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An infinite family of joint spectra

by

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Abstract. It is constructed an infinite family of uppersemicontinuous joint spectra having the spectral mapping property.

Introduction.

DEFINITION 0.1. Let X be a complex Banach space. We denote by $c_0(X)$ the family of all finite subsets S of $L(X)$, the algebra of all bounded linear operators from X into X. A *spectral system* σ^* *on* $c_0(X)$ (short: *spectrum on* X) is a map $S \to \sigma^*(S)$, where $S = (A_1, \ldots, A_n)$ belongs to $c_0(X)$ and $\sigma^*(S)$ is a subset of $\boldsymbol{C}^n$ satisfying the following conditions.

(a) For each S the set $\sigma^*(S)$ is a compact subset of $\boldsymbol{C}^n$.

(b) If S consists of a single operator A, then $\sigma^*(A) = \sigma^*(S)$ is a non-void subset of the usual spectrum $\sigma(A)$.

The definition as well as Definition 1.4(i), (ii), of spectral mapping property and projection property were for the tirst fime explicitly introduced in [4], and there the reader can find some background of these concepts.

The most important and natural spectral systems, e.g. the usual joint spectrum, the Taylor spectrum, σ_π and some others which are investigated in [4], satisfy the following additional conditions.

(i) σ^* possesses the spectral mapping property (Def. 1.4(i)).

(ii) σ^* is uppersemicontinuous (Def. 2.3).

DEFINITION 0.2. In this paper any joint spectrum which fulfils (i) and (ii) will be called a *joint spectrum*.

Up to now only finite number of so-defined joint spectra was known. The main object of this paper is to construct an infinite family of joint spectra. All constructed spectra are parts of Taylor spectrum. This result is a partial contribution to the problem of describing the family of all joint spectra in given Banach or, especially, Hilbert space.

In § 1 we shall construct spectral systems $\sigma_{\pi,h}$ and $\sigma_{\delta,l}$ and show that the latter have the projection property. As a by-product we obtain a new, relatively simple, proof that the Taylor spectrum has the projection property.

In § 2 the spectra of adjoint operators are expressed by some spectra of given operators. We apply these formulas in proving that the spectra are uppersemicontinuous and that $\sigma_{\pi,k}$ have the projection property. At the end we show that all spectra $\sigma_{\pi,k}$, $\sigma_{\delta,l}$, have the spectral mapping property and draw a conclusion that $\sigma_{\pi,k}\cup\sigma_{\delta,l}$ as well $\sigma_{\pi,h}$, $\sigma_{\delta,l}$ and Taylor's spectrum are joint spectra.

In § 3 we show that all introduced spectra are different in case of infinite-dimensional Hilbert space.

§ 1. The projection property. Let us recall that chain complex consists of linear spaces and endomorphisms

(1.1) $$0\to 0\to X_0\xrightarrow{d} X_1\xrightarrow{d} X_2\to \ldots \to$$

(finite or not) such that all compositions $dd = 0$. As a rule, we shall assume that spaces are Banach and coboundary maps are bounded linear operators.

Only for convenience, we will assume that X_i are left $\mathscr{A}$-modules, and for all $A\in\mathscr{A}$ the operator $x\to Ax$: $X\to X$ belongs to $L(X)$, but we do not assume that there is any topology in $\mathscr{A}$.

$Z(\mathscr{A})$ will denote the center of $\mathscr{A}$.

Chain complex (1.1) is exact at X_i if $\operatorname{Im}(X_{i-1}\xrightarrow{d} X_i) = \ker(X_i\xrightarrow{d} X_{i+1})$.

All spectra which we shall define will be subspectra of Taylor's spectrum. Thus we have to recall the definition of Koszul complex. This will the cohomological version (cf. § 1, [2]), but we shall introduce the complex in the inductive way. This will make proofs easier because it does not involve the machinery of homological algebra.

DEFINITION 1.1. We use induction on n, the number of operators, and assume that $A_1, \ldots, A_n$ belong to $Z(\mathscr{A})$ (for example, $\mathscr{A} = [A_1, \ldots, A_n]$; it will be easy to see that Definition 1.1 does not depend on the choice of $\mathscr{A}$ and Definition 1.3 does not depend on it either). For $n = 1$ by *Koszul complex* we mean the following chain complex:

$$\ldots\to 0\to X\xrightarrow{A_1} X\to 0\to\ldots$$

Let us assume that we have already defined the Koszul complex for any n-tuple from $c_0(X)$. Let $(A_1, \ldots, A_{n+1})\in c_0(X)$, $A_i\in Z(\mathscr{A})$. Let

$$\to 0\to X_0\xrightarrow{d} X_1\to\ldots\xrightarrow{d} X_n\to 0\to\ldots$$

denote the Koszul complex of $(A_1, \ldots, A_n)$. Then by the *Koszul complex of* $(A_1, \ldots, A_{n+1})$ we mean

$$\to 0\to Y_0\xrightarrow{d'} Y_1\xrightarrow{d'} Y_2\to\ldots\xrightarrow{d'} Y_{n+1}\to 0\to\ldots,$$

where

$$Y_p = X_p\oplus X_{p-1}$$

and the coboundary maps d': $Y_p\to Y_{p+1} = X_{p+1}\oplus X_p$ are given by the formula

$$d'(x_p, x_{n-1}) := (dx_p, dx_{p-1}+(-1)^p A_{n+1}X_p),\quad x_p\in X_p,\ x_{p-1}\in X_{p-1}.$$

It appears again that $\{Y_p\}$ is a chain complex, i.e. d''s are $\mathscr{A}$-module homomorphisms and all compositions $d'd'$ are zero. We omit the simple proof which uses the inductive assumption that $dd = 0$ and that d is an $\mathscr{A}$-module homomorphism (hence $dA_{n+1} = A_{n+1}d$).

DEFINITION 1.2. Let $A = (A_1, \ldots, A_n)\in c_0(X)$, ($X$ — $\mathscr{A}$-module, $A_i\in Z(\mathscr{A})$). We say that a tuple of complexes $c = (c_1, \ldots, c_n)\in \Sigma_p(A)$ iff the Koszul complex induced by the tuple $A-cI = (A_1-c_1I, \ldots, A_n-c_nI)$ is not exact at X_p.

Now we are ready to define spectral systems $\sigma_{\pi,k}$, $\sigma_{\delta,k}$, and σ_{T}.

DEFINITION 1.3.

(i) Let k be a non-negative integer and $A = (A_1, \ldots, A_n)\in c_0(X)$. Then

$$\sigma_{\delta,k}(A) := \bigcup_{n-k\leqslant p\leqslant n}\Sigma_p(A).$$

(ii) Let $k\geqslant 0$. We say that a *tuple* $c = (c_1, \ldots, c_n)$ *of complexes belongs to* $\sigma_{\pi,k}(A)$ iff c belongs to the sum $\bigcup_{0\leqslant p\leqslant k}\Sigma_p(A)$ or $\operatorname{Im}(X_k\xrightarrow{d} X_{k+1})$ is not closed, where d is the coboundary map in the Koszul complex induced by the tuple $A-cI := (A_1-c_1I, \ldots, A_n-c_nI)$.

(iii) The *Taylor spectrum* $\sigma_{\mathrm{T}}(A)$ is equal to the sum

$$\sigma_{\mathrm{T}}(A) := \bigcup_{p=-\infty}^{+\infty}\Sigma_p(A).$$

For $k = 0$ these spectra have already been known; $\sigma_{\pi,0}$ is the point approximate point spectrum (denoted by σ_π in [4]) and $\sigma_{\delta,0}$ is the defective spectrum, as it is called by Chandler Davis (denoted by σ_τ in [4]).

DEFINITION 1.4 ([4]).

(i) Let S be a complex Banach space and $S\to\sigma^*(S)$ a spectral system on it. We say that σ^* *has a spectral mapping property with respect to polynomial mappings* if for each S belonging to $c_0(X)$, $S = (A_1, \ldots, A_n)$ and for each polynomial mapping $p = (p_1, \ldots, p_m)$ from $\boldsymbol{C}^n$ into $\boldsymbol{C}^m$

$$\sigma^*(pS) = p\sigma^*(S).$$

Here pS is a member of $c_0(X)$ given by

$$pS = \big(p_1(A_1, \ldots, A_n), \ldots, p_m(A_1, \ldots, A_n)\big).$$

(ii) We say that a spectral system σ^* *has a projection property* if it has a spectral mapping property with respect to special polynomial mappings, namely the projections

$$p(z_1, \ldots, z_m) = (z_1, \ldots, z_n), \quad m \geqslant n.$$

Of course, the projection property is a necessary condition for having the spectral mapping property, so we shall first prove that $\sigma_{\delta,k'}$, σ_T, have a projection property. We need some preparations for that. Next lemma is a slight modification of Lemmas 1 and 2 in [3], or 2.6 and 2.7 in [4].

LEMMA 1.5. *Let X be a Banach space and A belong to $L(X)$. Let $[A]$ denote the subalgebra of $L(X)$ algebraically generated by A and I. Let p be a seminorm on $[A]$ such that*

$$(1.2) \qquad p(BC) \leqslant p(B)\|C\|, \quad B, C \in [A],$$

$$(1.3) \qquad \forall c \in C\ \exists\, \varepsilon_c > 0\ \forall B \in [A]\ \ p\big((A'-cI)B\big) \geqslant \varepsilon_c\, p(B).$$

Then $p \equiv 0$ on $[A]$.

Proof. Suppose that $p \not\equiv 0$. Let Y denote a complex Banach space which is the completion of non-zero quotient space $[A]/\ker p$ endowed with the norm

$$B + \ker p \to p(B).$$

This norm induces in Y the norm which we denote by $\| \ \|$. Let φ denote the canonical injection $[A]/\ker p \to Y$. The operator

$$B + \ker p \to AB + \ker p\colon [A]/\ker p \to [A]/\ker p$$

is continuous from (1.2). Let $T \in L(Y)$ denote its continuous extension. Then by (1.3) it holds that

$$\|(T-cI)\varphi(B+\ker p)\| \geqslant \varepsilon_c \|\varphi(B)\| \quad \text{for every } c \in C \text{ and } B \in [A].$$

As $\operatorname{Im}\varphi$ is dense in Y, we have

$$\|(T-cI)y\| \geqslant \varepsilon_c \|y\|, \quad c \in C, \quad y \in Y,$$

thus $\sigma_\pi(T) = \emptyset$ but this is impossible in non-zero Banach space because $\sigma_\pi(T) \supset \partial\sigma(T)$ (cf. e.g. [5]).

LEMMA 1.6. *Let*

$$\begin{array}{ccc} Z_0 & \xrightarrow{d} & Z \\ \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle A} \\ Z_0 & \xrightarrow{d} & Z \end{array}$$

be a commutative diagram consisting of complex Banach spaces and commuting operators. Let $d(Z_0) \neq Z$. Then there exists a complex number c such that

$$(1.4) \qquad d(Z_0) + (A-cI)(Z) \neq Z.$$

Proof. Put $A_c = A - cI$. Suppose that the conclusion does not hold, i.e. for every complex c there is

$$d(Z_0) + A_c(Z) = Z.$$

This means that the operator

$$(z, z_0) \to A_c z + d z_0\colon Z \oplus Z_0 \to Z$$

is an epimorphism, hence the adjoint operator

$$W \to (A_c^* w, d^* w)\colon Z^* \to Z^* \times Z_0^*$$

is an isomorphical embedding, i.e. there is $\varepsilon_c > 0$ such that

$$(1.5) \qquad \|A_c^* w\| + \|d^* w\| \geqslant \varepsilon_c \|w\| \quad \text{for all } w \in Z^*.$$

From the assumption $d(Z_0) \neq Z$, so that d^* is not an isomorphical embedding. Hence there exists a sequence $(w_k)_{k=1}^{\infty} \subset Z^*$ such that

$$(1.6) \qquad \|w_k\| = 1, \quad k = 1, 2, \ldots,$$

$$(1.7) \qquad \lim_k d^* w_k = 0.$$

Let $[A^*]$ denote the subalgebra of $L(Z^*)$ algebraically generated by A^* and I. Put

$$(1.8) \qquad p(b) = \limsup_k \|Bw_k\| \quad \text{for all } B \text{ in } [A^*].$$

From inequality (1.5) we see that

$$(1.9) \qquad \|A_c^* Bw_k\| + \|d^* Bw_k\| \geqslant \varepsilon_c \|Bw_k\|, \quad B \in [A^*].$$

Every B in A^* is of the form $B = v(A^*)$, where v is a polynomial. So the following holds

$$(1.10) \qquad \lim_k d^* Bw_k = \lim_k d^* v(A^*) w_k = \lim_k v(A_c^*) d^* w_k = 0$$

because of (1.7) and $A_0^* d^* = d^* A^*$. Relation (1.9) implies

$$\limsup_k \|A_c^* Bw_k\| + \limsup_k \|d^* Bw_k\| \geqslant \varepsilon_c \limsup_k \|Bw_k\|.$$

So using (1.10) we obtain

$$(1.11) \qquad p\big((A^*-cI)B\big) \geqslant \varepsilon_c p(B).$$

The seminorm p is defined on $[A^*]$ and satisfies (1.2), (1.3), hence from Lemma 1.5 $p \equiv 0$. But $p(I) = \limsup_k \|Iw_k\| = 1$ (see (1.6)). The contradiction shows that for some complex number c (1.4) holds.

The following theorem is essentially new result of the paper. It is the main tool used in proving the projection property.

THEOREM 1.7. *Let X be a complex Banach space, $A' = (A_1, A_2, \ldots, A_n, A_{n+1}) \in c_0(X)$, $A = (A_1, \ldots, A_n)$. If $P\colon C^{n+1} \to C^n$ is the canonical projection, then for every integer p the following hold:*

(i) $\Sigma_{p-1}(A) \cup \Sigma_p(A) \supset P\Sigma_p(A')$,

(ii) $\Sigma_p(A) \subset P\Sigma_{p+1}(A')$.

Proof. (i) Let

$$\ldots \to 0 \to X_0 \xrightarrow{d} X_1 \to \ldots \xrightarrow{d} X_n \to 0 \to \ldots$$

and

$$\ldots \to 0 \to Y_0 \xrightarrow{d} Y_1 \to \ldots \xrightarrow{d} Y_{n+1} \to 0 \to \ldots$$

denote Koszul complexes of A and A', respectively. Then it is sufficient to show: if

$$\begin{aligned} &\operatorname{Im}(X_{p-2} \xrightarrow{d} X_{p-1}) = \ker(X_{p-1} \xrightarrow{d} X_p),\\ &\operatorname{Im}(X_{p-1} \xrightarrow{d} X_p) = \ker(X_p \xrightarrow{d} X_{p+1}), \end{aligned} \tag{1.12}$$

then

$$\operatorname{Im}(Y_{p-1} \xrightarrow{d} Y_p) = \ker(Y_p \xrightarrow{d} Y_{p+1}). \tag{1.13}$$

As $dd = 0$, we only have to prove the inclusion $\supset$ in (1.13).

If $(x, y) \in Y_p = X_p \oplus X_{p-1}$ and $d(x, y) = 0$, then (cf. Def. 1.1, 1.2)

$$dx = 0,$$

$$dy + (-1)^p A_{n+1} x = 0.$$

There exists $x' \in X_{p-1}$ such that $x = dx'$ (from (1.12)). Then $dy + (-1)^p A_{n+1} x = d(y + (-1)^p A_{n+1} x') = 0$ and from (1.12) there is $x'' \in X_{p-2}$ satisfying $y + (-1)^p A_{n+1} x' = dx''$. The pair (x', x'') belonging to Y_{p-1} fulfils

$$d(x', x'') = (x, y).$$

(ii) Let (X_i) denote the Koszul complex of $(A_1, \ldots, A_n)$. It is sufficient to show: if

$$\operatorname{Im}(X_{p-1} \xrightarrow{d} X_p) \neq \ker(X_p \xrightarrow{d} X_{p+1}), \tag{1.14}$$

then there exists a complex number c such that

$$\operatorname{Im}(Y_p \xrightarrow{d} Y_{p+1}) \neq \ker(Y_{p+1} \xrightarrow{d} Y_{p+2}), \tag{1.15}$$

where (Y_p) denotes the Koszul complex of the tuple $(A_1, \ldots, A_n, A_{n+1} - cI)$. If c does not fulfil (1.15), then for every $x \in \ker(X_p \xrightarrow{d} X_{p+1})$ we have $(0, x) \in \ker(Y_{p+1} \to Y_{p+2})$ (cf. Def. 1.1) and there exists $(x', x'') \in Y_p = X_p \oplus X_{p-1}$ satisfying $(0, x) = d(x', x'')$. Hence $dx' = 0$ and $x = dx'' + (-1)^p (A_{n+1} - cI) x'$, what means that

$$(A_{p+1} - cI)\big(\ker(X_p \xrightarrow{d} X_{p+1})\big) + \operatorname{Im}(X_{p-1} \xrightarrow{d} X_p) = \ker(X_p \xrightarrow{d} X_{p+1}). \tag{1.16}$$

Since d's are $\mathscr{A}$-module homomorphisms, the diagram

$$\begin{array}{ccc} Z_0 & \xrightarrow{d} & Z \\ {\scriptstyle A_{n+1}}\uparrow & & \uparrow{\scriptstyle A_{n+1}} \\ Z & \xrightarrow{d} & Z \end{array}$$

(where $Z = \ker(X_p \xrightarrow{d} X_{p+1})$ and $Z_0 = X_{p-1}$) commutes and from (1.16) assumptions of Lemma 1.6 are fulfilled. Applying the lemma we obtain a complex number c, such that equality (1.16) does not hold and hence (1.15) is true.

COROLLARY 1.8. *All spectral systems $\sigma_{\delta,k}$, $k \geqslant 0$, as well as the Taylor spectrum σ_T, have the projection property.*

The corollary follows immediately from Theorem 1.7 and Definition 1.3 (i) and (iii). To obtain the projection property for $\sigma_{\pi,k}$ we need some information on spectra of adjoint operators.

§ 2. Application of spectra of adjoint operators. In this section we shall prove the following

THEOREM 2.0. *Let X be a Banach space and $A_1, \ldots, A_n$ belong to $L(X)$. Then for every integer k the following hold:*

(i) $\sigma_{\pi,k}(A_1^*, \ldots, A_n^*) = \sigma_{\delta,k}(A_1, \ldots, A_n)$,

(ii) $\sigma_{\delta,k}(A_1^*, \ldots, A_n^*) = \sigma_{\pi,k}(A_1, \ldots, A_n)$.

Though the theorem is not difficult, we need some lemmas to prove it.

LEMMA 2.1. *Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be a chain complex (of Banach spaces and linear operators, as usual). Then the following conditions are equivalent:*

(i) $\operatorname{Im}\alpha = \ker\beta$ *and* $\operatorname{Im}\beta$ *is closed;*

(ii) $\operatorname{Im}\beta^* = \ker\alpha$ *and* $\operatorname{Im}\alpha^*$ *is closed.*

This lemma belongs to so-called mathematical folklore, but as we cannot point out the place where it is proved, we shall prove it ourselves.

Proof. (i)$\Rightarrow$(ii) $\operatorname{Im}\alpha$ is closed, hence from the well-known property of adjoint operators, $\operatorname{Im}\alpha^*$ is closed too. Let $y^* \in \ker\alpha^*$, i.e. $y^*(\operatorname{Im}\alpha) = 0$. Let us define the functional $z_1^* \in (\operatorname{Im}\beta)^*$ by the formula

$$z_1^*(\beta y) = y^*(y). \tag{2.1}$$

As $y^*(y) = 0$ if $\beta(y) = 0$, i.e. $y \in \operatorname{Im}\alpha$, this function is well defined, and seeing that $y \to \beta y\colon Y \to \beta(Y)$ is open we may conclude that z_1^* is continuous. From the Hahn–Banach theorem we may extend it to z^* on the whole space Z. Hence and from (2.1) $y^* = \beta^* z^*$.

(ii) $\Rightarrow$ (i) From the implication we have just proved it follows that $\operatorname{Im}\alpha^{**} = \ker\beta^{**}$ and $\operatorname{Im}\beta^{**}$ is closed. Let us identify Y, Z in the canonical way with subspaces of Y^{**}, Z^{**}, respectively. Then

$$(\operatorname{Im}\alpha^{**})\cap Y = \operatorname{Im}\alpha,$$

$$(\ker\beta^{**})\cap Y = \ker\beta,$$

$$\operatorname{Im}\beta^{**}\cap Z = \operatorname{Im}\beta,$$

and the implication is evident.

Lemma 2.2. *Let X be an $\mathscr{A}$-module, $A_1, \ldots, A_n \in Z(\mathscr{A})$ and let*

$$\ldots \to X_0 \xrightarrow{d} X_1 \xrightarrow{d} X_2 \to \ldots \xrightarrow{d} X_n \to 0 \to \ldots$$

denote their Koszul complex. Let

$$\ldots \to 0 \to Z_0 \xrightarrow{d} Z_1 \xrightarrow{d} \ldots \xrightarrow{d} Z_n \to 0 \to \ldots$$

denote the Koszul complex of $(A_1^*, \ldots, A_n^*)$. Then this complex is isomorphic with the complex adjoint to the previous one, i.e. there exists a linear $\mathscr{A}$-module isomorphism $\varphi_p\colon Z_p \to X_{n-p}^*$ such that the following diagram commutes

$$\begin{array}{ccccccccc} \ldots \to 0 \to & Z_0 & \xrightarrow{d} & Z_1 & \xrightarrow{d} & \ldots & \to & Z_n & \to 0 \to \ldots \\ & \downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_1} & & & & \downarrow{\scriptstyle\varphi_n} & \\ \ldots \to 0 \to & X_n^* & \xrightarrow{d^*} & X_{n-1}^* & \xrightarrow{d^*} & \ldots & \xrightarrow{d^*} & X_0^* & \to 0 \to \ldots \end{array} \tag{2.2}$$

Proof. We shall prove the lemma by induction on n.

For $n = 1$ both complexes are

$$\ldots \to 0 \to X^* \xrightarrow{A_1^*} X^* \to 0 \to \ldots$$

So the statement is trivial. Let us assume that we have proved it for n. Let

$$\ldots \to 0 \to W_0 \xrightarrow{d} W_1 \to \ldots \xrightarrow{d} W_{n+1} \to 0 \to \ldots$$

denote the Koszul complex of the tuple $(A_1, \ldots, A_{n+1})$.

Now we shall construct required isomorphisms $\psi_0, \ldots, \psi_{n+1}$, where $\psi_p\colon W_p \to Y_{n+1-p}^*$. By Definition 1.1, $W_p = Z_p \oplus Z_{p-1}$ and $Y_{n+1-p}^* = X_{n+1-p}^* \oplus X_{n-p}^*$. We put

$$\psi_p(z_p, z_{p-1}) := \big(\varphi_{p-1}z_{p-1}, (-1)^{n+1}\varphi_p z_p\big).$$

It is obvious that ψ_p is an $\mathscr{A}$-module isomorphism if φ_{p-1}, and φ_p are. Form Definition 1.1 the map $Y_{n-p}(= X_{n-p}\oplus X_{n-p-1}) \to Y_{n-p+1}(= X_{n-p+1}\oplus X_{n-p})$ is given by the formula

$$d(x_{n-p}, x_{n-p-1}) = \big(dx_{n-p}, dx_{n-p-1} + (-1)^{n-p}A_{n+1}x_{n-p}\big).$$

An easy computation shows that

$$d^*(x_{n-p+1}^*, x_{n-p}^*) = \big(d^*x_{n-p+1}^* + (-1)^{n-p}A_{n+1}^*x_{n-p}^*, d^*x_{n-p}^*\big).$$

Let us remember that $d\colon W_p \to W_{p+1}(= Z_{p+1}\oplus Z_p)$ is defined by

$$d(z_p, z_{p-1}) = \big(dz_p, dz_{p-1} + (-1)^p A_{n+1}z_p\big)$$

and

$$\psi_{p+1}(z_{p+1}, z_p) = \big(\varphi_p z_p, (-1)^{n+1}\varphi_{p+1}z_{p+1}\big),$$

and diagram (2.2) commutes. Now checking that diagram

$$\begin{array}{ccc} W_p & \xrightarrow{d} & W_{p+1} \\ \downarrow{\scriptstyle\psi_p} & & \downarrow{\scriptstyle\psi_{p+1}} \\ Y_{n-p+1}^* & \xrightarrow{d^*} & Y_{n-p}^* \end{array}$$

commutes is only a matter of a plain computation which we omit.

Proof of Theorem 2.0. As proofs of identities (i) and (ii) are almost the same, we will prove only one, for instance (ii). For that it is sufficient to show that $(0, \ldots, 0) \notin \sigma_{\pi,k}(A_1, \ldots, A_n)$ iff $(0, \ldots, 0) \notin \sigma_{\delta,k}(A_1^*, \ldots, A_n^*)$. The first condition holds by Definition 1.3 (ii) if and only if the Koszul complex of $A_1, \ldots, A_n$, denoted by

$$\ldots \to 0 \to X_0 \xrightarrow{d} X_1 \xrightarrow{d} \ldots \xrightarrow{d} X_k \xrightarrow{d} X_{k+1} \xrightarrow{d} \ldots \xrightarrow{d} X_n \to 0 \to \ldots,$$

is exact at $X_0, \ldots, X_k$ and $\operatorname{Im}(X_k \xrightarrow{d} X_{k+1})$ is closed. This by Lemma 2.1 is equivalent to the condition that the complex

$$\ldots \to 0 \to X_n^* \xrightarrow{d} X_{n-1}^* \xrightarrow{d^*} \ldots \xrightarrow{d^*} X_0^* \to 0 \to \ldots$$

is exact at $X_k^*, \ldots, X_0^*$ and this by Lemma 2.2 is equivalent to the condition that the Koszul complex of $A_1^*, \ldots, A_n^*$, denoted by

$$\ldots \to 0 \to Z_0 \to Z_1 \to \ldots \to Z \to Z_n \to 0 \to \ldots,$$

is exact in $Z_{n-k}, Z_{n-k+1}, \ldots, Z_n$, what means exactly that $(0, \ldots, 0)$ does not belong to $\sigma_{\delta,k}(A_1^*, \ldots, A_n^*)$.

Now we consider uppersemicontinuity of our spectra.

Definition 2.3. Let X be a Banach space and let σ^* be a spectral system on it. We shall say that σ^* *is uppersemicontinuous* if for every integer $n \geqslant 1$ the set

$$\{(\lambda_1, \ldots, \lambda_n;\ A_1, \ldots, A_n) \in C^n \times [L(X)^n \cap c_0(X)]\colon (\lambda_1, \ldots, \lambda_n) \notin \sigma^*(A_1, \ldots, A_n)\} \tag{2.3}$$

is open in $C^n \times [L(X)^n \cap c_0(X)]$.

This definition means exactly that for every $n \geqslant 1$ the set-valued function

$$(A_1, \ldots, A_n) \to \sigma^*(A_1, \ldots, A_n): \ c_0(X) \cap L(X)^n \to \mathscr{F}(C^n)$$

(where $\mathscr{F}(C^n)$ is the space of all compact subsets of C^n) is uppersemicontinuous. The definition says that the graph of the function is closed.

If one looks at the proof ot Lemma 2.1 in [1], one can see that what Taylor has really proved may be stated as follows.

Lemma 2.4. *Let*

$$X \xrightarrow{\alpha_0} Y \xrightarrow{\beta_0} Z \xrightarrow{\gamma_0} W$$

be an exact chain complex of Banach spaces and operators. Then there exists $\varepsilon > 0$ *such that if*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} W$$

is a chain complex satisfying $\max(\|\alpha-\alpha_0\|, \|\beta-\beta_0\|, \|\gamma-\gamma_0\|) < \varepsilon$, *then it is exact at* Y (*i.e.* $\operatorname{Im}\alpha = \ker\beta$).

Let

$$(A_1^0, \ldots, A_n^0), (A_1, \ldots, A_n) \in c_0(X) \cap L(X)^n,$$

$$(\gamma_1^0, \ldots, \gamma_n^0) \notin \sigma_{\delta,k}(A_1^0, \ldots, A_n^0) \quad \text{and} \quad (\gamma_1, \ldots, \gamma_n) \in C^n.$$

Let

$$\ldots \rightleftarrows 0 \rightleftarrows X_0 \underset{d}{\overset{d^0}{\rightleftarrows}} X_1 \underset{d}{\overset{d^0}{\rightleftarrows}} X_2 \rightleftarrows \ldots \underset{d}{\overset{d^0}{\rightleftarrows}} X_n \rightleftarrows 0 \rightleftarrows \ldots$$

denote Koszul complexes with d^0 for $(A_1^0-\gamma_1^0 I, \ldots, A_n^0-\gamma_n^0 I)$ and d for $(A_1-\gamma_1 I, \ldots, A_1-\gamma_n I)$, respectively. From Lemma 2.4 it follows that if "d^0-complex" is exact in $X_{n-k}, \ldots, X_n$, then $\exists\, \varepsilon > 0$ such that if $\|d-d^0\| < \varepsilon$, then d-complex is exact in $X_{n-k}, \ldots, X_n$. As boundary operators d depend in continuous way on $(\gamma_1, \ldots, \gamma_n, A_1, \ldots, A_n)$, there exists a neighbourhood of $(\gamma_1^0, \ldots, \gamma_n^0, A_1^0, \ldots, A_n^0)$ such that for $(\gamma_1, \ldots, \gamma_n, A_1, \ldots, A_n)$ in it we have $(\gamma_1, \ldots, \gamma_n) \notin \sigma_{\delta,k}(A_1, \ldots, A_n)$; thus $\sigma_{\delta,k}$ is uppersemicontinuous.

From Theorem 2.0

$$\{(\gamma_1, \ldots, \gamma_n, A_1, \ldots, A_n): (\gamma_1, \ldots, \gamma_n) \notin \sigma_{\delta,k}(A_1, \ldots, A_n)\}$$
$$= \{(\gamma_1, \ldots, \gamma_n, A_1, \ldots, A_n): (\gamma_1, \ldots, \gamma_n) \notin \sigma_{\delta,k}(A_1^*, \ldots, A_n^*)\}.$$

The latter set is open from uppersemicontinuity of $\sigma_{\delta,k}$ and continuity of "$*$": $L(X) \to L(X^*)$, and so such is the former; thus $\sigma_{\pi,k}$ is uppersemicontinuous. Such is σ_T because for every n

$$\sigma_T(A_1, \ldots, A_n) = \sigma_{\delta,n}(A_1, \ldots, A_n).$$

Corollary 2.5. *All spectral systems* $\sigma_{\pi,k}, \sigma_{\delta,k}$ *and* σ_T *are uppersemicontinuous.*

Corollary 2.6. *All spectral systems* $\sigma_{\pi,k}$ *possess the projection property.*

Proof. We apply Theorem 2.0 (ii). Let $P: C^{n+1} \to C^n$ denote the canonical projection. Let $A_1, \ldots, A_{n+1}$ commute. Then

$$P(\sigma_{\pi,k}(A_1, \ldots, A_n, A_{n+1})) = P(\sigma_{\delta,k}(A_1^*, \ldots, A_n^*, A_{n+1}^*))$$
$$= \sigma_{\delta,k}(A_1^*, \ldots, A_n^*) = \sigma_{\pi,k}(A_1, \ldots, A_n)$$

(from Corollary 1.8).

We need one more lemma.

Lemma 2.7. *Let* (A_1, A_2, A_3) *commute. If* $[A_1, A_2, A_3]$ *denotes the subalgebra of* $L(X)$ *algebraically generated by* I, A_1, A_2, A_3, *then*

$$\sigma_T(A_1, A_2, A_3) \subset \sigma_{[A_1,A_2,A_3]}(A_1, A_2, A_3), \tag{2.4}$$

where $\sigma_{[\ldots]}$ *denotes the usual joint spectrum computed with respect to the commutative algebra* $[A_1, A_2, A_3]$.

This lemma is a special case of Lemma 1.1 in [1].

Now we apply Theorem 3.3 from [4] which states that if a spectral system has the projection property and satisfies the condition of Lemma 2.7 than it has the spectral mapping property. All spectra $\sigma_{\delta,k}, \sigma_{\pi,k}$, are parts of σ_T, so they satisfy condition (2.4) and from the mentioned theorem all these spectra have the spectral mapping property (cf. Corollary 2.6 and Corollary 1.8).

Proposition 2.8. *If* σ^* *and* $\tilde\sigma$ *are joint spectra* (Def. 0.2), *then the spectral system* $\sigma^* \cup \tilde\sigma$ *defined by*

$$(\sigma^* \cup \tilde\sigma)(A_1, \ldots, A_n) := \sigma^*(A_1, \ldots, A_n) \cup \tilde\sigma(A_1, \ldots, A_n)$$

is a joint spectrum.

Proof. We omit a trivial proof that $\sigma^* \cup \tilde\sigma$ possesses a spectral mapping property if σ^* and $\tilde\sigma$ do. As to uppersemicontinuity notice that set (2.3) considered for $\sigma^* \cup \tilde\sigma$ is an interesection of such sets taken for σ^* and $\tilde\sigma$; thus it is open.

We will sum up the above remarks in

Theorem 2.9. *All spectral systems in the family*

$$\{\sigma_{\pi,k}: k \geqslant 0\} \cup \{\sigma_{\delta,k}: k \geqslant 0\} \cup \{\sigma_{\pi,k} \cup \sigma_{\delta,l}: k, \ : k, l \geqslant 0\} \cup \{\sigma_T\}$$

are joint spectra.

§ 3. All introduced spectra are different. All spaces in this paragraph are Hilbert and separable.

Notations 3.0. If X is a Hilbert space, then

(i) $l_2(X) = \{(x_i)_{i=0}^{\infty}: x_i \in X, \sum \|x_i\|^2 < +\infty\}$ is a Hilbert space with usual norm and operations;

(ii) if $A \in L(X)$, then by $\tilde{A}$ we mean the operator $l_2(X) \to l_2(X)$ defined by the formula

$$(\tilde{A}\tilde{x})_i = Ax_i, \quad \text{where } i \geqslant 0 \text{ and } \tilde{x} = (x_i)_{i \geqslant 0};$$

(iii) by $V \in L(l_2(X))$ and $D \in L(l_2(X))$ we denote the operators

$$(V\tilde{x})_i = \begin{cases} 0, & i = 0, \\ x_{i-1}, & i > 0, \end{cases}$$

$$(D\tilde{x}) = x_{i+1}, \quad \text{where } \tilde{x} = (x_i)_{i \geqslant 0}.$$

LEMMA 3.1. *Let X be a Hilbert space and $(A_1, \ldots, A_n) \in c_0(X)$; if*

$$\ldots \to 0 \to X_0 \xrightarrow{d} X_1 \to \ldots \xrightarrow{d} X_n \to 0 \to \ldots$$

denotes the Koszul complex of $(A_1, \ldots, A_n)$, then the Koszul complex of the commutative tuple $(\tilde{A}_1, \ldots, \tilde{A}_n) \in c_0(l_2(X))$ is isomorphic with the following complex

$$\ldots \to 0 \to l_2(X_0) \xrightarrow{\tilde{d}} l_2(X_1) \xrightarrow{\tilde{d}} \ldots \xrightarrow{\tilde{d}} l_2(X_n) \to 0 \to \ldots$$

Proof. We shall prove it by induction. For $n = 1$ both complexes are of the form $0 \to l_2(X) \xrightarrow{\tilde{A}_1} l_2(X) \to 0$ so this case is trivial. Let us assume that the lemma holds for some n, i.e. there exist $\varphi_0, \varphi_1, \ldots, \varphi_n$; $\varphi_p\colon Z_n \to l_2(X_p)$ such that the diagram

$$\text{(3.1)} \qquad \begin{array}{ccccccccccccc} \ldots \to 0 \to & Z_0 & \xrightarrow{d'} & Z_1 & \xrightarrow{d'} & Z_2 \to & \ldots \to Z_{n-1} & \xrightarrow{d'} & Z_n & \to 0 \to \ldots \\ & \downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & & \downarrow{\scriptstyle\varphi_n} & \\ \ldots \to 0 \to & l_2(X_0) & \xrightarrow{\tilde{d}} & l_2(X_1) & \xrightarrow{\tilde{d}} & l_2(X_2) \to & \ldots & \xrightarrow{\tilde{d}} & l_2(X_n) & \to 0 \to \ldots \end{array}$$

commutes, where higher line denotes the Koszul complex of the operators $\tilde{A}_1, \ldots, \tilde{A}_n$. Let $A_1, \ldots, A_{n+1} \in Z(\mathscr{A})$ and let

$$\to 0 \to Y_0 \xrightarrow{d} Y_1 \to \ldots \xrightarrow{p} Y_{n+1} \to 0 \to \ldots$$

and

$$\to 0 \to W_0 \xrightarrow{d'} W_1 \xrightarrow{d'} \ldots \xrightarrow{d'} W_{n+1} \to 0 \to \ldots$$

denote Koszul complexes of $A_1, \ldots, A_{n+1}$ and $\tilde{A}_1, \ldots, \tilde{A}_{n+1}$, respectively. We shall now define an isomorphism of these complexes consisting of the operators

$$\psi_p\colon W_p \to l_2(Y_p), \quad p = 0, \ldots, n+1.$$

By Definition 1.1, $Y_p = X_p \oplus X_{p-1}$ and $W_p = Z_p \oplus Z_{p-1}$. We put

$$\psi_p(w_p) = \psi_p(z_{p'}, z_{p-1}) := \big((\varphi_p z_p)_i \oplus (\varphi_{p-1}, z_{p-1})_i\big)_{i=0}^{\infty} \in l_2(Y_p).$$

For completeness of the proof if remains to check that the diagram

$$\begin{array}{ccc} W_p & \xrightarrow{d'} & W_{p+1} \\ \downarrow{\scriptstyle\psi_p} & & \downarrow{\scriptstyle\psi_{p+1}} \\ l_2(Y_p) & \xrightarrow{\tilde{d}} & l_2(Y_{p+1}) \end{array}$$

commutes. It is only an easy computation in which one must use commutativity of diagram (3.1) and the formulas

$$d'(w_p) = d'(z_p, z_{p-1}) = \big(d'z_p, d'z_{p-1} + (-1)^p \tilde{A}_{n+1} z_p\big)$$

and

$$\tilde{d}\big((x_i^p \oplus x_i^{p-1})_{i=0}^{\infty}\big) = \big((dx_i^p, dx_{i-1}^p + (-1)^{p+1} A_{n+1} x_i^p)\big)_{i=0}^{\infty}.$$

We omit the details.

LEMMA 3.2. *Let*

$$\text{(3.2)} \qquad X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} W$$

be a chain complex of Banach $\mathscr{A}$-modules. Then

$$\text{(3.3)} \qquad l_2(Y) \oplus l_2(X) \xrightarrow{\delta} l_2(Z) \oplus l_2(X) \xrightarrow{\varepsilon} l_2(W) \oplus l_2(Z)$$

is a chain complex if

$$\delta(\tilde{y}, \tilde{x}) = \big(\tilde{\beta}\tilde{y}, \tilde{\alpha}\tilde{x} + (-1)^p A\tilde{y}\big),$$

$$\varepsilon(\tilde{z}, \tilde{y}) = \big(\tilde{\gamma}\tilde{z}, \tilde{\beta}\tilde{y} + (-1)^{p+1} A\tilde{z}\big),$$

where p is an arbitrary integer and operators A acting on $l_2(Y)$ and $l_2(Z)$ are either both of the V-form, or both of the D-form (see Notations 3.0 (iii)). *In addition*

(i) *if A is of the V-form, then the mapping φ such that*

$$(\tilde{z}, \tilde{y}) + \operatorname{Im}\delta \to y_0 + \operatorname{Im}\alpha\colon \ker\varepsilon/\operatorname{Im}\delta \xrightarrow{\varphi} \ker\beta/\operatorname{Im}\alpha$$

is a linear isomorphism;

(ii) *if A is of the D-form, the mapping ψ such that*

$$(\tilde{z}, \tilde{y}) + \operatorname{Im}\delta \to z_0 + \operatorname{Im}\beta\colon \ker\varepsilon/\operatorname{Im}\delta \xrightarrow{\psi} \ker\delta/\operatorname{Im}\beta$$

is a linear isomorphism.

Proof. We omit the proof that (3.3) is a chain complex.

(i) If $y \in \ker\beta$ and $\tilde{y} = (y, 0, 0, \ldots)$, $\tilde{z} = 0$, then

$$\varepsilon(\tilde{z}, \tilde{y}) = 0 \quad \text{and} \quad \varphi\big((\tilde{z}, \tilde{y}) + \operatorname{Im}\delta\big) = y + \operatorname{Im}\alpha;$$

hence φ is an epimorphism. Why φ is a monomorphism? Let $\varphi\big((\tilde{z}, \tilde{y}) + \operatorname{Im}\delta\big) = 0$, i.e.

$$\text{(3.4)} \qquad y_0 = \alpha x,$$

where $\tilde{z} = (z_i)_{i=0}^{\infty}$, $\tilde{y} = (y_i)_{i=0}^{\infty}$. As $(\tilde{z}, \tilde{y}) \in \ker \varepsilon$, we have

(3.5) $$\gamma z_i = 0, \quad i \geqslant 0,$$

(3.6) $$\beta y_i + (-1)^{p+1} z_{i-1} = 0, \quad i > 0.$$

We have to show that $(\tilde{z}, \tilde{y}) \in \operatorname{Im} \delta$. Let us put $y' = (y'_i)_{i\geqslant 0}$, $y'_i = (-1)^p y_{i+1}$, $i \geqslant 0$, and $\tilde{x} = (x_i)_{i=0}^{\infty}$, $x_0 = x$ (cf. (3.4)) and $x_i = 0$ for $i > 0$. Then

$$\big(\delta(\tilde{y}', \tilde{x})\big)_i = \begin{cases} (\beta y'_0, \alpha x), & i = 0 \\ \beta y'_i, 0 + (-1) y'_{i-1}, & i \geqslant 1 \end{cases}$$
$$= \begin{cases} (z_0, y_0), & i = 0 \\ (z_i, y_i), & i > 0 \end{cases} = (\tilde{z}, \tilde{y}).$$

(ii) If $z \in \ker \gamma$ and $\tilde{z} = (z, 0, 0, \ldots)$, $\tilde{y} = 0$, then $(\tilde{z}, \tilde{y}) \in \ker \varepsilon$ and $\psi\big((\tilde{z}, \tilde{y}) + \operatorname{Im} \delta\big) = z + \operatorname{Im} \beta$; thus ψ is an epimorphism. Let $\psi\big((\tilde{z}, \tilde{y}) + \operatorname{Im} \delta\big) = 0$, i.e.

(3.7) $$z_0 = \beta y \quad \text{for some } y.$$

To prove that ψ is a monomorphism we should prove that $(\tilde{z}, \tilde{y}) \in \operatorname{Im} \delta$. As $(\tilde{z}, \tilde{y}) \in \ker \varepsilon$, we have

(3.8) $$\gamma z_i = 0, \quad i \geqslant 0,$$

(3.9) $$\beta y_i + (-1)^{p+1} z_{i+1} = 0, \quad i \geqslant 0.$$

Let us put $\tilde{x} = 0$ and $\tilde{y}' = (y'_i)_{i\geqslant 0}$, where $y'_i = (-1)^p y_{i-1}$, $i > 0$, $y'_0 = y$ (see (3.7)). Then

$$\big(\delta(\tilde{y}, \tilde{x})\big)_i = \begin{cases} \big(\beta y, 0 + (-1)^p y'\big), & i = 0 \\ \big(\beta y'_i, 0 + (-1)^p y'_{i+1}\big), & i > 0 \end{cases}$$
$$= \begin{cases} (z_0, y_0), & i = 0 \\ \big((-1)^p \beta y_i, y_i\big), & i > 0 \end{cases} = \begin{cases} (z_0, y_0) \\ (z_i, y_i)_{i>0} \end{cases}$$
$$= (\tilde{z}, \tilde{y}) \quad \text{(see (3.9))}.$$

LEMMA 3.3. *Let n and k be integers such that $n \geqslant 1$, $0 \leqslant k \leqslant n$. Then there exist a separable Hilbert space X and n commuting linear operators $A = (A_1, \ldots, A_n)$ on it such that*

(i) $(0, \ldots, 0) \notin \Sigma_p(A)$ *for* $p \neq k$;

(ii) $(0, \ldots, 0) \in \Sigma_k(A)$;

and moreover if

$$\to 0 \to X_0 \to X_1 \to \ldots \to X_n \to 0 \to \ldots$$

denotes the Koszul complex of the tuple A, then

(iii) *the cohomology space*

$$\ker(X_k \xrightarrow{d} X_{k+1}) / \operatorname{Im}(X_{k-1} \xrightarrow{d} X_k)$$

is exactly one-dimensional;

(iv) *every coboundary operator has closed image.*

Proof. We shall prove the lemma by induction on n. For $n = 1$, $k = 0$, we put $A_1 = V\colon l_2 \to l_2$, namely

$$(V\tilde{x})_i = \begin{cases} 0, & i = 0, \\ X_{i-1}, & i > 0, \end{cases} \quad \text{where} \quad \tilde{x} = (x_i)_{i=0}^{\infty} \ (\textit{pure isometry}),$$

and for $k = 1$ (n still one) $A_1 = D\colon l_2 \to l_2$,

$$(D\tilde{x})_i = x_{i+1} \quad (\textit{unilateral shift}).$$

All statements of the lemma are easy to see. We omit the details. Let us assume that we have proved the lemma for a given n (and all $0 \leqslant k \leqslant n$). Let $n+1 \geqslant k \geqslant 0$. We must consider two cases.

1° $k \geqslant 1$. We use the inductive hypothesis for the pair $(n, k-1)$. Let $(A_1, \ldots, A_n)$ satisfy (i)–(iii). We shall prove that $(n+1, k)$-hypothesis is satisfied by the space $Y = l_2(X)$ and the tuple $B_1, \ldots, B_{n+1}$, where $B_i = \tilde{A}_i$ (cf. Notations (3.0) (ii)) for $1 \leqslant i \leqslant n$ and

(3.10) $$B_{n+1}(\tilde{x}) = V(\tilde{x}) \ \text{(see Notations 3.0 (iii))}, \quad \tilde{x} = (\tilde{x}_i)_{i=0}^{\infty}.$$

We omit an easy checking that $(B_1, \ldots, B_{n+1})$ is a commuting tuple of bounded linear operators.

Let

(3.11) $$\to 0 \to X_0 \xrightarrow{d_0} X_1 \xrightarrow{d_1} \ldots \to X_{n-1} \xrightarrow{d_{n-1}} X_n \to 0 \to \ldots$$

denote the Koszul complex of $(A_1, \ldots, A_n)$. By Lemma 3.1, the Koszul complex of $(B_1, \ldots, B_n) = (\tilde{A}_1, \ldots, \tilde{A}_n)$ is isomorphic to the complex

$$\ldots \to 0 \to l_2(X_0) \xrightarrow{\tilde{d}_0} l_2(X_1) \xrightarrow{\tilde{d}_1} \ldots \xrightarrow{\tilde{d}_{n-1}} l_2(X_n) \to 0 \to \ldots$$

Now, by Definition 1.1, the Koszul complex of $(B_1, \ldots, B_{n+1})$ is isomorphic to the complex

(3.12) $$\to 0 \to l_2(X_0) \oplus 0 \xrightarrow{d'_0} l_2(X_1) \oplus l_2(X_0) \xrightarrow{d'_1} \ldots \to 0 \oplus l_2(X_{n+1}) \to 0 \to \ldots,$$

where

$$d'_p\colon l_2(X_p) \oplus l_2(X_{p-1}) \to l_2(X_{p+1}) \oplus l_2(X_p)$$

is defined by the formula

$$d'_p(\tilde{x}^p, \tilde{x}^{p-1}) = (\tilde{d}_p\tilde{x}_p, \tilde{d}_{p-1}\tilde{x}_{p-1} + (-1)^p V\tilde{x}^{p-1}).$$

Now we come to the situation of Lemma 3.2 (i) and hence

$$\dim(\ker d'_{p+1}/\mathrm{Im}\, d'_p) = \dim(\ker d_p/\mathrm{Im}\, d_{p-1}).$$

We have obtained it applying the lemma for $X = X_{p-1}$, $Y = X_p$, $Z = X_{p+1}$, $W = X_{p+2}$. As the second number is equal to 1 for $p = k-1$ and 0 otherwise, the hypothesis is proved for the pair $(n+1, k)$, $k \geqslant 1$.

2° $k \leqslant n$. We shall use the inductive hypothesis for the pair (n, k). Almost whole previous construction remains valid but for the definition of B_{n+1} (see (3.10)) we put

$$B_{n+1}(\tilde{x}) = D(\tilde{x}) \text{ (see Notations 3.0 (iii))}.$$

We are once more in the situation of Lemma 3.2, but as it is (ii) case this time, we get (applying the lemma to $X = X_{p-2}$, $Y = X_{p-1}$, $Z = X_p$, $W = X_{p+1}$) that

$$\dim(\ker d'_p/\ker d'_{p-1}) = \dim(\ker d_p/\ker d'_{p-1}).$$

COROLLARY 3.4. *Every two spectra from the family*

$$\{\sigma_{\pi,k}\colon k \geqslant 0\} \cup \{\sigma_{\delta,k}\colon k \geqslant 0\} \cup \{\sigma_{\pi,k} \cup \sigma_{\delta,l}\colon k, l \geqslant 0\} \cup \{\sigma_T\}$$

are different in case of infinite-dimensional Hilbert space.

Proof. $\sigma_{\pi,k_1} \cup \sigma_{\delta,l_1}$ is different from $\sigma_{\pi,k} \cup \sigma_{\delta,l}$ as well as from $\sigma_{\pi,k}$ and $\sigma_{\delta,l}$ when $k_1 > k$. To see this take a tuple $(A_1, \ldots, A_n)$ satisfying Lemma 3.3 with the pair (n, k_1), where $n > k + \max(l, l_1)$. Then only $\Sigma_{k_1}(A_1, \ldots, A_n) \ni (0, \ldots, 0)$; hence $(0, \ldots, 0) \in (\sigma_{\pi,k_1} \cup \sigma_{\delta,l_1})(A_1, \ldots, A_n)$ and $(0, \ldots, 0) \in \sigma_{\pi,k_1}(A_1, \ldots, A_n)$ but $(0, \ldots, 0) \notin (\sigma_{\pi,k} \cup \sigma_{\delta,l})(A_1, \ldots, A_n)$, $(0, \ldots, 0) \notin \sigma_{\pi,k}(A_1, \ldots, A_n)$ and $(0, \ldots, 0) \notin \sigma_{\delta,l}(A_1, \ldots, A_n)$, either.

The same example of operators shows that σ_{π,k_1} is different from $\sigma_{\pi,k}$, $\sigma_{\delta,l}$, and $\sigma_{\pi,k} \cup \sigma_{\delta,l}$, when $k_1 > k$ and l arbitrary. Similarly, but using a tuple $(A_1, \ldots, A_n)$ satisfying Lemma 3.3 with the pair $(n, n-l_1)$, $n > \max(k, k_1) + l_1$, one may obtain that if $l_1 > l$ than each of $\sigma_{\pi,k_1} \cup \sigma_{\delta,l_1}$ and σ_{δ,l_1} is different from $\sigma_{\pi,k} \cup \sigma_{\delta,l}$, $\sigma_{\pi,k}$, and $\sigma_{\delta,l}$. So we have covered all the possibilities.

To see that every such spectrum is different from Taylor's one it suffices to notice that $\sigma_{\pi,k}$, $\sigma_{\delta,l}$, and $\sigma_{\pi,k} \cup \sigma_{\delta,l}$, are included in, and different from, $\sigma_{\pi,k+1} \cup \sigma_{\delta,l}$ which in turn is included in Taylor's spectrum so it is different from every spectrum in question.

Remark. We cannot prove this corollary in case of an arbitrary Banach space.

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