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Received August 20, 1975 (1059)

STUDIA MATHEMATICA, T. LXI. (1977)

Integrability of seminorms, the 0-1 law and the affine kernel for product measures

by

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Abstract. Let (X_n) be a sequence of random variables taking values in a measurable linear space E, and let q be a quasi-convex subadditive function on E^∞. The first part of the paper deals with the problem of finding conditions, which assures that $\mathbf{E}(e^{\varepsilon M})$ is finite for some positive ε, where $M = \sup_n q(X_1, \ldots, X_n, 0, 0, \ldots)$. In the second and third part of the paper we take $E = \boldsymbol{R}$, and we show that if X_n has no mass points, then every linear subspace of $\boldsymbol{R}^\infty$ has probability 0 or 1. Finally, we study the affine kernel of (X_n), i.e. the intersection of all affine subspaces of probability 1, and we give an analytic expression for this.

1. Introduction. If μ is a Gaussian measure on a locally convex space, there are three main results which have proved to be useful.

The first is the result of Fernique stating that, if φ is a measurable a.e. finite seminorm, then $\mathbf{E}\big(\exp(\varepsilon\varphi^2)\big) < \infty$ where $\mathbf{E}$ denotes expectation with respect to μ. A similar result has been proved by C. Borel ([1]) for certain other classes of measures. In Section 2 of this paper we shall prove some results in this direction when μ is a product measure on $\big(\prod_1^\infty E_n, \bigotimes_1^n \boldsymbol{B}_n\big)$ and $(E_n, \boldsymbol{B}_n)$ is a measurable linear space. Here we define a measurable linear space, $(E, \boldsymbol{B})$, to be a linear space E equipped with a σ-algebra $\boldsymbol{B}$ satisfying

$$(1.1)\qquad (x, y) \to x+y \text{ is measurable: } (E\times E, \boldsymbol{B}\otimes\boldsymbol{B}) \to (E, \boldsymbol{B}),$$

$$(1.2)\qquad (\lambda, y) \to \lambda y \text{ is measurable: } \big(E\times \boldsymbol{R}, \boldsymbol{B}\otimes\boldsymbol{B}(R)\big) \to (E, \boldsymbol{B}).$$

The methods and the results of that section are closely related to the results of Marcus and Jain in [9] and to the results in [2] and [3].

The second result is the 0-1 law by Kallianpur in [5]. C. Borel has in [1] shown that the same result holds for certain other classes of measure. In Section 3 we show that, if μ is a product measure on $\boldsymbol{R}^\infty$ with non-atomic factors, then $\mu(A) = 0$ or 1 for all μ-measurable affine subsets.

The third result on Gaussian measures, which has proved to be a very powerful tool, is the reproducing kernel Hilbert space which, in case μ

has mean 0, can be characterized as the intersection of all μ-measurable linear subsets of measure 1. In Section 4 we shall show that, if μ is a product measure on $\boldsymbol{R}^\infty$ with non-degenerated factors, then $\boldsymbol{R}_0^\infty$ (the set of sequences with atmost finitely many non-zero coordinates) is contained in the intersection of all μ-measurable linear sets with measure 1. We shall also give an analytic expression for this intersection under some restrictions on μ.

2. Integrability of quasi-convex functions. Let $(E_n, \boldsymbol{B}_n)$ be a measurable linear space, and (X_n) a sequence of independent random vectors so that X_n takes its values in E_n for all $n \geqslant 1$. Let φ be a map from E_∞ into $\overline{\boldsymbol{R}}_+ = [0, \infty]$, where

$$E_\infty = \prod_{n=1}^\infty E_n \quad \text{and} \quad \boldsymbol{B}_\infty = \bigotimes_{n=1}^\infty \boldsymbol{B}_n$$

and suppose that

(2.1) $$\varphi \text{ is } \boldsymbol{B}_\infty\text{-measurable},$$

(2.2) $$\varphi(\tfrac{1}{2}x+\tfrac{1}{2}y) \leqslant \max\{\varphi(x), \varphi(y)\} \quad \forall x, y \in E_\infty,$$

(2.3) $$\varphi(x+y) \leqslant \varphi(x)+\varphi(y) \quad \forall x, y \in E_\infty,$$

that is, φ is a measurable quasi-convex subadditive function on E_∞. We shall say that φ is *symmetric* if

(2.4) $$\varphi(x) = \varphi(-x) \quad \forall x.$$

Let us define the random variables

$$N_j = \varphi(0, \ldots, 0, X_j, 0, \ldots), \qquad N = \sup_j N_j,$$
$$M_j = \varphi(X_1, \ldots, X_j, 0, \ldots), \qquad M = \sup_j M_j;$$

we shall consider their tail probabilities

$$G_j(t) = \mathbf{P}(N_j > t), \qquad G(t) = \mathbf{P}(N > t),$$
$$F_j(t) = \mathbf{P}(M_j > t), \qquad F(t) = \mathbf{P}(M > t).$$

We shall use these notations throughout this section.

LEMMA 2.1. *Suppose that X_n is symmetric for all $n \geqslant 1$; then we have*

(2.1.1) $$\mathbf{P}(\max_{1\leqslant j\leqslant n} M_j > t) \leqslant 2\mathbf{P}(M_n > t) \quad \forall t \geqslant 0,$$

(2.1.2) $$\mathbf{P}(M > t) \leqslant 2\liminf_{n\to\infty}\mathbf{P}(M_n > t) \quad \forall t \geqslant 0.$$

Proof. The proof is fairly standard and actually only requires quasi-convexity of φ. Let

$$T = \inf\{1 \leqslant j \leqslant n \mid M_j > t\} \qquad (\inf(\emptyset) = \infty),$$
$$Y_n = (X_1, \ldots, X_n, 0, \ldots) \qquad \text{for } n \geqslant 1,$$
$$Y_{nj} = (X_1, \ldots, X_j, -X_{j+1}, \ldots, -X_n, 0, \ldots) \qquad \text{for } n \geqslant j \geqslant 0;$$

then $Y_j = \frac{1}{2}Y_n + \frac{1}{2}Y_{nj}$ and so

$$M_j = \varphi(Y_j) \leqslant \max\{\varphi(Y_n), \varphi(Y_{nj})\}.$$

If $T = j$, then $M_j > t$, and so either $\varphi(Y_n) > t$ or $\varphi(Y_{nj}) > t$, however, Y_n and Y_{nj} are equidistributed and so

$$\mathbf{P}(T = j) \leqslant \mathbf{P}\big(T = j, \varphi(Y_n) > t\big) + \mathbf{P}\big(T = j, \varphi(Y_{nj}) > t\big)$$
$$= 2\mathbf{P}(T = j, M_n > t).$$

Hence we have

$$\mathbf{P}(\max_{1\leqslant j\leqslant n} M_j > t) = \mathbf{P}(T \leqslant n) \leqslant 2\mathbf{P}(M_n > t).$$

Now let $a > \liminf_{n\to\infty} F_n(t)$; then there exist integers $n_1 < n_2 < \ldots$ so that $F_{n_j}(t) \leqslant a$ for all $j \geqslant 1$. From the first part of the lemma we know that

$$\mathbf{P}(\max_{1\leqslant v\leqslant n_j} M_v > t) \leqslant 2a \quad \forall j \geqslant 1$$

and since $\max_{1\leqslant v\leqslant n_j} M_v$ increases to M, we have $\mathbf{P}(M > t) \leqslant 2a$, and (2.1.2) follows.

LEMMA 2.2. *Let $(E, \boldsymbol{B})$ be measurable linear space and ψ a $\boldsymbol{B}$-measurable function from E into $\overline{\mathbf{R}}_+$, so that ψ is symmetric quasi-convex and subadditive. If Z is an E-valued random vector and $\mathbf{P}\big(\psi(Z) \leqslant a\big) \geqslant \frac{1}{2}$ for some $a \geqslant 0$, then*

(2.2.1) $$\mathbf{P}\big(\psi(Z^*) > 2t\big) \leqslant 2\mathbf{P}\big(\psi(Z) > t\big) \quad \forall t \geqslant 0,$$

(2.2.2) $$\mathbf{P}\big(\psi(Z) > t+a\big) \leqslant 2\mathbf{P}\big(\psi(Z^*) > t\big) \quad \forall t \geqslant 0,$$

where Z^* is a symmetrization of Z.

Proof. (2.2.1) is obvious. If μ is the distribution law of Z, then

$$\mathbf{P}\big(\psi(Z^*) > t\big) = \int_E \mathbf{P}\big(\psi(Z-x) > t\big)\mu(dx) \geqslant \int_A \mathbf{P}\big(\psi(Z-x) > t\big)\mu(dx),$$

where $A = \{x \in E \mid \psi(x) \leqslant a\}$. Then $\mu(A) \geqslant \frac{1}{2}$ and so there exists $x_t \in A$ for all $t \geqslant 0$ so that

$$\mathbf{P}\big(\psi(Z-x_t) > t\big) \leqslant 2\mathbf{P}\big(\psi(Z^*) > t\big)$$

and since $\psi(Z) \leqslant \psi(Z-x_t)+a$, (2.2.2) follows.

THEOREM 2.3. *Suppose that X_n is symmetric for all $n \geqslant 1$; then we have*

$$F(2t+s) \leqslant 2G(s)+4F(t)^2 \quad \forall t, s \geqslant 0. \tag{2.3.1}$$

If we drop the assumption about symmetry of X_n, but assume that φ is symmetric, then we have

$$F(4t+s+a) \geqslant 8G(s)+32F(t)^2 \quad \forall s, t \geqslant 0, \tag{2.3.2}$$

where $a \geqslant 0$ is determined by $\mathbf{P}(M \leqslant a) \geqslant \frac{1}{2}$.

Proof. Suppose first that (X_n) is symmetric and define

$$T = \inf\{n \geqslant 1 \mid M_n > t\};$$

then $M_k > 2t+s$ implies that $T \leqslant k$ and so

$$F_k(2t+s) = \sum_{j=1}^{k} \mathbf{P}(T = j,\, M_k > 2t+s).$$

Now let $Z_j = (0, \ldots, 0, X_j, 0, \ldots)$ and $Y_j = (X, \ldots, X_j, 0, \ldots)$; then we have

$$Y_k = (Y_k - Y_j) + Y_{j-1} + Z_j$$

and so

$$M_k \leqslant \varphi(Y_k - Y_j) + M_{j-1} + N.$$

Hence, if $T = j$, $N \leqslant s$ and $M_k > 2t+s$, then $M_{j-1} \leqslant t$, and $\varphi(Y_k - Y_j) > t$, and so we have

$$\begin{aligned}\mathbf{P}(T = j,\, M_k > 2t+s) &\leqslant \mathbf{P}(T = j,\, N > s) + \mathbf{P}\big(T = j,\, \varphi(Y_k - Y_j) > t\big)\\ &= \mathbf{P}(T = j,\, N > s) + \mathbf{P}(T = j)\ \mathbf{P}\big(\varphi(Y_k - Y_j) > t\big)\end{aligned}$$

since $\{T = j\}$ and $Y_k - Y_j$ are independent. Now we observe that by Lemma 2.1 we have

$$\mathbf{P}\big(\varphi(Y_k - Y_j) > t\big) \leqslant 2\mathbf{P}\big(\varphi(Y_k) > t\big) \leqslant 2\mathbf{P}(M > t).$$

Hence we find

$$F_k(2t+s) \leqslant \sum_{j=1}^{k} \mathbf{P}(T = j, N > s) + 2\sum_{j=1}^{n} \mathbf{P}(T = j)\ \mathbf{P}(M > t) \leqslant G(s) + 2F(t)^2$$

and from Lemma 2.1, we find

$$F(2t+s) \leqslant 2 \liminf_{k\to\infty} F_k(2t+s) \leqslant 2G(s) + 4F(t)^2$$

which proves (2.3.1).

Now suppose that φ is symmetric, and let (X_n^*) be a symmetrization. Define M^* and N^* in the obvious way and let F^*, respectively, G^* be their tail probabilities; then by Lemma 2.2 and (2.3.1) we find

$$F(4t+2s+a) \leqslant 2F^*(4t+2s) \leqslant 4G^*(2s) + 8F^*(2t)^2 \leqslant 8G(s) + 32F(t)^2.$$

THEOREM 2.4. *If (X_n) is symmetric, $M_n < \infty$ a.s. for $n \geqslant 1$, and $\{M_n\}$ stochastically bounded, then $M < \infty$ a.s.*

If φ is symmetric, $M_n < \infty$ a.s. for all $n \geqslant 1$, and $\{M_n\}$ is stochastically bounded, then $M < \infty$ a.s.

Proof. The first case is an immediate consequence of Lemma 2.1. So suppose that φ is symmetric, and let (X_n^*) be a symmetrization of (X_n). Let M_n^* and M^* be defined in the obvious way. Then (2.2.2) shows that $\{M_n^*\}$ is stochastically bounded, and so $M^* < \infty$ a.s. Hence we find

$$1 = \mathbf{P}(M^* < \infty) = \int_{E_\infty} \mathbf{P}\big(\varphi_0(X - x) < \infty\big)\,\mu(dx),$$

where μ is the distribution of X, and

$$\varphi_0(x) = \sup_n \varphi(x_1, \ldots, x_n, 0, \ldots) \qquad \forall x = (x_j) \in E_\infty.$$

So for some $x \in E_\infty$ we have $\varphi_0(X - x) < \infty$ a.s. Let $y_n = (x_1, \ldots, x_n, 0, \ldots)$ and $Y_n = (X_1, \ldots, X_n, 0, \ldots)$, then we have

$$\varphi(y_n) \leqslant \varphi(Y_n - y_n) + \varphi(Y_n) \leqslant \varphi_0(X - x) + M_n.$$

Now we choose $a \geqslant 0$ so that $\mathbf{P}(M_n \leqslant a) > \frac{1}{2}$ and $\mathbf{P}\big(\varphi_0(X - x) \leqslant a\big) > \frac{1}{2}$, then for each $n \geqslant 1$ we have

$$\{M_n \leqslant a\} \cap \{\varphi_0(X - x) \leqslant a\} \neq \emptyset$$

and so $\varphi(y_n) \leqslant 2a$ for all $n \geqslant 1$. That is

$$M = \sup_n \varphi(Y_n) \leqslant \varphi_0(X - x) + 2a < \infty \text{ a.s.}$$

and the theorem is proved.

THEOREM 2.5. *Suppose that $M < \infty$ a.s., and let g be an increasing function from $[t_0, \infty)$ into $\mathbf{R}_+$ so that*

$$\lim_{t\to\infty} g(t) = \infty, \tag{2.5.1}$$

$$G(t) \leqslant K \exp\big(-g(\varepsilon t)\big) \quad \forall t \geqslant t_0, \tag{2.5.2}$$

for some constants $K > 0$ and $\varepsilon > 0$. Let $c > 1$ and put $\lambda = (\log c)/(\log 2)$, and $\nu = \lambda^{-1}$.

(A) *If g satisfies:*

$$\lim_{t\to\infty} \big(g(ct) - 2g(t)\big) = -\infty, \tag{2.5.3}$$

then there exists $K_1 > 0$, $\delta > 0$ *and* $t_1 \geqslant t_0$ *so that*

$$(2.5.4)\qquad F(t) \leqslant K_1 \exp\big(-g(\delta t)\big) \qquad \forall t \geqslant t_1 \qquad \text{if } c > 2,$$

$$(2.5.5)\qquad F(t) \leqslant K_1 \exp\big(-g(\delta t^{\lambda})\big) \qquad \forall t \geqslant t_1 \qquad \text{if } c < 2,$$

$$(2.5.6)\qquad F(t) \leqslant K_1 \exp\left(-g\left(\frac{\delta t}{\log t}\right)\right) \quad \forall t \geqslant t_1 \qquad \text{if } c = 2.$$

(B) *If* g *satisfies*

$$(2.5.7)\qquad g(ct) \geqslant 2g(t) \qquad \forall t \geqslant t_1,$$

then there exist $K_2 > 0$, $\delta > 0$ *and* $t_1 \geqslant t_0$ *so that*

$$(2.5.8)\qquad F(t) \leqslant K_2 \exp(-\delta t) \qquad \forall t \geqslant t_1 \quad \text{if } c < 2,$$

$$(2.5.9)\qquad F(t) \leqslant K_2 \exp(-\delta t^{\gamma}) \qquad \forall t \geqslant t_1 \quad \text{if } c > 2,$$

$$(2.5.10)\qquad F(t) \leqslant K_2 \exp\left(-\frac{\delta t}{\log t}\right) \quad \forall t \geqslant t_1 \quad \text{if } c = 2.$$

(C) *If* g *satisfies*

$$(2.5.11)\qquad \int_{t_0}^{\infty} g(t)^{-1}\,dt < \infty,$$

then there exist $K_3 > 0$ *and* $\delta > 0$ *so that*

$$(2.5.12)\qquad F(t) \leqslant K_3 \exp(-\delta t) \qquad \forall t \geqslant t_1.$$

Proof. Let (X_n^*) be a symmetrization of (X_n) and let M^* and N^* be defined in the obvious way. Let $F^*(t) = \mathbf{P}(M^* > t)$ and $G^*(t) = \mathbf{P}(N^* > t)$, then we have

$$F^*(2t+s) \leqslant 2G^*(s) + 4F^*(t)^2 \qquad \forall t, s \geqslant 0,$$

$$G^*(2s) \leqslant 2G(s),$$

$$F(t+a) \leqslant 2F^*(t),$$

where $a \geqslant 0$ is determined by $\mathbf{P}(M \leqslant a) \geqslant \frac{1}{2}$. Now let

$$F_0(t) = 8F^*(t-a), \qquad G_0(t) = 32G(2t),$$

then we have

(i) $F_0(2t+2s+3a) \leqslant \frac{1}{2}G_0(s) + \frac{1}{2}F_0(t)^2 \leqslant \max\{G_0(s), F_0(t)^2\}$,

(ii) $F(t) \leqslant 8F_0(t)$,

(iii) $G_0(t) \leqslant \exp\big(-g_0(t)\big)$ $\forall t \geqslant t_0$ where $g_0(t) = g(\frac{1}{2}\varepsilon t) - \log(32K)$.

Now suppose that σ_n $(n \geqslant 0)$ and τ_0 is given; then we define

$$\tau_n = 2^n \tau_0 + \sum_{j=0}^{n-1} 2^{n-j-1}(2\sigma_j + 3a) = 2\tau_{n-1} + 2\sigma_{n-1} + 3a.$$

Then we have

(iv) $\tau_n \leqslant K_1 2^n + 2^n \sum_{j=0}^{n-1} 2^{-j}\sigma_j$ $\forall n \geqslant 0$,

(v) $F_0(\tau_n) \leqslant \max\{G_0(\sigma_{n-1}), G_0(\sigma_{n-2})^2, \ldots, G_0(\sigma_0)^{2^{n-1}}, F_0(\tau_0)^{2^n}\}$.

Case (A), $c \neq 2$. Now we choose $\sigma_0 \geqslant t_0$ from (2.5.3) so that

$$g(ct) \leqslant 2g(t) - 3\log(32K) \qquad \forall t \geqslant 2\varepsilon^{-1}\sigma_0,$$

then $g_0(ct) \leqslant 2g_0(t)$ for all $t \geqslant \sigma_0$. Then we put $\sigma_n = c^n \sigma_0$ for $n \geqslant 0$ and choose $\tau_0 \geqslant \sigma_0$, so that $\log F(\tau_0) \leqslant -\frac{1}{2}g_0(\sigma_0)$. Now since $2^{-j}g_0(\sigma_j)$ decreases in j we have

$$G_0(\sigma_j)^{2^{n-j-1}} \leqslant \exp\big(-2^{n-j-1}g_0(\sigma_j)\big) \leqslant \exp\big(-g_0(\sigma_{n-1})\big)$$

for all $0 \leqslant j \leqslant n-1$, and since $g_0(\sigma_{n-1}) \leqslant 2^{n-1}g_0(\sigma_0)$, we have

$$F_0(\tau_0)^{2^n} \leqslant \exp\big(-2^{n-1}g_0(\sigma_0)\big) \leqslant \exp\big(-g_0(\sigma_{n-1})\big).$$

So by (v) we find

$$F_0(\tau_n) \leqslant \exp\big(-g_0(c^{n-1}\sigma_0)\big) \qquad \forall n \geqslant 0.$$

And from (iv) we find

$$\tau_n \leqslant 2^n K_1 + 2^n \sigma_0 \sum_{j=0}^{n-1} (\tfrac{1}{2}c)^j \leqslant K_2 d^n \qquad \forall n \geqslant 0,$$

where $d = \max\{2, c\}$.

Now let $t_1 = K_2$, and $t \geqslant t_1$; then there exists an integer $n \geqslant 0$ so that $K_2 d^n \leqslant t \leqslant K_2 d^{n+1}$, and so

$$F_0(t) \leqslant F_0(\tau_n) \leqslant \exp\big(-g_0(c^{n-1}\sigma_0)\big).$$

However, $c = d^{\gamma}$, where $\gamma = \min\{1, \lambda\}$, and so

$$c^{n-1}\sigma_0 = c^{-2}\sigma_0 d^{\gamma(n+1)} \geqslant at^{\gamma}$$

where $a = K_2^{-\gamma}c^{-2}\sigma_0 > 0$, and so

$$F(t) \leqslant 8F_0(t) \leqslant 8\exp\big(-g_0(at^{\gamma})\big) = 256 \cdot K\exp\big(-g(\delta t^{\gamma})\big)$$

where $\delta = \frac{1}{2}\varepsilon a$. This proves case (A) $c \neq 2$.

Case (A), $c = 2$. σ_0 and τ_0 is chosen as before and we find that

$$F_0(\tau_n) \leqslant \exp\big(-g_0(\sigma_{n-1})\big).$$

However, in this case we have

$$\tau_n \leqslant 2^n K_1 + n\sigma_0 2^n \leqslant n 2^n K_3 \qquad \forall n \geqslant 1.$$

Now let $t_1 = 2K_3$, and let $t \geqslant t_1$; then there exists $n \geqslant 1$ so that $n2^n K_3 \leqslant t \leqslant (n+1)2^{n+1}K_3$, then

$$F_0(t) \leqslant F(\tau_n) \leqslant \exp\big(-g_0(2^{n-1}\sigma_0)\big).$$

Now we observe that

$$\frac{\beta t}{\log t} \leqslant \frac{\beta(n+1)2^{n+1}K_3}{(n+1)+\log(n+1)+\log K_3} \leqslant \beta 2^{n+1}K_3 = 2^{n-1}\sigma_0$$

if $\beta = K_3^{-1}2^{-2}\sigma_0$. Hence,

$$F(t) \leqslant 8F_0(t) \leqslant 8\exp\left(-g_0\left(\frac{\beta t}{\log t}\right)\right) = 256\cdot K\exp\left(-g\left(\frac{\delta t}{\log t}\right)\right),$$

where $\delta = \frac{1}{2}\varepsilon\beta$.

Case (B). In this case we choose $\sigma_n = c^n t_0$, and τ_0 so that $\log F(\tau_0) \leqslant \frac{1}{2}g_0(t_0)$. Now $g_0(ct) \geqslant 2g_0(t)$ for all $t \geqslant t_0$ by (2.5.7) and the definition of g_0, hence we have that $2^{-j}g_0(\sigma_j)$ increases with j, and so

$$G_0(\sigma_j)^{2^{n-j-2}} \leqslant \exp\left(-2^{n-j-1}g_0(\sigma_0)\right) \leqslant \exp\left(-2^{n-1}g_0(t_0)\right),$$

$$F_0(\tau_0)^{2^n} \leqslant \exp\left(-2^{n-1}g_0(t_0)\right).$$

From (v) we find that

$$F_0(\tau_n) \leqslant \exp\left(-2^{n-1}g_0(t_0)\right)$$

and as in case (A) we find

$$\tau_n \leqslant \begin{cases} 2^n K_4, & \text{if } c<2, \\ n2^n K_5, & \text{if } c=2, \\ c^n K_6, & \text{if } c>2. \end{cases}$$

Case (B) now follows by an argument similar to the proof of case (A).

Case (C). Let σ_n be defined by

$$\sigma_n = \inf\{t \geqslant t_0 \mid g_0(t) \geqslant 2^n g_0(t_0)\} \quad \forall n \geqslant 0;$$

then $g_0(\sigma_n) = 2^n g_0(t_0)$, and we have

$$G_0(\sigma_j)^{2^{n-j-1}} \leqslant \exp\left(-2^{n-j-1}g_0(\sigma_j)\right) = \exp\left(-2^{n-1}g_0(t_0)\right).$$

So if τ_0 is chosen so that $\log F(\tau_0) \leqslant -\frac{1}{2}g_0(t_0)$, then we have by (v)

$$F_0(\tau_n) \leqslant \exp\left(-2^{n-1}g_0(t_0)\right).$$

Now let $b_j = \sigma_j g_0(\sigma_j)^{-1} = k\sigma_j 2^{-j}$, then we have

$$a_j = \int_{\sigma_{j-1}}^{\sigma_j} g_0(t)^{-1}dt \geqslant (\sigma_j - \sigma_{j-1})g_0(\sigma_j)^{-1} = b_j - \tfrac{1}{2}b_{j-1},$$

$$b \leqslant \tfrac{1}{2}b_{j-1} + a_j \leqslant (\tfrac{1}{2})^j b_0 + \sum_{i=1}^{j} a_i(\tfrac{1}{2})^{j-i}.$$

So using (2.5.11) we find that

$$\tau_n \leqslant K_1 2^n + 2^n \sum_{j=0}^{n-1} 2^{-j}\sigma_j \leqslant K_1 2^n + 2^n\left(\sigma_0 + \sum_{j=1}^{\infty} k^{-1}b_j\right)$$

$$\leqslant K_1 2^n + 2^n\left(\sigma_0 + b_0 + \sum_{j=1}^{\infty}\sum_{i=1}^{j} k^{-1}a_i(\tfrac{1}{2})^{j-i}\right)$$

$$= 2^n\left(K_1 + b_0 + \sigma_0 + k^{-1}\sum_{i=1}^{\infty} a_i\right)$$

$$\leqslant 2^n\left(K_1 + b_0 + \sigma_0 + k^{-1}\int_{t_0}^{\infty} g_0(t)^{-1}dt\right) = K_7 2^n$$

from which (2.5.11) follows as above.

Examples. (a) $G(t) \leqslant K_1 f(t)^{-1}$, where f satisfies

(2.4) $\quad f$ is increasing on $[t_0, \infty)$ and $\lim_{t\to\infty} f(t) = \infty$.

(2.5) $\quad f(2t) \leqslant Mf(t) \ \ \forall t \geqslant t_0, \quad$ for some $M > 0$,

then $g(t) = \log f(t)$ satisfies (2.5.3) for all $c > 1$, and so we find

(2.6) $$F(t) \leqslant K_2 f(t)^{-1}.$$

Note that $f(t) = t^p$ satisfies (2.4) and (2.5) for all $p > 0$.

A simple computation, using integration by parts, shows that if f satisfies (2.4) and (2.5), then we have

(2.7) $$\mathbf{E}f(N) < \infty \quad \text{implies} \quad \mathbf{E}f(M) < \infty.$$

(b) $G(t) \leqslant K_1\exp(-\varepsilon_1 t^p)$, where $p > 0$, then $g(t) = t^p$ satisfies (2.5.7) for $c = 2^{1/p}$, hence we have

(2.8) $$F(t) \leqslant K_2\exp(-\varepsilon_2 t) \quad \text{if} \quad p > 1,$$

(2.9) $$F(t) \leqslant K_2\exp(-\varepsilon_2 t^p) \quad \text{if} \quad p < 1,$$

(2.10) $$F(t) \leqslant K_2\exp\left(-\varepsilon_2 t(\log t)^{-1}\right) \quad \text{if} \quad p = 1.$$

(c) $G(t) \leqslant K_1\exp\left(-\varepsilon_1 t(\log t)^p\right)$ for some $p > 1$, then $g(t) = t(\log t)^p$ satisfies (2.5.11), and so we have

(2.11) $$F(t) \leqslant K_2\exp(-\varepsilon_2 t).$$

3. The zero-one law for product measures. Let μ_n be a probability measure on $(\boldsymbol{R}, \boldsymbol{B})$, and let

$$\boldsymbol{R}^\infty = \prod_{n=1}^{\infty}\boldsymbol{R}, \quad \boldsymbol{B}^\infty = \bigotimes_{n=1}^{\infty}\boldsymbol{B}, \quad \mu = \prod_{n=1}^{\infty}\mu_n.$$

A set $A \subseteq \boldsymbol{R}^\infty$ is called a *tail event* if A satisfies

(3.1) *If $x \in \mathbf{R}^\infty$ and $x_j = y_j$ for all $j \geqslant n$, for some $y \in A$ and some $n \geqslant 1$, then $X \in A$.*

If $\mathbf{R}_0^\infty$ is the set of $x = (x_j) \in \mathbf{R}^\infty$ with almost finitely many non-zero coordinates, then it is easily checked that (3.1) is equivalent to

(3.2) $$\mathbf{R}_0^\infty + A \subseteq A.$$

It is well know that $\mu(A)$ is either 0 or 1 if A is a μ-measurable tail event. We shall in this section show that the 0-1 law also holds for linear subspaces of $\mathbf{R}^\infty$.

THEOREM 3.1. *Suppose that μ_n is non-atomic for all $n \geqslant 1$, and let L be a μ-measurable linear subset of $\mathbf{R}^\infty$. Then either $\mu(L) = 0$ or $\mu(L) = 1$ and in the latter case we have $\mathbf{R}_0^\infty \subseteq L$.*

Proof. Since μ is a Radon measure on $\mathbf{R}^\infty$, we can find a σ-compact linear subset $L_0 \subseteq L$, so that $\mu(L_0) = \mu(L)$, so we may as well assume that L itself is σ-compact.

Now suppose that $\mu(L) > 0$, and that $\mathbf{R}_0^\infty \nsubseteq L$. Then for some $n \geqslant 1$ we have $e_n \notin L$, where e_n is the nth unit vector in $\mathbf{R}^\infty$. Let

$$F = L \cap \mathbf{R}_0^\infty$$

and let G be a linear complement to F in $\mathbf{R}_0^\infty$, so that $e_n \in G$. Then G is σ-compact (note that all linear subsets of $\mathbf{R}_0^\infty$ are σ-compact), and so $L_1 = L + G$ is σ-compact. Moreover, we have

$$L_1 \supseteq F + G = \mathbf{R}_0^\infty, \qquad \mu(L_1) \geqslant \mu(L) > 0.$$

Since L_1 is linear, we find that L_1 is a tail event, and so $\mu(L_1) = 1$. Now we observe that $L \cap G = (0)$, and so

$$(x, y) \frown x + y$$

is a continuous bijection from $L \times G$ to $L + G$, hence, by Theorem III.8.4 in [4], we see that there exist linear Borel maps p and q so that

$$p\colon L+G \to L, \qquad q\colon L+G \to G,$$
$$x = p(x) + q(x) \qquad \forall x \in L_1.$$

Let π_n be the projection of $\mathbf{R}^\infty$ onto the nth coordinate, and put $p_0 = \pi_n \circ p$ and $q_0 = \pi_n \circ q$. Then p_0 and q_0 are Borel measurable linear functionals which are defined μ-a.e., and we have

$$q_0(x) = 0 \qquad \text{for } x \in L,$$
$$q_0(e_n) = 1, \qquad p_0(e_n) = 0.$$

Hence we have

$$p_0(x) = p_0(x_1, \ldots, x_{n-1}, 0, x_{n+1}, \ldots) \qquad \forall x = (x_j) \in L_1$$

so p_0 and π_n are independent and $p_0 + q_0 = \pi_n$ a.s. Hence if ν is the distribution law of p_0, we have

$$\mu(L) \leqslant \mu(q_0 = 0) = \mu(p_0 = \pi_n)$$
$$= \int_{-\infty}^{\infty} \mu(\pi_n = x)\,\nu(dx) = \int_{-\infty}^{\infty} \mu_n(\{x\})\,\nu(dx) = 0$$

since μ_n is non-atomic. So we have derived a contradiction, and we must have that $\mathbf{R}_0^\infty \subseteq L$ whenever $\mu(L) > 0$. However, since L is linear, we have that $\mathbf{R}_0^\infty \subseteq L$ implies that L is a tail event and so $\mu(L) = 1$.

4. The affine kernel of a product measure. Let $\mu = \prod_{n=1}^{\infty} \mu_n$ be a product probability measure on $\mathbf{R}^\infty$, then the *affine kernel* of μ, $A(\mu)$, is defined by

$$A(\mu) = \bigcap \{A \mid A \ \mu\text{-measurable affine set with } \mu(A) = 1\}.$$

The measurable affine subsets of $\mathbf{R}^\infty$ with full measure are often too badly behaved, and we shall mainly restrict ourselves to consider only those affine sets, $A \subseteq \mathbf{R}^\infty$ which satisfy

(4.1) *$\forall \varepsilon > 0\ \exists K$ compact convex, so that $K \subseteq A$ and $\mu(K) > 1 - \varepsilon$.*

An affine space satisfying (4.1) will be called a *μ-Lusin affine space*, and we define the *Lusin affine kernel* of μ, $A_L(\mu)$, by

$$A_L(\mu) = \bigcap \{A \mid A \text{ is a } \mu\text{-Lusin affine space}\}.$$

Let $\Lambda(\mu)$ denote the set of μ-measurable linear functions, $f\colon \mathbf{R}^\infty \to \mathbf{R}$. Again these functions are often too badly behaved, and we shall call $f \in \Lambda(\mu)$ a *μ-Lusin functional*, if f satisfies

(4.2) *$\forall \varepsilon > 0\ \exists K$ compact convex, so that $\mu(K) > 1 - \varepsilon$ and $f \mid K$ is continuous.*

And we denote the set of μ-Lusin functionals by $\Lambda_L(\mu)$. The space $\Lambda(\mu)$ and $\Lambda_L(\mu)$ may be considered as subspaces of $L^0(\mu)$, where $L^0(\mu)$ is the space of all μ-measurable functions, X, equipped with the metric

$$\|X\|_\mu = \int_{\mathbf{R}^\infty} \varphi(|x|)\,d\mu$$

where $\varphi(x) = x/(1+x)$. It is well known (see Theorem 1 in [10]) that

(4.3) *$\Lambda(\mu)$ and $\Lambda_L(\mu)$ are closed subspaces of $L^0(\mu)$ and $\Lambda_L(\mu) = \mathrm{cl}(\mathbf{R}_0^\infty)$ (closure in $L^0(\mu)$), where $\mathbf{R}_0^\infty$ is identified with the dual of $\mathbf{R}^\infty$.*

The main results of this section state under certain conditions on μ that every $f \in A_L(\mu)$ admit a representation of the form

$$f(x) = \sum_{j=1}^{\infty} a_j x_j,$$

where $a = (a_j)$ belongs to the set:

$$C(\mu) = \Big\{(a_j) \mid \sum_{j=1}^{\infty} a_j x_j \text{ converges } \mu\text{-a.e.}\Big\}$$

and that the Lusin affine kernel $A_L(\mu)$, is equal to the sequence dual, $C^*(\mu)$, of $C(\mu)$, that is

$$C^*(\mu) = \Big\{(b_j) \in \boldsymbol{R}^\infty \mid \sum_{j=1}^{\infty} a_j b_j \text{ converges } \forall a \in C(\mu)\Big\}.$$

The condition, which assures the validity of these two results, is that $0 \in A_L(\mu)$, and we shall say that μ *is centered at* 0, if $0 \in A_L(\mu)$.

If $a \in C(\mu)$, then we define

$$|a|_\mu = \sup_n \int_{\boldsymbol{R}^\infty} \varphi\Big(\Big|\sum_{j=1}^{n} a_j x_j\Big|\Big)\mu(dx).$$

A slight modification of the proof of Theorem 5.2 in [2] shows that, if μ_n is non-degenerated for all $n \geqslant 1$, then

(4.4) $(C(\mu), |\cdot|_\mu)$ *is a Fréchet space (in general non-locally convex).*

THEOREM 4.1. *If μ_n is non-degenerated for all $n \geqslant 1$, and A is a μ-measurable affine set with $\mu(A) = 1$, then $A + \boldsymbol{R}_0^\infty \subseteq A$.*

Proof. Let μ^* be the symmetrization of μ. Then

$$\mu^*(B) = \int_{\boldsymbol{R}^\infty} \mu(B+x)\mu(dx) \qquad \forall B \in \boldsymbol{B}^\infty,$$

and, in particular, we have

$$\mu^*(A-x_0) = \int_{\boldsymbol{R}^\infty} \mu(A-x_0+x)\mu(dx) = \int_A \mu(A-x_0+x)\mu(dx).$$

So if $x_0 \in A$, then $A - x_0 + x = A$ for all $x \in A$, and so

$$\mu^*(A-x_0) = 1 \qquad \forall x_0 \in A.$$

Let $L = A - x_0$, where $x_0 \in A$, then L is a linear set and $\mu^*(L) = 1$. Let Σ be the set of linear transformations, σ: $\boldsymbol{R}^\infty \to \boldsymbol{R}^\infty$, of the form

$$\sigma x = (\varepsilon_n x_n) \quad \text{for} \quad x = (x_n) \in \boldsymbol{R}^\infty$$

where $\varepsilon_n = \pm 1$ and $\varepsilon_n = 1$ for all $n \geqslant N$ for some $N \geqslant 1$. Then Σ is a countable group and

$$\mu^*\big(\sigma(B)\big) = \mu^*(B) \qquad \forall B \in \boldsymbol{B}^\infty \ \forall \sigma \in \Sigma.$$

So if $L_0 = \bigcap_{\sigma\in\Sigma} \sigma(L)$, then $\mu^*(L_0) = 1$. Let $n \geqslant 1$, now since μ_n is non-degenerated we can find $y \in L_0$ so that $y_n \neq 0$. Now let σ be defined by

$$\sigma x = (x_1, \ldots, x_{n-1}, -x_n, x_{n+1}, \ldots);$$

then $\sigma \in \Sigma$ and since $\sigma(L_0) = L_0$, we have that $\sigma y \in L_0$. Moreover, since L_0 is linear, we have

$$e_n = (2y_n)^{-1}(y - \sigma y) \in L_0,$$

where e_n is the nth unit vector. Hence we have

$$\boldsymbol{R}_0^\infty \subseteq L_0 \subseteq L = A - x_0 \qquad \forall x_0 \in A$$

from which we find that $\boldsymbol{R}_0^\infty + A \subseteq A$.

THEOREM 4.2. *Suppose that μ_n is non-degenerated for all $n \geqslant 1$ and let μ^* be the symmetrization of μ, that is*

$$\mu^*(B) = \int_{\boldsymbol{R}^\infty} \mu(B+x)\mu(dx) \qquad \forall B \in \boldsymbol{B}^\infty.$$

Then μ is centered at 0 if and only if $C(\mu) = C(\mu^*)$.

Proof. Suppose that μ is centered at 0, and let $a \in C(\mu^*)$. Let

$$L = \Big\{(x_j) \in \boldsymbol{R}^\infty \mid \sum_{j=1}^{\infty} a_j x_j \text{ converges}\Big\};$$

then L is a linear Borel subset of $\boldsymbol{R}^\infty$ with $\mu^*(L) = 1$, that is

$$1 = \mu^*(L) = \int_{\boldsymbol{R}^\infty} \mu(L+x)\mu(dx)$$

and so $\mu(M) = 1$, where $M = \{x \mid \mu(L+x) = 1\}$. Now let $b = (b_j) \in M$, then

$$L + b = \Big\{(x_j) \in \boldsymbol{R}^\infty \mid \sum_{j=1}^{\infty} a_j(x_j - b_j) \text{ converges}\Big\},$$

$$\mu(L+b) = 1.$$

Let $\varepsilon > 0$ be given, then by Egoroff's theorem there exists $B \in \boldsymbol{B}^\infty$ and $n_1 < n_2 < \ldots$ so that

$$\Big|\sum_{j=n+1}^{n+m} a_j(x_j - b_j)\Big| \leqslant 2^{-k} \qquad \forall n \geqslant n_k \ \forall m, k \geqslant 1 \ \forall x \in B,$$

$$\mu(B) > 1 - \varepsilon.$$

Now let

$$C = \bigcap_{k=1}^{\infty} \bigcap_{n=n_k}^{\infty} \bigcap_{m=1}^{\infty} \{x \mid |\sum_{j=n+1}^{n+m} a_j(x_j - b_j)| \leqslant 2^{-k}\};$$

then $B \subseteq C \subseteq L+b$, and C is convex and closed. Now we choose a compact set $D \subseteq B$ so that $\mu(D) > 1-\varepsilon$, and put K equal to the closed convex hull of D. Then $D \subseteq K \subseteq C \subseteq L+b$, since C is convex and closed, and $\mu(K) > 1-\varepsilon$. This shows that $L+b$ is a μ-Lusin affine space and so by assumption $0 \in L+b$ for all $b \in M$, or $M \subseteq L$. Hence $\mu(M) = \mu(L) = 1$, and $a \in C(\mu)$ which proves that $C(\mu^*) \subseteq C(\mu)$, the other inclusion is trivial.

Now suppose that $C(\mu^*) = C(\mu)$, then $|\cdot|_\mu$ and $|\cdot|_{\mu^*}$ are Fréchet metrics on $C(\mu)$, so that

$$|a|_{\mu^*} \leqslant 2|a|_\mu \quad \forall a \in C(\mu).$$

Hence the two metrics are equivalent, and so we have

(i) $\forall \varepsilon > 0\ \exists \delta > 0$ so that $|a|_{\mu^*} \leqslant \delta$ implies $|a|_\mu \leqslant \varepsilon$.

Now we consider the linear operator, $T: C(\mu) \to L^0(\mu)$, defined by

$$Ta = \sum_{j=1}^{\infty} a_j x_j,$$

then $\|Ta\|_\mu \leqslant |a|_\mu$, so T is continuous. Let $\varepsilon > 0$ be given and choose $\delta > 0$ according to (i), if $a \in C(\mu)$ and $\|2Ta\|_\mu < \delta$, then by Theorem 2.6 in [3], we have

$$\int_{R^\infty} \varphi\Big(\Big|\sum_{j=1}^{n} a_j x_j\Big|\Big)\, \mu^*(dx) \leqslant \tfrac{1}{2}\|2Ta\|_{\mu^*} \leqslant \|2Ta\|_\mu \leqslant \delta$$

for all $n \geqslant 1$. Hence $|a|_{\mu^*} \leqslant \delta$ and so $|a|_\mu \leqslant \varepsilon$, that is $\|2Ta\|_\mu \leqslant \delta$ implies that $|a|_\mu \leqslant \varepsilon$. This shows that T is an isomorphism and $T(C(\mu))$ is a closed subspace of $L^0(\mu)$. So by (4.3) we have

(ii) $\Lambda_L(\mu) \subseteq T(C(\mu))$.

Let A be a μ-Lusin affine space, and suppose that $0 \notin A$, then there exists a linear functional, f, so that $f(x) = 1$ for $x \in A$. Obviously, we have that $f \in \Lambda_L(\mu)$, and so by (ii) we have $f = Ta$ for some $a \in C(\mu)$. That is

$$\sum_{j=1}^{\infty} x_j a_j = 1 \quad \mu\text{-a.e.}$$

and so $a \neq 0$. Let $n \geqslant 1$ be chosen, so that $a_n \neq 0$, then the random variables (defined on $(R^\infty, B^\infty, \mu)$)

$$X_n(x) = a_n x_n, \qquad Y_n = \sum_{j \neq n} a_j x_j,$$

are independent and $X_n + Y_n = 1$ a.s., but this implies that X_n is degenerated contrary to the assumption that μ_n is non-degenerated. Hence $0 \in A$ and μ is centered at 0.

THEOREM 4.3. *Suppose that μ is centered at 0, and μ_n is non-degenerated for all $n \geqslant 1$. Then there exists a linear functional, $\lambda: \Lambda(\mu) \to R$, so that*

(4.2.1) $$\lambda(f) = \lim_{n\to\infty} f(0, \ldots, 0, x_n, x_{n+1}, \ldots) \quad \textit{for a.a. } x \in R^\infty;$$

(4.2.2) $$(f(e_j)) \in C(\mu) \quad \textit{and} \quad f(x) = \lambda(f) + \sum_{j=1}^{\infty} x_j f(e_j) \text{ a.e.};$$

(4.2.3) $$\lambda(f) = 0 \quad \forall f \in \Lambda_L(\mu);$$

(4.2.4) $$\lambda \equiv 0 \quad \textit{if } \mu_n \textit{ is symmetric for all } n \geqslant 1.$$

Proof. Let $f \in \Lambda(\mu)$ and define $a_j = f(e_j)$ and

$$X_j(x) = a_j x_j,$$

$$Y_n(x) = f(0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots),$$

$$Z_n(x) = \sum_{j=1}^{n} x_j f(e_j) = \sum_{j=1}^{n} X_j(x) = f(x) - Y_n(x),$$

for $x = (x_j) \in R^\infty$. Then $X_1, \ldots, X_n, Y_n$ are independent as random variables on $(R^\infty, B^\infty, \mu)$ so if φ, φ_n and ψ_n denote the characteristic functions of f, X_n and Y_n, we have

$$\varphi(t) = \psi_n(t) \prod_{j=1}^{n} \varphi_j(t)$$

and so

$$|\varphi(t)| \leqslant \lim_{n\to\infty} \prod_{j=1}^{n} |\varphi_j(t)|.$$

Hence by Corollary 2, p. 251, and Theorem b, p. 250 in [7], we find that $(f(e_j)) = a \in C(\mu^*)$, and so by Theorem 4.2 we have $a \in C(\mu)$. That is,

$$\lambda_0(f) = \lim_{n\to\infty} Y_n$$

exists a.e., and $\lambda_0(f)$ is measurable with respect to the tail σ-algebra, that is $\lambda_0(f)$ is constant a.s. Hence there exists $\lambda(f) \in R$ so that

$$\lambda(f) = \lim_{n\to\infty} f(0, \ldots, 0, x_n, x_{n+1}, \ldots) \text{ a.e.}$$

and, clearly, λ is linear on $\Lambda(\mu)$, and we have

$$f(x) = \lambda(f) + \sum_{1}^{\infty} x_j f(e_j) \text{ a.e.}$$

Now suppose that $f \in \Lambda_L(\mu)$, and let

$$L = \{x| \sum_{j=1}^{\infty} x_j f(e_j) \text{ converges}\},$$

$$L_0 = \{x \in L| f(x) - \sum_{j=1}^{\infty} x_j f(e_j) = \lambda(f)\};$$

then an argument similar to the first part of the proof of Theorem 4.2 shows that L_0 is a μ-Lusin affine space and so by assumption we have $0 \in L$. That is $\lambda(f) = 0$.

Now suppose that μ_n is symmetric for all $n > 1$, and let $f \in \Lambda(\mu)$. If we define

$$g(x) = f(x) - \sum_{j=1}^{\infty} x_j f(e_j),$$

then g and $-g$ has the same distribution law, and $g = \lambda(f)$ a.e. Hence $\lambda(f) = 0$.

THEOREM 4.4. *Suppose that μ is centered at 0, and μ_n is non-degenerated for all $n \geqslant 1$. Then we have that $C^*(\mu) = \Lambda_L(\mu)$.*

Proof. Let $b \in C^*(\mu)$ and let A be a μ-Lusin affine space. Then A is linear, since μ is centered at 0, and we can find compact convex symmetric sets, K_n, so that

$$K_n \subseteq A, \quad K_n + K_n \subseteq K_{n+1}, \quad \mu(K_n) > 1 - 2^{-n}.$$

Let $L = \bigcup_{n=1}^{\infty} K_n$; then L is a linear space $L \subseteq A$ and $\mu(L) = 1$. Now let

$$F = \{f| f \text{ linear}: L \to \mathbf{R},\ f|K_n \text{ is continuous } \forall n\},$$

$$\|f\|_n = \sup_{x \in K_n} |f(x)| \quad \forall f \in F,$$

$$\varrho(f, g) = \sum_{n=1}^{\infty} 2^{-n} \|f - g\|_n \quad \forall f, g \in F;$$

then it is easily seen that (F, ϱ) is a locally convex Fréchet space. Moreover, it is clear that the ϱ-topology is stronger than the $\sigma(F, L)$-topology (note that (F, L) is a dual pair).

If $y \notin K_n$, then there exists $f \in (\mathbf{R}^\infty)'$, so that $f(y) > 1$ and $|f(x)| \leqslant 1$ for all $x \in K_n$, that is $f \in K_n^0$, where K_n^0 is the polar of K_n in F. Since $f(y) > 1$, this shows that $y \notin K_n^{00}$, where K_n^{00} is the polar of K_n^0 in L. Hence $K_n^{00} \subseteq K_n$, and since the converse inclusion is obvious, we have $K_n = K_n^{00}$. So K_n is $\sigma(L, F)$-compact and convex.

Since the ϱ-topology is the topology of uniform convergence on each K_n, we find that the ϱ-topology is weaker than the Mackey topology: $\tau(F, L)$. Hence we have

(i) $$(F, \varrho)' = L.$$

Now let $f \in F$ and define

$$S_n f = \sum_{j=1}^{n} b_j f(e_j);$$

then S_n is a linear functional on F, and S_n is continuous since $e_j \in L$ for all $j \geqslant 1$ by Theorem 4.1. Moreover, we have

$$\lim_{n\to\infty} S_n f = Sf = \sum_{j=1}^{\infty} b_j f(e_j)$$

since $(f(e_j)) \in C(\mu)$ by Theorem 4.3 and $b \in C^*(\mu)$. But (F, ϱ) is a Fréchet space and so $S \in F'$. So by (i) there exist $x \in L$ with

$$\sum_{j=1}^{\infty} b_j f(e_j) = f(x) \quad \forall f \in F.$$

Now the functional $f_j(x) = x_j$ for $x \in L$ belongs to F and so $b_j = x_j$ for all $j \geqslant 1$. Hence $b = x \in L \subseteq A$ and so $C^*(\mu) \subseteq \Lambda_L(\mu)$.

Now suppose that $b \notin C^*(\mu)$, then there exists $a \in C(\mu)$ so that $\sum_1^\infty a_j b_j$ diverges. Let

$$L = \{x \in \mathbf{R}^\infty| \sum_1^\infty a_j x_j \text{ converges}\};$$

then as in the proof of Theorem 4.2 one finds that L is a μ-Lusin affine space, and since $b \notin L$, we have $b \notin \Lambda_L(\mu)$, and so $C^*(\mu) \supseteq \Lambda_L(\mu)$.

THEOREM 4.5. *If μ_n is symmetric for all $n \geqslant 1$, then μ is centered at 0, and actually we have $0 \in \Lambda(\mu)$.*

Proof. Let A be a μ-measurable affine set, then $\mu(A) = \mu(-A)$, so if $\mu(A) = 1$, then $A \cap (-A) \neq \emptyset$ and so $0 \in A$.

LEMMA 4.6. *Let μ_n^* be the symmetrization of μ_n; then the following five statements are equivalent:*

(4.6.1) $\exists a > 0$, *so that* $\sup_n \sup_{x \in \mathbf{R}} \mu_n([x-a, x+a]) < 1$;

(4.6.2) $\exists a > 0$, $\exists m_n$ *a median for* μ_n, *so that* $\sup_n \mu_n([m_n - a, m_n + a]) < 1$;

(4.6.3) $\exists a > 0$, *so that* $\sup_n \mu_n^*([-a, a]) < 1$;

(4.6.4) δ_0 *is not a weak limit point of* $\{\mu_n^*\}$;

(4.6.5) $$\inf_n \int_{-\infty}^{\infty} \frac{|x|}{1+|x|} \mu_n^*(dx) > 0.$$

Remark. If $\{\mu_n\}$ satisfies one of the five statements above, we shall say that μ is *totally non-degenerated*. Note that μ is totally non-degenerated if and only if μ^* is totally non-degenerated. Condition (4.6.3) appears in [8] as condition (4.2).

Proof. From Theorem A, p. 245 in [7], we have

$$\mu_n(x|\ |x-m_n|>a) \leqslant 2\mu_n^*(x|\ |x|>a) \leqslant 4\mu_n(x|\ |x-b|>\tfrac{1}{2}a)$$

if m_n is a median of μ_n. Hence it follows that (4.6.1), (4.6.2) and (4.6.3) are equivalent. It is obvious that (4.6.3) implies (4.6.4), and (4.6.4) implies (4.6.5).

Now suppose that (4.6.5) holds, then we have

$$b = \inf_n \int_{-\infty}^{\infty} \frac{|x|}{1+|x|}\mu_n^*(dx) > 0$$

and if $a = b(2-b)^{-1}$, then $a>0$ and

$$b \leqslant \mu_n^*(x|\ |x|>a) + \frac{a}{1+a} = \tfrac{1}{2}b + \mu_n^*(x|\ |x|>a)$$

and so $\mu_n^*([-a,a]) \leqslant 1-\frac{1}{2}b$. Hence (4.6.5) implies (4.6.3) and the lemma is proved.

Lemma 4.7. *Suppose that μ_n satisfies*

$$(4.7.1)\qquad \int_{-\infty}^{\infty} x\,\mu_n(dx) = 0 \quad \forall n,$$

$$(4.7.2)\qquad \inf_n \int_{-\infty}^{\infty} |x|\,\mu_n(dx) > 0,$$

$$(4.7.3)\qquad \sup_n \int_{-\infty}^{\infty} |x|^p\,\mu_n(dx) < \infty,$$

for some $p>1$, then μ is totally non-degenerated.

Proof. Let $1<q<\infty$ be taken so that $1/p+1/q=1$, then a slight modification of the proof of Inequality II, p. 6, in [6], shows that

(i) $\mathbf{P}(X \geqslant \lambda \mathbf{E}X) \geqslant (1-\lambda)^q(\mathbf{E}X)^q(\mathbf{E}X^p)^{-q/p}$ whenever X is a non-negative random variable with finite pth moment and $0\leqslant\lambda\leqslant 1$.

Let X_n be a random variable with distribution law μ_n, and let $X_n^* = X_n'-X_n''$ be a symmetrization of X_n. Then by Theorem 2.6 in [3], we have

$$\mathbf{E}|X_n| \leqslant \mathbf{E}|X_n^*|,$$
$$\mathbf{E}|X_n^*|^p \leqslant 2^p\mathbf{E}|X_n|.$$

Now let $a = \inf_n \mathbf{E}|X_n|$ and $A = \sup_n \mathbf{E}|X_n|^p$, then $a>0$ and $A<\infty$ by (4.7.2) and (4.7.3), hence by (i) we have

$$\mathbf{P}(|X_n^*|>\tfrac{1}{2}a) \geqslant \mathbf{P}(|X_n^*| \geqslant \tfrac{1}{2}\mathbf{E}|X_n^*|)$$
$$\geqslant 2^{-q}(\mathbf{E}|X_n^*|)^q(\mathbf{E}|X_n^*|^p)^{-q/p} \geqslant 2^{-2q}a^qA^{-q/p}$$

and so (4.6.3) is satisfied.

Theorem 4.8. *If μ is totally non-degenerated, then $C(\mu)\subseteq l^2$ (here l^2 is the space of square-summable sequences).*

Proof. We have $C(\mu)\subseteq C(\mu^*)$, where μ^* is a symmetrization of μ. Let $\varepsilon>0$ be chosen so that $\mu_n^*([-\varepsilon,\varepsilon]) \leqslant 1-\varepsilon$. Now we can find Borel functions f_n from $I=[0,1]$ into $\boldsymbol{R}$ so that

$$\lambda(f_n^{-1}(A)) = \mu_n^*(A) \qquad \forall n\geqslant 1\ \forall A\in \boldsymbol{B}(\boldsymbol{R}),$$

where λ is the Lebesgue measure on I. Let $\Omega = I^\infty$ and $\mathbf{P} = \prod_1^\infty \lambda$, and let

$$X_n(\omega) = f_n(\omega_n) \quad \text{for} \quad \omega = (\omega_n)\in\Omega.$$

Then $X_1, X_2, \ldots$ are independent random variables and X_n has distribution law μ_n^*. Since we have that

$$\lambda(|f_n|>\varepsilon) \geqslant \varepsilon$$

we can find $a_n\geqslant\varepsilon$ so that

$$\lambda(|f_n|>a_n) \leqslant \varepsilon \leqslant \lambda(|f_n|\geqslant a_n)$$

and so there exist a Borel set A_n so that $\lambda(A_n)=\varepsilon$ and

$$\{|f_n|>a_n\} \subseteq A_n \subseteq \{|f_n|\geqslant a_n\}.$$

Now let

$$Y_n(\omega) = \begin{cases} 1 & \text{if } \omega_n\in A_n, \\ 0 & \text{if } \omega_n\notin A_n; \end{cases}$$

then $Y_1, Y_2, \ldots$ are independent identically distributed random variables, so that

$$\mathbf{P}(Y_n=1)=\varepsilon, \qquad \mathbf{P}(Y_n=0)=1-\varepsilon.$$

Moreover, we have $0\leqslant Y_n\leqslant a_n^{-1}|X_n|\leqslant\varepsilon^{-1}|X_n|$.

Now let $b=(b_j)\in C(\mu^*)$; then by Proposition 2.8 in [3] we have that

$$\sum_{j=1}^{\infty} b_j^2 X_j(\omega)^2 < \infty \text{ a.s.}$$

and so

$$\sum_{j=1}^{\infty} b_j^2 Y_j^2 < \infty \text{ a.s.}$$

Since $b_j X_j \to 0$ a.s. and μ is totally non-degenerated, we must have that $\{b_j\}$ is bounded. Hence

$$\sup_j |b_j^2 Y_j|^2 \leqslant \sup_j |b_j|^2 < \infty.$$

So from Corollary 3.3 in [3] we find that

$$\mathbf{E}\Big(\sum_{j=1}^{\infty} b_j^2 Y_j^2\Big) = \varepsilon \sum_{j=1}^{\infty} b_j^2 < \infty$$

and so $b \in l^2$. Hence $C(\mu) \subseteq C(\mu^*) \subseteq l^2$.

THEOREM 4.9. *Suppose that* μ *satisfies*

(4.9.1) $$\int_{-\infty}^{\infty} x\,\mu_n(dx) = 0 \quad \forall n \geqslant 1,$$

(4.9.2) $$\sup_n \int_{-\infty}^{\infty} x^2 \mu_n(dx) < \infty;$$

then $l^2 \subseteq C(\mu)$ *and* $A_L(\mu) \subseteq l^2$.

Proof. Let $a = (a_j) \in l^2$, then

$$\mathbf{E}\Big|\sum_{j=n+1}^{n+m} a_j X_j\Big|^2 \leqslant \sup_j \mathbf{E}|X_j|^2 \sum_{j=n+1}^{n+m} |a_j|^2$$

if $X_1, X_2, \ldots$ are independent and X_j has distribution law μ_j. Hence $\sum_{j=1}^{\infty} a_j X_j$ converges in L^2 and so $a \in C(\mu)$. That is $l^2 \subseteq C(\mu)$.

It is easily seen that $A_L(\mu) \subseteq C^*(\mu)$, and since $l^2 \subseteq C(\mu)$ we have $C^*(\mu) \subseteq l^2$, and so the theorem is proved.

THEOREM 4.10. *Suppose that* μ *satisfies*

(4.10.1) $$\int_{-\infty}^{\infty} x\,\mu_n(dx) = 0 \quad \forall n,$$

(4.10.2) $$\sup_n \int_{-\infty}^{\infty} x^2 \mu_n(dx) < \infty,$$

(4.10.3) μ *is totally non-degenerated*,

then μ *is centered at* 0 *and we have*

(4.10.4) $$A(\mu) \subseteq L^2(\mu),$$

(4.10.5) $$C(\mu) = C(\mu^*) = C^*(\mu) = A_L(\mu) = l^2.$$

Remark. Note that under (4.10.1) and (4.10.2) we have that

(4.10.3)* $$\inf \int_{-\infty}^{\infty} |x|\,\mu_n(dx) > 0$$

implies (4.10.3) by Lemma 4.7.

Proof. By Theorem 4.8 and Theorem 4.10, we have $C(\mu) = C(\mu^*) = l^2$ and so μ is centered at 0. Moreover, it is clear that

$$\sum_{j=1}^{\infty} a_j x_j \in L^2(\mu) \qquad \forall a = (a_j) \in l^2.$$

Hence by Theorem 4.3 we have that $A(\mu) \subseteq L^2(\mu)$, and from Theorem 4.4 it follows that $A_L(\mu) = C^*(\mu) = l^2$.

THEOREM 4.11. *Let* g *be a non-negative function on* $\boldsymbol{R}_+$ *and* $p \geqslant 1$. *If we have*

(4.11.1) $$\int_{-\infty}^{\infty} x\,\mu_n(dx) = 0 \quad \forall n,$$

(4.11.2) $$R_n(ts) \leqslant g(t) R_n(s) \quad \forall t, s \geqslant a \ \forall n \geqslant 1,$$

(4.11.3) μ *is totally non-degenerated*,

(4.11.4) $$\int_{-\infty}^{\infty} t^{p-1} g(t)\,dt < \infty,$$

where $R_n(t) = \mu_n(x \in \boldsymbol{R} \mid |x| > t)$ *and* a *is a positive number, then* μ *is centered at* 0, *and we have*

(4.11.5) $$A(\mu) \subseteq L^p(\mu).$$

Remark. From Lemma 4.7 it follows easily (4.11.1), (4.11.2), (4.11.4) and

(4.11.3)* $$\inf \int_{-\infty}^{\infty} |x|\,\mu_j(dx) > 0$$

implies (4.11.3), whenever $p > 1$.

Proof. Let $X_1, X_2, \ldots$ be independent random variables with distributions $\mu_1, \mu_2, \ldots$ If $a \in C(\mu)$, then $\sum_1^{\infty} a_j X_j$ converges a.s., and so $\sup_j |a_j X_j| < \infty$ a.s. Hence by Lemma 3.1 in [9] we have that

$$\sum_{j=1}^{\infty} R_j(a_j^{-1} t_0) < \infty$$

for some $t_0 > 0$, moreover, since $a \in l^2$ by Theorem 4.8, we have that $b = \inf_j \{t_0 a_j^{-1}\} > 0$, and we may assume that t_0 is taken so large that $t_0 a_j^{-1} \geqslant a$ for all $j \geqslant 1$. Then we have

$$\mathbf{P}(\sup_j |a_j X_j| > t) \leqslant \sum_{j=1}^{\infty} R_j(t a_j^{-1}) \leqslant g(t t_0^{-1}) \sum_{j=1}^{\infty} R_j(t_0 a_j^{-1})$$

for $t \geqslant a t_0$, and so $\sup_j |a_j X_j| \in L^p$ by (4.11.4). Hence from Corollary 3.3 in [3] we have that

(i) $\sum_{j=1}^{\infty} a_j x_j \in L^p(\mu)$ $\forall a = (a_j) \in C(\mu)$.

Now we may define

$$|a|_p = \sup_n \Big\{ \int \Big| \sum_{j=1}^{n} a_j x_j \Big|^p \mu(dx) \Big\}^{1/p} \qquad \forall a \in C(\mu),$$

$$Ta = \sum_{1}^{\infty} a_j x_j \qquad \forall a \in C(\mu).$$

Then by Theorem 2.6 in [3] we have

(ii) $\big\| Ta \big\|_p = \Big\{ \int \Big| \sum_{1}^{\infty} a_j x_j \Big|^p \mu(dx) \Big\}^{1/p} = |a|_p$ $\forall a \in C(\mu)$.

Moreover, a slight modification of the proof of Theorem 5.2 in [2] shows that

(iii) $\big(C(\mu), |\cdot|_p\big)$ is a Banach space.

Since the identity map, $I: \big(C(\mu), |\cdot|_p\big) \to \big(C(\mu), |\cdot|_\mu\big)$, obviously is continuous, we have that $|\cdot|_p$ and $|\cdot|_\mu$ give the same topology on $C(\mu)$ by the closed graph theorem. Hence by (ii) we find that T is an isomorphism of $\big(C(\mu), |\cdot|_\mu\big)$ onto a closed subspace of $L^p(\mu)$. Then $T\big(C(\mu)\big)$ is certainly also closed in $L^0(\mu)$ and by (4.3) we have

(iv) $\Lambda_L(\mu) \subseteq T\big(C(\mu)\big)$.

Now let A be a μ-Lusin affine space. If $0 \notin A$, then there exists a linear functional, f, so that $f(x) = 1$ for all $x \in A$. Then $f \in \Lambda_L(\mu)$ and so we can find $a \in C(\mu)$, such that

$$\sum_{j=1}^{\infty} a_j x_j = 1 \text{ a.e.}$$

But the sum converges in $L^p(\mu)$ and $p \geqslant 1$, and so

$$1 = \int d\mu = \sum_{j=1}^{\infty} a_j \int_{-\infty}^{\infty} x \mu_j(dx) = 0.$$

Hence the assumption of $0 \notin A$ leads to a contradiction, and so we must have that μ is centered at 0. Finally, (4.11.5) follows from (iv) and Theorem 4.3.

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Received November 24, 1975 (1095)