasi-nilpotent;*

3° *A is commutative.*

Thus, in the class of C^*-algebras, the result of I. Kaplansky seems to be more powerful than our general characterization.

However, in general, the equality $N = \operatorname{rad} A$ alone, which would be a weaker condition than the subadditivity or submultiplicativity of the spectral radius, does not ensure the commutativity of the algebra $A/\operatorname{rad} A$. For this there is an elegant example, originally due to A. S. Nemirovskij (see a footnote on p. 294 in the Russian edition of the book [12]; cf. also [2], p. 254, where essentially the same example is attributed to J. Duncan and A. W. Tullo). But is it possible to complete appropriately this natural necessary condition, $N = \operatorname{rad} A$, to obtain a commutativity criterion? In [6], Problem 2, R. A. Hirschfeld and W. Żelazko proposed continuity of the spectral radius. Therefore it would be worthwhile to investigate continuity properties of the spectral radius in examples of the mentioned type.

To conclude the paper, the most important remark should be made. The algebra $A/\operatorname{rad} A$ being semi-simple, its topology is uniquely determined (according to a well-known theorem of B. E. Johnson, see [2], p. 130) by the purely algebraic structure of A. Therefore, the continuity or the uniform continuity of the spectral radius (or of the spectrum) on A, which is the same as on $A/\operatorname{rad} A$, is, in fact, also an algebraic property. Thus we have some justification to hope that our main results on the spectral radius may be closely related to the problems discussed in this last section. Moreover, this observation shows that any spectral characterization of commutativity, in the sense of the present paper, should be essentially rather algebraic. Among them, the subadditivity or the submultiplicativity of the spectral radius are, of course, the simplest ones. Nevertheless, it would be interesting to express various kinds of continuity of the spectrum explicitly by means of the algebraic structure.

Added in proof. The conjecture concerning uniform continuity of the spectral radius can be confirmed by similar ideas as in Lemma 5. For another elementary approach (including, in particular, a direct proof of the implication 1° ⇒ 2°) see [18].

In the meantime B. Aupetit [16] has independently published proofs of these and other similar results based on certain deeper properties of subharmonic functions. Some partial results (including the implication 2° ⇒ 3°) had B. Aupetit announced without proof already in [15]; we regret we have learned about this notice only from the paper [16] when it has appeared.

The conjecture concerning the stability of the set N under addition or multiplication has been confirmed in [19]. Indeed, in any Banach algebra the following three conditions are equivalent: 1° $x, y \in N \Rightarrow x+y \in N$; 2° $x, y \in N \Rightarrow xy \in N$; 3° $N = \operatorname{rad} A$.

Further related results can be found in [14], [17] and [20]–[23].

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Received February 5, 1976 (1118)

STUDIA MATHEMATICA, T. LXI. (1977)

On the Fejér–F. Riesz inequality in L^p

by

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Abstract. Using the Lions–Peetre interpolation theory, various generalizations of a classical theorem of Fejér and F. Riesz, are proved.

Introduction. The inequality under consideration is:

$$\Big(\int_0^1 |f(r,\theta)|^p\,dr\Big)^{1/p} \leqslant A_p\Big(\int_{-\pi}^{\pi} |f(\theta)|^p\,d\theta\Big)^{1/p}, \qquad 1 < p \leqslant \infty,$$

where $f(r,\theta)$ is the harmonic function in $r<1$ whose boundary values are $f(\theta)$. See [3]. Fejér and Riesz proved the inequality using complex function theory so that their methods do not extend to $\boldsymbol{R}^n$.

N. du Plessis was the first to generalize the theorem to $\boldsymbol{R}^n$. Another proof and a somewhat stronger generalization (for $n = 3$ only) was given by F. R. Keogh. See [5].

Using interpolation theory we shall present a method for proving strong versions of the various theorems. The proofs are considerably simpler, and the results apply not only to the Poisson kernel, but to others as well. Even in the classical case we get, without any added difficulty, a stronger inequality:

$$\Big(\int_0^1 \operatorname*{Max}_{0\leqslant\varrho\leqslant r} |f(\varrho,\theta)|^p\,dr\Big)^{1/p} \leqslant A_p\Big(\int_{-\pi}^{\pi} |f(\theta)|^p\,d\theta\Big)^{1/p}.$$

The note is divided into two sections. In the first, we shall prove the spherical Fejér–Riesz inequalities, and in the second, the half-space versions.

We shall use freely the language and results of interpolation theory. For an outline of the theory, see for example [4], [6]. An interesting aspect of the application of interpolation theory we make here is that $L(p,q)$ spaces with $p<1$ are used in a natural way, to get results for L^p with $1<p$.