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H^2 spaces of generalized half-planes*

by

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Abstract. New proofs are given of the following assertions about the Hardy space H^2 on Siegel domains of type II: H^2 is a Hilbert space and has a reproducing kernel. Elements of H^2 have "L^2-boundary values", and admit a Paley–Wiener type representation formula.

1. Introduction. The basic elementary facts in the theory of H^2 spaces are: (a) that these spaces are Hilbert spaces, (b) that H^2-functions have "L^2-boundary values", (c) that a Paley–Wiener type representation formula holds, and (d) that H^2 spaces have reproducing kernels. For tube domains over regular cones these results were proved by S. Bochner [2]. For Siegel domains of type II they were obtained by S. G. Gindikin [4]. Gindikin's arguments, however, were not conclusive, and the first complete derivation of his results was given — using methods different from his — by A. Korányi and E. M. Stein [7]. The purpose of this paper is to present, for Siegel domains of type II, a new and, maybe, simpler approach to the proof of the four facts listed above.

2. Definitions, notation, and statement of results. Let W and V be finite dimensional complex vector spaces of positive dimensions with $\dim V = n$. Let U be a real form of W chosen once and for all. Elements of U and W will be, usually, denoted by l.c. *latin characters*, elements of V always by ζ and ω. The conjugate of $z \in W$ relative to U will be written as $\bar{z}$. The value of an element λ of the dual space U' of U at the vector a of U or W will be denoted by $\langle \lambda, a\rangle$. We select once and for all Haar measures dx and $d\zeta$ on the vector groups U and V. The Fourier transform on $L^1(U)$ is defined by $\hat{f}(\lambda) = \int_U \exp(-2\pi i\langle \lambda, x\rangle) f(x)\,dx$. The Haar measure $d\lambda$ on U' is normalized so that the Fourier inversion formula reads $f(x) = \int_{U'} \exp(2\pi i\langle\lambda, x\rangle)\hat{f}(\lambda)\,d\lambda$. If Γ is an inner product space, the inner product will be denoted, usually, by $\langle\cdot|\cdot\rangle_\Gamma$.

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A *regular cone* Ω in U is an open convex cone with vertex at the origin which contains no affine line. Let $\overline{\Omega}$ denote the closure of Ω. If Ω is regular, then so is its dual cone $\Omega' = \{\lambda \in U': \langle\lambda, y\rangle > 0,\ \forall y \in \overline{\Omega}-\{0\}\}$. A Hermitean bilinear map $\Phi: V\times V\to W$ is said to be *Ω-positive* if for all $\zeta \in V$, $\Phi(\zeta,\zeta)\in\overline{\Omega}$, and if $\Phi(\zeta,\zeta) = 0$ implies that $\zeta = 0$. For $\lambda \in \Omega'$ define a positive definite Hermitean form on $V\times V$ by $H_\lambda(\zeta,\omega) = 4\langle\lambda, \Phi(\zeta,\omega)\rangle$, and set $\varrho(\lambda) = \det H_\lambda$.

The *tube domain* over Ω in W is $T_\Omega = \{z\in W: \operatorname{Im} z \in \Omega\}$. The *Siegel domain of type II* determined in $W\times V$ by Ω and Φ is the set $D = \{(z,\zeta)\in W\times V: \operatorname{Im} z - \Phi(\zeta,\zeta)\in\Omega\}$ The *distinguished bonudary* of D is the subset $B = \{(z,\zeta): \operatorname{Im} z-\Phi(\zeta,\zeta) = 0\}$ of the topological boundary of D. The map $(x,\zeta)\mapsto\big(x+i\Phi(\zeta,\zeta),\zeta\big)$ is a homeomorphism of $U\times V$ onto B The topological and measure theoretic structures of B are those of $U\times V$, transferred to B by the above map. Now L^p spaces for $1\leqslant p<\infty$ can be defined on B. The $L^p(B)$ norm of a measurable function on B is explicitly

$$\|f\|_{L^p(B)} = \Big(\int\limits_{U\times V} \big|f\big(x+i\Phi(\zeta,\zeta),\zeta\big)\big|^p dx\,d\zeta\Big)^{\frac{1}{p}}.$$

If $F: D\to \boldsymbol{C}$ and $t\in\Omega$, then the function $F_t: D\cup B\to\boldsymbol{C}$ is defined by $F_t(z,\zeta) = F(z+it,\zeta)$. Finally, for $1\leqslant p<\infty$ the space $H^p = H^p(D)$ is defined as the set of all holomorphic functions $F: D\to\boldsymbol{C}$ such that

$$\|F\|_{H^p} = \sup_{t\in\Omega}\|F_t|_B\|_{L^p(B)} < \infty.$$

The function $F\mapsto\|F\|_{H^p}$ is a norm on H^p. By abuse of notation we shall write $\|F_t|_B\|_{L^p(B)}$ as $\|F_t\|_{L^p(B)}$. General references about the facts reviewed here are [5], [6], and [8].

We now introduce a function space which will play an important part in our proofs. Consider the set of functions $\hat{F}: \Omega'\times V\to\boldsymbol{C}$ subject to the following two conditions:

(A) For every $\zeta\in V$ $\hat{F}(\cdot,\zeta)$ is measurable on Ω'.

(B) For every $\lambda\in\Omega'$ $\hat{F}(\lambda,\cdot)$ is a holomorphic entire function on V.

This set clearly forms a linear space. By a result of H. D. Ursell ([12], Theorem 8), the following statement is true: (we record it for future reference as)

Remark 1. A function $f: \Omega'\times V\to\boldsymbol{C}$ satisfying conditions (A) and (B) is measurable on $\Omega'\times V$.

In view of the remark it is meaningful to impose the following, third condition on our functions:

(C) $\|\hat{F}\|_{\hat{H}^2} = \int\limits_{\Omega'\times V} e^{-\pi H_\lambda(\zeta,\zeta)}|\hat{F}(\lambda,\zeta)|^2 d\lambda\,d\zeta < \infty.$

We define the space $\hat{H}^2_0$ to be the set $\{\hat{F}: \Omega'\times V\to\boldsymbol{C}: \hat{F}$ satisfies (A), (B), (C)$\}$. If $\hat{F}$ and $\hat{G}$ belong to $\hat{H}^2_0$, we say that $\hat{F}$ and $\hat{G}$ are *equivalent* ($\hat{F}\sim\hat{G}$) if $\hat{F}(\lambda,\zeta) = \hat{G}(\lambda,\zeta)$ for almost all $(\lambda,\zeta)\in\Omega'\times V$. In view of condition (C) we have

Remark 2. If $\hat{F},\hat{G}\in\hat{H}^2_0$, then

$$\hat{F}\sim\hat{G} \Leftrightarrow \int\limits_{\Omega'}\exp\big(-\pi H_\lambda(\zeta,\zeta)\big)|\hat{F}(\lambda,\zeta)-\hat{G}(\lambda,\zeta)|^2 d\lambda = 0$$

for almost every $\zeta\in V$

$$\Leftrightarrow \int\limits_{V}\exp\big(-\pi H_\lambda(\zeta,\zeta)\big)|\hat{F}(\lambda,\zeta)-\hat{G}(\lambda,\zeta)|^2 d\zeta = 0$$

for almost every $\lambda\in\Omega'$.

Now $\hat{H}^2$ is defined as the set of equivalence classes (relative to $\sim$) of elements of $\hat{H}^2_0$. Clearly, $\hat{H}^2$ is an inner product space with norm defined by (C). We can now state our results.

LEMMA 1. *The space $\hat{H}^2$ is a Hilbert space.*

THEOREM. (i) *Let $\hat{F}\in\hat{H}^2_0$, and let $(z,\zeta)\in D$. Define $\boldsymbol{U}\hat{F}(z,\zeta)$* by

(1) $$\boldsymbol{U}\hat{F}(z,\zeta) = \int\limits_{\Omega'} e^{2\pi i\langle\lambda,z\rangle}\hat{F}(\lambda,\zeta)\,d\lambda.$$

The integral in (1) *is absolutely convergent, $\boldsymbol{U}\hat{F}$ belongs to H^2, and if $\hat{G}\in\hat{H}^2_0$ is equivalent to $\hat{F}$, then $\boldsymbol{U}\hat{F} = \boldsymbol{U}\hat{G}$.*

(ii) *The space H^2 is a Hilbert space, and the map $\boldsymbol{U}: \hat{H}^2\to H^2$ defined in* (i) *maps $\hat{H}^2$ unitarily onto H^2.*

(iii) *If $F\in H^2$, then for $t\in\Omega$ tending to* 0, *$F_t|_B$ converges in the norm of $L^2(B)$ to an element $\tilde{F}$ of $L^2(B)$, and $\|F\|_{H_2} = \|\tilde{F}\|_{L^2(B)}$.*

(iv) *If (w,ω), $(z,\zeta)\in D$, then the function $(z,\zeta)\mapsto S_{(w,\omega)}(z,\zeta)$ defined by*

(2) $$S_{(w,\omega)}(z,\zeta) = \int\limits_{\Omega'} e^{2\pi i\langle\lambda, z-\overline{w}-2i\Phi(\zeta,\omega)\rangle}\varrho(\lambda)\,d\lambda$$

belongs to H^2, and for every $F\in H^2$

(3) $$F(w,\omega) = \langle F|S_{(w,\omega)}\rangle_{H^2}.$$

Equation (3) states that $S_{(w,\omega)}$ is (a, and hence by general principles) the reproducing kernel of H^2, the so called *Szegö kernel* of D.

3. Proof of Lemma 1. Fix $\lambda\in\Omega'$. Define $\|f\|_\lambda$ for measurable functions on V by

$$\|f\|^2_\lambda = \int\limits_V e^{-\pi H_\lambda(\zeta,\zeta)}|f(\zeta)|^2 d\zeta,$$

and define $\mathscr{H}^\lambda$ to be the set of entire holomorphic functions on V for which $\|f_\lambda\|$ is finite. The space $\mathscr{H}^\lambda$ is an inner product space which obviously contains all the constants, and it is easily checked that it contains all polynomials. The proof of Lemma 1 consists in showing that $\mathscr{H}^\lambda$ is complete, and that $\hat{H}^2$ can be identified with the direct integral $\int_{\Omega'}^{\oplus} \mathscr{H}^\lambda d\lambda$. The basic facts about $\mathscr{H}^\lambda$, viz. the existence of a reproducing kernel and completeness are due to V. Bargmann [1]. For the sake of completeness we include simplified proofs of these facts.

For $\zeta \in V$, and $f \in \mathscr{H}^\lambda$ define $(A_\zeta f)(\omega)$ to be $\exp\big(\pi H_\lambda(\omega, \zeta) - \frac{1}{2}\pi H_\lambda(\zeta, \zeta)\big) \times f(\omega - \zeta)$. Clearly, $\omega \mapsto (A_\zeta f)(\omega)$ is an entire function on V. A simple calculation shows that for $f, g \in \mathscr{H}^\lambda$ one has (writing the inner product in $\mathscr{H}^\lambda$ as $\langle \cdot | \cdot \rangle_\lambda$)

$$(4) \qquad \langle A_\zeta f | A_\zeta g \rangle_\lambda = \langle f | g \rangle_\lambda,$$

and that in particular for $f \in \mathscr{H}^\lambda$ $\|A_\zeta f\|_\lambda = \|f\|_\lambda$. Another easy calculation checks that $A_{-\zeta}$ is the inverse of A_ζ. Therefore A_ζ is a unitary transformation of $\mathscr{H}^\lambda$ onto itself. Let now $\theta \in \boldsymbol{R}$, for $f \in \mathscr{H}^\lambda$ define f_θ by $f_\theta(\zeta) = f(e^{i\theta}\zeta)$. Clearly, $f_\theta \in \mathscr{H}^\lambda$. The change of variable $\zeta \mapsto e^{i\theta}\zeta$ and the fact that $H_\lambda(e^{-i\theta}\zeta, e^{-i\theta}\zeta) = H_\lambda(\zeta, \zeta)$ show that $\langle f_\theta | 1 \rangle_\lambda = \langle f | 1 \rangle_\lambda$. Therefore, using first Fubini's and then Cauchy's theorem we have

$$(5) \qquad \langle f | 1 \rangle_\lambda = \frac{1}{2\pi}\int_0^{2\pi} \langle f_\theta | 1 \rangle_\lambda d\theta = \Big\langle \frac{1}{2\pi}\int_0^{2\pi} f_\theta d\theta \,\Big|\, 1 \Big\rangle_\lambda = f(0)\langle 1 | 1 \rangle_\lambda.$$

By evaluating $\langle 1 | 1 \rangle_\lambda$ in a coordinate system in which H_λ is diagonal, we find that $\langle 1 | 1 \rangle_\lambda = \varrho(\lambda)^{-1}$. Using this in (5) we have

$$(6) \qquad f(0) = \varrho(\lambda)\langle f | 1 \rangle_\lambda.$$

Since $f(\zeta) = \exp\big(\frac{1}{2}\pi H_\lambda(\zeta, \zeta)\big)\,(A_{-\zeta} f)(0)$, we obtain from (6) that

$$f(\zeta) = \varrho(\lambda)\exp\big(\tfrac{1}{2}\pi H_\lambda(\zeta, \zeta)\big)\langle A_{-\zeta} f | 1 \rangle_\lambda.$$

Applying (4) to the right-hand side of the last equality we have

$$(7) \qquad f(\zeta) = \varrho(\lambda) e^{\frac{1}{2}\pi H_\lambda(\zeta, \zeta)} \langle f | A_\zeta 1 \rangle_\lambda.$$

Since $(A_\zeta 1)(\omega) = \exp\big(\pi H_\lambda(\omega, \zeta) - \frac{1}{2}\pi H(\zeta, \zeta)\big)$, we can rewrite (7) by setting $\varrho(\lambda)\exp\big(\pi H_\lambda(\omega, \zeta)\big) = K_\zeta^\lambda(\omega)$ as

$$(8) \qquad f(\zeta) = \langle f | K_\zeta^\lambda \rangle_\lambda.$$

(Note that since K_ζ^λ is a numerical multiple of $A_\zeta 1$, it is an element of $\mathscr{H}^\lambda$.)

We have proved that $\mathscr{H}^\lambda$ has a reproducing kernel given by K_ζ^λ. An easy calculation shows that $\|K_\zeta^\lambda\|_\lambda = \varrho(\lambda)^{\frac{1}{2}} \exp\big(\frac{1}{2}\pi H_\lambda(\zeta, \zeta)\big)$. Using this value of $\|K_\zeta^\lambda\|_\lambda$ and applying Schwarz's inequality to (8) we get

$$(9) \qquad |f(\zeta)| \leqslant \varrho(\lambda)^{\frac{1}{2}} e^{\frac{1}{2}\pi H_\lambda(\zeta, \zeta)} \|f\|_\lambda.$$

If $K \subset V$ is compact and $C_K = \sup\{\exp\big(\frac{1}{2}\pi H_\lambda(\zeta, \zeta)\big) : \zeta \in K\}$, then for $\zeta \in K$ (9) yields $|f(\zeta)| \leqslant \varrho(\lambda)^{\frac{1}{2}} C_K \|f\|_\lambda$. This inequality immediately implies the completeness of $\mathscr{H}^\lambda$. We shall now derive another consequence of (9) which will be needed in the proof of the theorem. Let $\hat{F} \in \hat{H}_0^2$, then, by condition (C), $\hat{F}(\lambda, \cdot)$ belongs to $\mathscr{H}^\lambda$ for almost every $\lambda \in \Omega'$. In view of (9) we therefore have

Remark 3. If $\hat{F} \in \hat{H}_0^2$, then, for every $\zeta \in V$, $\lambda \mapsto \varrho(\lambda)^{-\frac{1}{2}} e^{-\frac{1}{2}\pi H_\lambda(\zeta, \zeta)} \times \hat{F}(\lambda, \zeta)$ belongs to $L^2(\Omega')$.

Let us also observe the following fact: if ζ_j, $j = 1, 2, 3, \ldots$ is a dense sequence in V and $f \in \mathscr{H}^\lambda$ is such that $\langle f | K_{\zeta_j}^\lambda \rangle = 0$ for $j = 1, 2, 3, \ldots$, then by (8) $f = 0$. Consequently, we have the following

Remark 4. If ζ_j, $j = 1, 2, 3, \ldots$ is a dense sequence in V, then $K_{\zeta_j}^\lambda$, $j = 1, 2, 3, \ldots$ is a total sequence in $\mathscr{H}^\lambda$.

We now prove that $\hat{H}^2$ is complete. Let $\mathfrak{F} = \prod_{\lambda \in \Omega'} \mathscr{H}^\lambda$, and let $\mathfrak{G} = \{f : \Omega' \times V \to \boldsymbol{C} : f$ satisfies (A), and for every $\lambda \in \Omega'$ $f(\lambda, \cdot) \in \mathscr{H}^\lambda\}$. Note first that $\mathfrak{G}$ can be identified in an obvious way with a linear subspace of $\mathfrak{F}$. Also note that for fixed $\zeta \in V$ $(\lambda, \omega) \mapsto K_\zeta^\lambda(\omega)$ belongs to $\mathfrak{G}$. We shall now verify that the Hilbert spaces $\mathscr{H}^\lambda$ form a measurable field of Hilbert spaces ([3], p. 142). To this end, we must check three conditions.

(1) *If $f \in \mathfrak{G}$, then $\lambda \mapsto \|f(\lambda, \cdot)\|_\lambda$ is a measurable function on Ω'.*

To prove this, note that, by Remark 1, f is a measurable function on $\Omega' \times V$. Then approximate the integral giving $\|f(\lambda, \cdot)\|_\lambda^2$ to within $\varepsilon/2$ by an integral over a large cube in V. Now approximate the integral over the cube to within $\varepsilon/2$ by a Riemann sum. This Riemann sum is a measurable function of λ. Therefore $\|f(\lambda, \cdot)\|_\lambda$ is the pointwise limit of measurable functions, and hence measurable.

(2) *If $g \in \mathfrak{F}$ is such that $\lambda \mapsto \langle g | f \rangle_\lambda$ is measurable for every $f \in \mathfrak{G}$, then $g \in \mathfrak{G}$.*

Proof. $g(\lambda)(\zeta) = \langle g | K_\zeta^\lambda \rangle_\lambda$ is measurable for every $\zeta \in V$ because $K_\zeta^\lambda \in \mathfrak{G}$, then use Remark 1.

(3) *There is a sequence f_j of elements of $\mathfrak{G}$ such that for every $\lambda \in \Omega'$ the sequence $f_j(\lambda, \cdot)$ is total in $\mathscr{H}^\lambda$.*

Proof. Remark 4.

The elements of $\mathfrak{G}$ are called *measurable vector fields*. A measurable vector field f is said to be *square integrable* if $\int_{\Omega'} \|f(\lambda, \cdot)\|^2 d\lambda$ is finite. Two square integrable measurable vector fields f and g are *equivalent* if $\int_{\Omega'} \|f(\lambda, \cdot) - g(\lambda, \cdot)\|^2 d\lambda = 0$. The direct integral $\int_{\Omega'}^{\oplus} \mathscr{H}^\lambda d\lambda$ is defined as the set of equivalence classes of measurable, square integrable vector fields. The norm of $f \in \int_{\Omega'}^{\oplus} \mathscr{H}^\lambda d\lambda$ is $\big(\int_{\Omega'} \|f(\lambda, \cdot)\|^2 d\lambda\big)^{\frac{1}{2}}$.

If $f \in \mathfrak{G}$ is square integrable, then, clearly, f belongs to $\hat{H}^2_0$. If g is another square integrable element of $\mathfrak{G}$, and g is equivalent to f, then (Remark 2) f and g are also equivalent in $\hat{H}^2_0$. The norm of a square integrable $f \in \mathfrak{G}$ equals its $\hat{H}^2$-norm. So far we have shown that $\int_{\Omega'}^{\oplus} \mathscr{H}^\lambda d\lambda$ can be identified with a subspace of $\hat{H}^2$. To prove that this subspace is actually all of $\hat{H}^2$ let $\hat{F} \in H^2$, and select a representative $\hat{F}_1$ of $\hat{F}$ in $\hat{H}^2_0$. The set of λ's in Ω' for which $\hat{F}_1(\lambda, \cdot)$ does not belong to $\mathscr{H}^\lambda$ is of measure zero. Now define $\hat{F}_2$ as follows: $\hat{F}_2(\lambda, \zeta) = \hat{F}_1(\lambda, \zeta)$ if $\hat{F}_1(\lambda, \cdot) \in \mathscr{H}^\lambda$, and $\hat{F}_2(\lambda, \zeta) = 0$ otherwise. By Remark 1, $\hat{F}_2$ belongs to $\mathfrak{G}$, and hence to $\hat{F}^2_0$, and by Remark 2 it is equivalent to $\hat{F}_1$. This proves that $\hat{H}^2$ can be identified with the direct integral of the $\mathscr{H}^\lambda$'s. Since the direct integral of Hilbert spaces is a Hilbert space, Lemma 1 is proved.

4. Proof of the Theorem. In addition to Lemma 1 and Remark 3 two technical results will be needed which we now list.

LEMMA 2. *Let $F \in H^p$, $1 \leqslant p < \infty$. Let $\zeta \in V$, and $\delta \in \Omega$ such that $\delta - \Phi(\zeta, \zeta) \in \Omega$. Then $z \mapsto F_\delta(z, \zeta) = F(z+i\delta, \zeta)$ belongs to $H^p(T_\Omega)$.*

LEMMA 3. *Let $\varepsilon > 0$, $0 < a < \frac{1}{2}$, and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be a basis of U' contained in Ω' which is compatible with the Haar measure $d\lambda$ on U'. Then $G^\varepsilon(z, \zeta) = \exp\{-\varepsilon \sum_{j=1}^{m} <\lambda_j, z>^a\}$ belongs to $H^2(D)$, and is bounded and continuous on $\bar{D}$.*

Lemma 2 is actually true for all positive p, but we only need it for $p = 1, 2$. It is due to E. M. Stein [10]. Lemma 3 is from [9].

Let now $\hat{F} \in H^2_0$ and $(z, \zeta) \in D$, with $z = x+it+i\Phi(\zeta, \zeta)$, where $t \in \Omega$. In any coordinate system $\varrho(\lambda)$ is a homogeneous polynomial of degree n, and one can show readily that $\varrho(\lambda)\exp(-2\pi\langle\lambda, t\rangle)$ is square integrable on Ω'. Therefore, by Remark 3,

$$(10) \qquad \int_{\Omega'} e^{2\pi i\langle\lambda, z\rangle} \hat{F}(\lambda, \zeta)\, d\lambda = \int_{\Omega'} e^{2\pi i\langle\lambda, x\rangle} e^{-2\pi\langle\lambda, t\rangle} e^{-\frac{1}{2}\pi H_\lambda(\zeta,\zeta)} \hat{F}(\lambda, \zeta)\, d\lambda$$

is absolutely convergent for every $\zeta \in V$. Choosing coordinates in $W \times V$, and applying Morera's theorem in combination with Fubini's theorem one shows that $U\hat{F}$ is holomorphic in each coordinate of (z, ζ), and hence by Hartogs's theorem holomorphic in D. Since $\varrho(\lambda)\exp(-2\pi\langle\lambda, t\rangle)$ is a bounded function of λ on Ω', by Remark 3 one concludes that the quantity multiplying $\exp(2\pi i\langle\lambda, x\rangle)$ in (10) is square integrable on Ω'. Therefore, by Plancherel's theorem for every $\zeta \in V$

$$\int_U \left|U\hat{F}\big(x+it+i\Phi(\zeta, \zeta), \zeta\big)\right|^2 dx = \int_{\Omega'} e^{-4\pi\langle\lambda, t\rangle} e^{-\pi H_\lambda(\zeta,\zeta)} |\hat{F}(\lambda, \zeta)|^2 d\lambda.$$

Integrating this equality on V we get

$$(11) \qquad \|(U\hat{F})_t\|^2_{L^2(B)} = \int_{\Omega'\times V} e^{-4\pi\langle\lambda, t\rangle} e^{-\pi H_\lambda(\zeta,\zeta)} |\hat{F}(\lambda, \zeta)|^2 d\lambda\, d\zeta \leqslant \|\hat{F}\|^2_{\hat{H}^2}.$$

From (11) we conclude that $U\hat{F} \in H^2$. If $t_k \in \Omega$ is a sequence tending to 0, then by the dominated convergence theorem we have that $\|(U\hat{F})_t\|^2_{L^2(B)}$ converges to $\|\hat{F}\|^2_{\hat{H}^2}$, and that therefore

$$(12) \qquad \|U\hat{F}\|^2_{H^2} = \|\hat{F}\|^2_{\hat{H}^2}.$$

If $\hat{G} \in H^2_0$ and $\hat{F} \sim \hat{G}$, then (12) implies that $\|U\hat{F} - U\hat{G}\|_{H^2} = \|\hat{F} - \hat{G}\|_{\hat{H}^2} = 0$, i.e. that equivalent $\hat{F}$'s give rise to the same $U\hat{F}$. Therefore U defines a linear map from $\hat{H}^2$ to H^2 which we continue to write U. The equation (12) shows that U maps $\hat{H}^2$ isometrically into H^2. Now let $t_k \in \Omega$ be a sequence converging to 0, then (11) (with $(U\hat{F})_{t_k} - (U\hat{F})_{t_l}$ instead of $(U\hat{F})_t$) and the dominated convergence theorem show that $(UF)_{t_k}|_B$ is a Cauchy sequence in $L^2(B)$. Therefore $(U\hat{F})_{t_k}|_B$ converges in $L^2(B)$ norm to an element of $L^2(B)$. We omit the proof that the sequential limit can be replaced by $t \in \Omega'$ tending to 0. We therefore have

Remark 5. Assertion (iii) of the theorem holds for every $F \in H^2$ which admits the representation (1).

Let now $F \in H^2 \cap H^1$ and let $\zeta \in V$ be arbitrary but fixed. (By Lemma 3, $H^1 \cap H^2 \neq \{0\}$.) Set $\Omega_\zeta = \{\delta \in \Omega\colon \delta - \Phi(\zeta, \zeta) \in \Omega\}$. For $\delta \in \Omega_\zeta$ $z \mapsto F_\delta(z, \zeta)$ belongs to $(H^2 \cap H^1)(T_\Omega)$ by Lemma 2. By the theory of H^p spaces on tube domains ([11], Chapter 3) the boundary function of F_δ, viz. $x \mapsto F_\delta(x, \zeta)$ belongs to $(L^2 \cap L^1)(U)$. We can therefore define a function $\hat{F}_\delta(\lambda, \zeta)$ by

$$(13) \qquad \hat{F}_\delta(\lambda, \zeta) = \int_U e^{-2\pi i\langle\lambda, x\rangle} F_\delta(x, \zeta)\, dx.$$

Remark 6. By the H^2 theory for tube domains $\hat{F}_\delta(\cdot, \zeta)$ is supported in Ω'. Since $F_\delta(\cdot, \zeta) \in L^1(U) \cap C^\infty(U)$, $\hat{F}_\delta(\cdot, \zeta)$ is continuous and integrable. Therefore Fourier inversion can be applied to (13) everywhere.

If δ' is another element of Ω_ζ, then for $\lambda \in \Omega'$, by the H^2 theory for tube domains,

$$\hat{F}_{\delta+\delta'}(\lambda, \zeta) = \exp(-2\pi\langle\lambda, \delta\rangle)\hat{F}_{\delta'}(\lambda, \zeta) = \exp(-2\pi\langle\lambda, \delta'\rangle)\hat{F}_\delta(\lambda, \zeta).$$

Therefore for $\delta \in \Omega_\zeta$ $\exp(2\pi\langle\lambda, \delta\rangle)\hat{F}_\delta(\lambda, \zeta)$ is independent of δ. Denote this function by $\hat{F}(\lambda, \zeta)$.

Now let $z = x+iy \in W$ be such that $(z, \zeta) \in D$, i.e. $y \in \Omega_\zeta$. Note that $F(z, \zeta) = F_y(x, \zeta)$. By Remark 6, we can apply Fourier inversion to (13). If we now express $\hat{F}_y$ in terms of F in the Fourier inversion formula, we get

$$(14)\qquad F(z,\zeta)=\int_{\Omega'}e^{2\pi i\langle\lambda,z\rangle}\hat{F}(\lambda,\zeta)\,d\lambda=U\hat{F}(z,\zeta).$$

Since ζ was arbitrary, (14) holds for every $(z,\zeta)\in D$.

We now prove that $\hat{F}\in\hat{H}^2$. Again fix $\zeta_0\in V$, and also $\lambda\in\Omega'$. If $\delta\in\Omega_{\zeta_0}$, then there is a polydisc, $\Delta\subset V$ centered at ζ_0 such that $\delta\in\Omega_\zeta$ for $\zeta\in\Delta$. Now by (13) and by the definition of $\hat{F}$ we have for $\zeta\in\Delta$ that

$$\hat{F}(\lambda,\zeta)=e^{2\pi\langle\lambda,\delta\rangle}\int_U e^{-2\pi i\langle\lambda,x\rangle}F_\delta(x,\zeta)\,dx.$$

Exactly as before, by combining the theorems of Fubini, Morera and Hartogs, we can show that $\zeta\mapsto\hat{F}(\lambda,\zeta)$ is holomorphic in Δ. Since ζ_0 was arbitrary in V, it follows that $\hat{F}(\lambda,\cdot)$ is an entire function. By Remark 6, we know that $\hat{F}(\cdot,\zeta)$ is continuous for every $\zeta\in V$, therefore by Remark 1, $\hat{F}$ is measurable on $\Omega'\times V$.

Now let $t\in\Omega$, then Plancherel's theorem applied to (13) gives for every $\zeta\in V$

$$\int_U\big|F\big(x+it+i\Phi(\zeta,\zeta),\zeta\big)\big|^2dx=\int_{\Omega'}e^{-4\pi\langle\lambda,t+\Phi(\zeta,\zeta)\rangle}|\hat{F}(\lambda,\zeta)|^2\,d\lambda.$$

Integrating this equality on V we have

$$\|F_t\|^2_{L^2(B)}=\int_{\Omega'\times V}e^{-4\pi\langle\lambda,t\rangle}e^{-\pi H_\lambda(\zeta,\zeta)}|\hat{F}(\lambda,\zeta)|^2\,d\lambda\,d\zeta\leqslant\|F\|^2_{H^2}.$$

By Fatou's lemma it follows that $\hat{F}\in\hat{H}^2$. Taking suprema over Ω we see that $\|F\|_{H^2}=\|\hat{F}\|_{\hat{H}^2}$. We conclude that the map $F\mapsto\hat{F}$ maps the subspace $H^2\cap H^1$ of H^2 isometrically into $\hat{H}^2$. By Lemma 1, the range of this map is contained in a complete space, and therefore, if we denote by M the closure in H^2 of $H^2\cap H^1$, it extends uniquely to an isometry V of M into $\hat{H}^2$. Now for $F\in H^2\cap H^1$ (14) holds, and therefore for such F, $UVF=F$, i.e., UV is the identity of $H^2\cap H^1$. By continuity it follows that UV is the identity on all of M, and hence if $F\in M$, then $F=U(VF)$, i.e., U maps $\hat{H}^2$ isometrically *onto* M: M is a Hilbert space, and the unitary maps $U|_{VM}$ and V are inverses of each other. By Remark 5 it follows that assertion (iii) holds for every $F\in M$.

We now prove that $M=H^2$. Let $F\in H^2$, and let G^ε be the function introduced in Lemma 3. By that lemma and Schwarz's inequality $G^\varepsilon F\in H^2\cap H^1$. Since assertion (iii) of the theorem holds in M, there exists an element $(G^\varepsilon F)^\sim$ of $L^2(B)$ such that $(G^\varepsilon F)_t|_B$ tends to $(G^\varepsilon F)^\sim$ in $L^2(B)$ as $t\in\Omega$ tends to zero. Consider first the case $\varepsilon=1$. For some sequence $t_k\in\Omega$, $t_k\to0$ (fixed once and for all in this proof) $(G^1F)_{t_k}|_B\to(G^1F)^\sim$ almost everywhere on B. Since $G^1_{t_k}|_B\to G^1|_B=\tilde{G}^1$ everywhere on B, and $\tilde{G}^1$ does not vanish anywhere, we can conclude that $F_{t_k}|_B$ converges almost everywhere on B to a limit $\tilde{F}$. Since $\|F_{t_k}\|_{L^2}\leqslant\|F\|_{H^2}$, it follows from Fatou's lemma that $\tilde{F}\in L^2(B)$. Now let ε be arbitrary positive. Since $(G^\varepsilon F)_{t_k}|_B=G^\varepsilon_{t_k}|_BF_{t_k}|_B\to\tilde{G}^\varepsilon\tilde{F}$ almost everywhere, and $(G^\varepsilon F)_{t_k}|_B\to(G^\varepsilon F)^\sim$ in $L^2(B)$ it follows that $(G^\varepsilon F)^\sim=\tilde{G}^\varepsilon\tilde{F}$ almost everywhere on B.

Now let $\varepsilon_\nu\to0$, then

$$\|G^{\varepsilon_\nu}F-G^{\varepsilon_\mu}F\|^2_{H^2}=\int_{U\times V}|\tilde{G}^{\varepsilon_\nu}-\tilde{G}^{\varepsilon_\mu}|^2\,|\tilde{F}|^2\,dx\,d\zeta$$

because (iii) holds in M. Since $\tilde{G}^{\varepsilon_\nu}-\tilde{G}^{\varepsilon_\mu}$ tends to zero boundedly, we have that $G^{\varepsilon_\nu}F$ is a Cauchy sequence in the complete space M, and therefore tends in H^2 to an element H of M. Now let $t\in\Omega$ be arbitrary but fixed, then

$$\|(G^{\varepsilon_\nu}F)_t-H_t\|_{L^2(B)}\leqslant\|G^{\varepsilon_\nu}F-H\|_{H^2}$$

and therefore $(G^{\varepsilon_\nu}F)_t|_B\to H_t|_B$ in $L^2(B)$ norm. On the other hand $G^{\varepsilon_\nu}_t|_B\to1$ everywhere on B. Consequently $F_t|_B=H_t|_B$ because both functions are continuous. Since $t\in\Omega$ was arbitrary, it follows that $F=H$, and hence, that $M=H^2$. Therefore $U\hat{H}^2=H^2$.

To prove (iv) let $(w,\omega)\in D$ and $F\in H^2$. By the assertions of the theorem already proved, we have

$$(15)\qquad F(w,\omega)=\int_{\Omega'}e^{2\pi i\langle\lambda,w\rangle}\hat{F}(\lambda,\omega)\,d\lambda$$

where $\hat{F}=U^{-1}F\in\hat{H}^2$. Since $\hat{F}(\lambda,\cdot)\in\mathscr{H}^\lambda$ for almost every $\lambda\in\Omega'$, we have $\hat{F}(\lambda,\cdot)=\langle\hat{F}(\lambda,\cdot)|K^\lambda_\omega\rangle_\lambda$ for almost every $\lambda\in\Omega'$. Introducing this into (15) and rewriting the integral formally as a double integral we have (only formally, so far)

$$(16)\qquad F(w,\omega)=\int_{\Omega'\times V}e^{-\pi H_\lambda(\zeta,\zeta)}\hat{F}(\lambda,\zeta)\,\overline{\{e^{-2\pi i\langle\lambda,\overline{w}\rangle}e^{\pi H_\lambda(\zeta,\omega)}\varrho(\lambda)\}}\,d\lambda\,d\zeta.$$

Denote the quantity in curly brackets by $T_{(w,\omega)}(\lambda,\zeta)$. A straightforward check verifies that $(\lambda,\zeta)\mapsto T_{(w,\omega)}(\lambda,\zeta)$ belongs to $\hat{H}^2$. Therefore the double integral in (16) is absolutely convergent (this justifies the passage from (15) to (16)) and equal to $\langle\hat{F}|T_{(w,\omega)}\rangle_{\hat{H}^2}$. Consequently,

$$(17)\qquad F(w,\omega)=\langle\hat{F}|T_{(w,\omega)}\rangle_{\hat{H}^2}.$$

Now a simple calculation shows that $UT_{(w,\omega)}$ is the function $S_{(w,\omega)}$ defined by (2), hence in view of the fact that U is unitary, (17) yields

$$F(w,\omega)=\langle F|S_{(w,\omega)}\rangle_{H^2}.$$

But this is equation (3) in assertion (iv) of the theorem whose proof is now complete.

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Integrability of seminorms, the 0-1 law and the affine kernel for product measures

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Abstract. Let (X_n) be a sequence of random variables taking values in a measurable linear space E, and let q be a quasi-convex subadditive function on E^∞. The first part of the paper deals with the problem of finding conditions, which assures that $\mathbf{E}(e^{\varepsilon M})$ is finite for some positive ε, where $M = \sup_n q(X_1, \ldots, X_n, 0, 0, \ldots)$. In the second and third part of the paper we take $E = \boldsymbol{R}$, and we show that if X_n has no mass points, then every linear subspace of $\boldsymbol{R}^\infty$ has probability 0 or 1. Finally, we study the affine kernel of (X_n), i.e. the intersection of all affine subspaces of probability 1, and we give an analytic expression for this.

1. Introduction. If μ is a Gaussian measure on a locally convex space, there are three main results which have proved to be useful.

The first is the result of Fernique stating that, if φ is a measurable a.e. finite seminorm, then $\mathbf{E}\big(\exp(\varepsilon\varphi^2)\big) < \infty$ where $\mathbf{E}$ denotes expectation with respect to μ. A similar result has been proved by C. Borel ([1]) for certain other classes of measures. In Section 2 of this paper we shall prove some results in this direction when μ is a product measure on $\big(\prod_1^\infty E_n, \bigotimes_1^n \boldsymbol{B}_n\big)$ and $(E_n, \boldsymbol{B}_n)$ is a measurable linear space. Here we define a measurable linear space, $(E, \boldsymbol{B})$, to be a linear space E equipped with a σ-algebra $\boldsymbol{B}$ satisfying

(1.1) $\quad (x, y) \to x + y$ is measurable: $(E \times E, \boldsymbol{B} \otimes \boldsymbol{B}) \to (E, \boldsymbol{B})$,

(1.2) $\quad (\lambda, y) \to \lambda y$ is measurable: $\big(E \times \boldsymbol{R}, \boldsymbol{B} \otimes \boldsymbol{B}(R)\big) \to (E, \boldsymbol{B})$.

The methods and the results of that section are closely related to the results of Marcus and Jain in [9] and to the results in [2] and [3].

The second result is the 0-1 law by Kallianpur in [5]. C. Borel has in [1] shown that the same result holds for certain other classes of measure. In Section 3 we show that, if μ is a product measure on $\boldsymbol{R}^\infty$ with non-atomic factors, then $\mu(A) = 0$ or 1 for all μ-measurable affine subsets.

The third result on Gaussian measures, which has proved to be a very powerful tool, is the reproducing kernel Hilbert space which, in case μ