H1 spaces of generalized half-planes

by

STEPHEN VÁGI (Chicago, Ill.)

Abstract. New proofs are given of the following assertions about the Hardy space H1 on Siegel domains of type II: H1 is a Hilbert space and has a reproducing kernel. Elements of H1 have "L1-boundary values", and admit a Paley–Wiener type representation formula.

1. Introduction. The basic elementary facts in the theory of H1 spaces are: (a) that these spaces are Hilbert spaces, (b) that H1-functions have "L1-boundary values", (c) that a Paley–Wiener type representation formula holds, and (d) that H1 spaces have reproducing kernels. For tube domains over regular cones these results were proved by S. Bochner [2]. For Siegel domains of type II they were obtained by S. G. Gindikin [4]. Gindikin’s arguments, however, were not conclusive, and the first complete derivation of his results was given — using methods different from his — by A. Korányi and E. M. Stein [7]. The purpose of this paper is to present, for Siegel domains of type II, a new and, maybe, simpler approach to the proof of the four facts listed above.

2. Definitions, notation, and statement of results. Let W and V be finite dimensional complex vector spaces of positive dimensions with \( \dim V = n \). Let U be a real form of W chosen once and for all. Elements of U and W will be, usually, denoted by \( \mathfrak{c} \) and \( w \). The conjugate of \( z \in W \) relative to \( U \) will be written as \( \overline{z} \). The value of an element \( \lambda \) of the dual space \( U’ \) of \( U \) at the vector \( a \) of \( U \) or \( W \) will be denoted by \( \langle \lambda, a \rangle \). We select once and for all Haar measures \( ds \) and \( d\sigma \) on the vector groups U and V. The Fourier transform on \( L^1(U) \) is defined by \( \hat{f}(\lambda) = \int f(x) \exp(-2\pi i \langle \lambda, x \rangle) dx d\sigma. \) The Haar measure \( d\lambda \) on \( U’ \) is normalized so that the Fourier inversion formula reads \( f(x) = \int_{U’} \exp(2\pi i \langle \lambda, x \rangle) \hat{f}(\lambda) d\lambda. \) If \( L^1 \) is an inner product space, the inner product will be denoted, usually, by \( \langle \cdot, \cdot \rangle \).
A regular cone $Q$ in $U$ is an open convex cone with vertex at the origin which contains no affine line. Let $D$ denote the closure of $Q$. If $Q$ is regular, then so is its dual cone $Q' = \{ x \in U : \langle x, y \rangle > 0, y \in \overline{Q} = \{ 0 \} \}$. A Hermitian bilinear map $\Phi : V \times V \to W$ is said to be $Q$-positive if for all $z \in V$, $\Phi(z, z) \in Q$, and if $\Phi(z, z) = 0$ implies that $z = 0$. For $\lambda \in Q'$ define a positive definite Hermitian form on $V \times V$ by $H_\lambda(z, w) = \langle \lambda, \Phi(z, w) \rangle$, and set $\rho(\lambda) = \det H_\lambda$.

The tube domain over $Q$ in $W$ is $T_Q(z) = \{ w \in W : \Im w \in Q \}$. The Siegel domain of type II determined in $W \times V$ by $\Phi$ is the set $D = \{ (z, \zeta) \in W \times V : \Im z - \Phi(z, \zeta) \in Q \}$. The distinguished boundary of $D$ is the subset $B = \{ (z, \zeta) : \Im z - \Phi(z, \zeta) = 0 \}$ of the topological boundary of $D$. The map $(\alpha \zeta) \mapsto (\alpha + \i I \Phi(\alpha \zeta), \alpha \zeta)$ is a homeomorphism of $U \times V$ onto $B$. A theorem of measure theoretic structures of $B$ is those of $U \times V$, transferred to $B$ by the above map. Now $L^p$ spaces for $1 \leqslant p < \infty$ can be defined on $B$. The $L^p(B)$ norm of a measurable function on $B$ is explicitly

$$\|f\|_{L^p(B)} = \left( \int_{U \times V} |f(z - i \Phi(\zeta, \zeta), \zeta)|^p \, dx \, df \right)^{1/p}.$$ 

If $F : D \to C$ and $t \in Q$, then the function $F_t : D \to C$ is defined by $F_t(z) = F(z + it, \zeta)$. Finally, for $1 \leqslant p < \infty$ the space $H^p = H^p(D)$ is defined as the set of all holomorphic functions $F : D \to C$ such that

$$\|F\|_{H^p} = \sup_{t \in Q} \|F_t\|_{L^p(B)} < \infty.$$

The function $F \mapsto \|F\|_{H^p}$ is a norm on $H^p$. By abuse of notation we shall write $\|F\|_{H^p} \in L^p(B)$ as $\|F\|_{L^p(B)}$. General references about the facts reviewed here are [5], [6], and [8].

We now introduce a function space which will play an important part in our proofs. Consider the set of functions $\hat{F} : Q' \times V \to C$ subject to the following two conditions:

(A) For every $z \in V$, $\hat{F}(\cdot, z)$ is measurable on $Q'$.

(B) For every $\lambda \in Q'$, $\hat{F}(\lambda, \cdot)$ is a holomorphic entire function on $V$.

This set clearly forms a linear space. By a result of H. D. Ursell ([15], Theorem 4), the following statement is true: (we record it for future reference as)

Remak 1. A function $f : Q' \times V \to C$ satisfying conditions (A) and (B) is measurable on $Q' \times V$.

In view of the remark it is meaningful to impose the following, third condition on our functions:

(C) $\|\hat{F}\|_{H^p} = \int_{Q' \times V} \exp(-mH_{Q', Q}(F(\lambda, z), \zeta) \, d\lambda \, dz < \infty.$

We define the space $\hat{H}^p$ to be the set $\{ \hat{F} : Q' \times V \to C : \hat{F} \text{ satisfies (A), (B), (C)} \}$. If $\hat{F}$ and $\hat{G}$ belong to $\hat{H}^p$, we say that $\hat{F}$ and $\hat{G}$ are equivalent ($\hat{F} \sim \hat{G}$) if $\hat{F}(\lambda, z) = \hat{G}(\lambda, z)$ for almost all $(\lambda, z) \in Q' \times V$. In view of condition (C) we have

Remark 2. If $\hat{F}, \hat{G} \in \hat{H}^p$, then

$$\hat{F} \sim \hat{G} \iff \int_{Q'} \exp(-mH_{Q', Q}(\hat{F}(\lambda, z), \zeta) \, d\lambda \, dz < \infty,$$

for almost every $z \in V$.

Now $\hat{H}^p$ is defined as the set of equivalence classes (relative to $\sim$) of elements of $\hat{H}^p$. Clearly, $\hat{H}^p$ is an inner product space with norm defined by $\langle \cdot, \cdot \rangle$. We can now state our results.

Lemma 1. The space $\hat{H}^2$ is a Hilbert space.

Theorem (i) Let $\hat{F} \in \hat{H}^1$ and let $(x, \zeta) \in D$. Define $U \hat{F}(x, \zeta)$ by

$$U \hat{F}(x, \zeta) = \langle \hat{F}(\lambda, \cdot), \rho(\lambda) \rangle \, d\lambda.$$

The integral in (1) is absolutely convergent, $U \hat{F}$ belongs to $H^2$, and if $\hat{G} \in \hat{H}^2$ is equivalent to $\hat{F}$ then $U \hat{F} = U \hat{G}$.

(ii) The space $H^2 \in \hat{H}^2$. The map $U : \hat{H}^2 \to H^2$ is continuous, but $U$ is not continuous.

(iii) If $F \in H^2$, then for $t \in Q$ tending to 0, $F \mapsto \hat{F}$ converges in the norm of $L^1(B)$.

(iv) If $(x, \omega, z) \in D$, then the function $(x, \omega) \mapsto S_{\rho, \omega}(x, \zeta)$ defined by

$$S_{\rho, \omega}(x, \zeta) = \int_{Q'} \exp(-mH_{Q', Q}(\hat{F}(\lambda, z), \zeta) \, d\lambda \, dz$$

belongs to $H^2$, and for every $F \in H^2$

$$F(x, \omega) = \langle U \hat{F}(x, \zeta), \rho(\lambda) \rangle \, d\lambda.$$

Equality (3) states that $S_{\rho, \omega}(x, \zeta)$ is $(\omega, \rho)$, and hence by general principles the reproducing kernel of $H^2$, the so called Segal kernel of $D$.

Proof of Lemma 1. Fix $\lambda \in Q'$. Define $\|f\|_1$ for measurable functions on $V$ by

$$\|f\|_1 = \int_{Q'} \exp(-mH_{Q', Q}(f(\lambda), \zeta) \, d\lambda \, dz.$$
and define $\mathfrak{M}$ to be the set of entire holomorphic functions on $V$ for which $\|f\|_\alpha$ is finite. The space $\mathfrak{M}$ is an inner product space which obviously contains all the constants, and it is easily checked that it contains all polynomials. The proof of Lemma 1 consists in showing that $\mathfrak{M}$ is complete, and that $\hat{H}^\alpha$ can be identified with the direct integral $\int_d \mathfrak{M}^\alpha d\lambda$. The basic facts about $\mathfrak{M}^\alpha$ via the existence of a reproducing kernel and completeness are due to V. Bargmann [1]. For the sake of completeness we include simplified proofs of these facts.

For $\zeta \in V$, and $f \in \mathfrak{M}$ define $\langle A_\zeta f \rangle (\zeta)$ to be $\exp\{\pi H_\zeta(\zeta, \xi) - |\pi H_\zeta(\zeta, \xi)|^2\} \times f(\xi - \zeta)$. Clearly, $\omega \mapsto \langle A_\zeta f \rangle (\omega)$ is an entire function on $V$. A simple calculation shows that for $f, g \in \mathfrak{M}$ one has (writing the inner product in $\mathfrak{M}$ as $\langle \cdot, \cdot \rangle$)

$$\langle A_\zeta f | A_\zeta g \rangle_\alpha = \langle f | g \rangle_\alpha,$$

and that in particular for $f \in \mathfrak{M}$, $\|A_\zeta f\|_\alpha = \|f\|_\alpha$. Another easy calculation checks that $A_\zeta^\dagger$ is the inverse of $A_\zeta$. Therefore $A_\zeta$ is a unitary transformation of $\mathfrak{M}$ onto itself. Let now $g \in \mathfrak{M}$, for $f \in \mathfrak{M}$ define $f_g$ by $f_g(\zeta) = (f(\zeta), \zeta)$. Clearly, $f_g \in \mathfrak{M}$. The change of variable $\xi \mapsto \xi + \zeta$ and the fact that $H_\zeta(\zeta, \xi + \zeta) = H_\zeta(\zeta, \xi)$ show that $\langle f_g | f \rangle_\alpha = \langle f | f \rangle_\alpha$. Therefore, using first Fubini's and then Cauchy's theorem we have

$$\langle f | g \rangle_\alpha = \frac{1}{2\pi} \int_{\mathbb{C}} f_g(\zeta) d\zeta = \frac{1}{2\pi} \int_{\mathbb{C}} f(\zeta) d\zeta = \langle f | g \rangle_\alpha.$$  

By evaluating $\langle 1, 1 \rangle$, in a coordinate system in which $H_\zeta$ is diagonal, we find that $\langle 1, 1 \rangle_\alpha = \langle \omega | \omega \rangle_\alpha$. Using this in (6) we have

$$\langle f | f \rangle_\alpha = \langle \omega | \omega \rangle_\alpha.$$

Since $f(\zeta) = \exp\{\pi H_\zeta(\zeta, \xi)\} \langle A_{-\xi} f \rangle(\xi)$, we obtain from (6) that

$$f(\zeta) = \exp\{\pi H_\zeta(\zeta, \xi)\} \langle A_{-\xi} f \rangle(\xi).$$

Applying (4) to the right-hand side of the last equality we have

$$f(\zeta) = \exp\{\pi H_\zeta(\zeta, \xi)\} \langle A_{-\xi} f \rangle(\xi) = \exp\{\pi H_\zeta(\zeta, \xi)\} \langle A_{-\xi} f \rangle(\xi).$$

(Note that since $H_\zeta$ is a numerical multiple of $A_\zeta$, it is an element of $\mathfrak{M}^\alpha$.)

We have proved that $\mathfrak{M}$ has a reproducing kernel given by $K_\zeta$. An easy calculation shows that $\|K_\zeta\|_\alpha = \exp\{\pi H_\zeta(\zeta, \xi)\}$. Using this value of $\|K_\zeta\|_\alpha$ and applying Schwarz's inequality to (8) we get

$$\|K_\zeta\|_\alpha \leq \exp\{\pi H_\zeta(\zeta, \xi)\} \|f\|_\alpha.$$

If $K \subset V$ is compact and $G_K = \sup\{\exp\{\pi H(\zeta, \xi)\} : \xi \in K\}$, then for $\zeta \in K$ (9) yields $\|K_\zeta\|_\alpha \leq G_K \|f\|_\alpha$. This inequality immediately implies the completeness of $\mathfrak{M}$. We shall now derive another consequence of (9) which will be needed in the proof of the theorem. Let $\hat{f} \in \hat{H}^\alpha$, then, by condition (C), $\hat{f}(\lambda, \cdot)$ belongs to $\mathfrak{M}$ for almost every $\lambda \in \Omega'$. In view of (9) we therefore have

**Remark 3.** If $\hat{f} \in \hat{H}^\alpha$, then, for every $\xi \in V$, $\lambda \mapsto \hat{f}(\lambda, \xi)^{-k} e^{-inf(\xi, \cdot) \xi} \times \hat{f}(\lambda, \cdot)$ belongs to $\mathcal{V}(\Omega')$.

Let us also observe the following fact: if $\zeta_j, j = 1, 2, 3, \ldots$ is a dense sequence in $V$ and $f \in \mathfrak{M}$ then such that $\langle f | K_\zeta_j \rangle_\alpha = 0$ for $j = 1, 2, 3, \ldots$, then by (8) $\|f\|_\alpha = 0$. Consequently, we have the following

**Remark 4.** If $\zeta_j, j = 1, 2, 3, \ldots$ is a dense sequence in $V$, then $K_\zeta_j, j = 1, 2, 3, \ldots$ is a total sequence in $\mathfrak{M}$.

We now prove that $\hat{H}^\alpha$ is complete. Let $\hat{g} = \int_d \mathfrak{M}^\alpha d\lambda$, and let $\mathfrak{G} = \{f : \Omega' \times V \to C : f$ satisfies (A), and for every $\lambda \in \Omega'$, $f(\lambda, \cdot) \in \mathfrak{M}\}$. Note first that every $\mathfrak{G}$ can be identified in an obvious way with a linear subspace of $\mathfrak{Y}$. Also note that for fixed $J \subset V$ ($\lambda, \omega) \mapsto \mathfrak{K}_\zeta(\omega, J)$ belongs to $\mathfrak{G}$. We shall now verify that the Hilbert spaces $\mathfrak{M}$ form a measurable field of Hilbert spaces ($\Omega'$, p. 149). To this end, we must check three conditions.

1. If $f \in \mathfrak{G}$, then $\lambda \mapsto \langle f(\lambda, \cdot) \rangle_\alpha$ is a measurable function on $\Omega'$.

To prove this, note that, by Remark 1, $f$ is a measurable function on $\Omega' \times V$. Then approximate the integral giving $\|f(\lambda, \cdot)\|_\alpha$ to within $\epsilon/2$ by an integral over a large cube in $V$. Now approximate the integral over the cube to within $\epsilon/2$ by a Riemann sum. This Riemann sum is a measurable function of $\lambda$. Therefore $\|f(\lambda, \cdot)\|_\alpha$ is the pointwise limit of measurable functions, and hence measurable.

2. If $g \in \mathfrak{Y}$ is such that $\lambda \mapsto \langle g(\lambda, \cdot) \rangle_\alpha$ is measurable for every $f \in \mathfrak{G}$, then $g \in \mathfrak{Y}$.

Proof. $\langle g(\lambda, \cdot) \rangle_\alpha = \langle g | K_\zeta \rangle_\alpha$ is measurable for every $\lambda \in \Omega'$ because $K_\zeta \in \mathfrak{G}$, then use Remark 1.

3. There is a sequence $f_j$ of elements of $\mathfrak{G}$ such that for every $\lambda \in \Omega'$ the sequences $f_j(\lambda, \cdot)$ is total in $\mathfrak{M}$.


The elements of $\mathfrak{G}$ are called measurable vector fields. A measurable vector field $f$ is said to be square integrable if $\int_{\Omega'} |f(\lambda, \cdot)|^2 d\lambda$ is finite. Two square integrable measurable vector fields $f$ and $g$ are equivalent if $\int_{\Omega'} |f(\lambda, \cdot) - g(\lambda, \cdot)|^2 d\lambda = 0$. The direct integral $\int_d \mathfrak{M}^\alpha d\lambda$ is defined as the set of equivalence classes of measurable, square integrable vector fields.

The norm of $f \in \int_d \mathfrak{M}^\alpha d\lambda$ is $\|f(\lambda, \cdot)|^2 d\lambda\|^1$.  

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If \( f \in G \) is square integrable, then, clearly, \( f \) belongs to \( H_1^1 \). If \( g \) is another square integrable element of \( G \), and \( g \) is equivalent to \( f \), then (Remark 2) \( f \) and \( g \) are also equivalent in \( H_1^1 \). The norm of a square integrable \( f \in G \) equals its \( H_1^1 \)-norm. So far we have shown that \( \int |f^2| d\lambda \) can be identified with a subspace of \( H_1^1 \). To prove that this subspace is actually all of \( H_1^1 \), let \( f \in H_1^1 \), and select a representative \( \hat{f}_1 \) of \( f \) in \( H_1 \). The set of all \( \lambda \)'s in \( \Omega \) for which \( \hat{f}_1(\lambda, \cdot) \) does not belong to \( H_1^1 \) is of measure zero.

Now define \( \hat{f}_2 \) as follows: \( \hat{f}_2(\lambda, \cdot) = \hat{f}_1(\lambda, \cdot) \) if \( \hat{f}_1(\lambda, \cdot) \in H_1^1 \), and \( \hat{f}_2(\lambda, \cdot) = 0 \) otherwise. By Remark 1, \( \hat{f}_2 \) belongs to \( G \), and hence to \( H_1^1 \), and by Remark 2 it is equivalent to \( \hat{f}_1 \). This proves that \( H_1^1 \) can be identified with the direct integral of the \( \sigma \)-duals. Since the direct integral of Hilbert spaces is a Hilbert space, Lemma 1 is proved.

4. Proof of the Theorem. In addition to Lemma 1 and Remark 3 two technical results will be needed which we now list.

**Lemma 2.** Let \( F \in H_2, 1 \leq p < \infty \). Let \( \xi \in V \), and \( \delta \in \Omega \) such that \( \delta - \Phi(\xi, \cdot) \in \Omega \). Then \( z \mapsto \int F(z + it) d\lambda \) belongs to \( H^p(T_\delta) \).

**Lemma 3.** Let \( \varepsilon > 0 \), \( 0 \leq a < \frac{1}{4} \), and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be a basis of \( U \) contained in \( \Omega \) which is compatible with the Haar measure \( d\lambda \) on \( \Omega \). Then \( G^n(\varepsilon, \xi) = \exp \left( -\varepsilon \sum_{j=1}^n \lambda_j \zeta_j^n \right) \) belongs to \( H^1(\Omega) \), and is bounded and continuous on \( D \).

Lemma 2 is actually true for all positive \( p \), but we only need it for \( p = 1, 2 \). It is due to E. M. Stein [9]. Lemma 3 is from [9].

Let now \( \hat{f} \in H_2 \) and \( (z, \xi) \in D \), with \( z = x + it + i\Phi(\xi, \cdot) \), where \( t \in \Omega \). In any coordinate system \( \bar{\phi}(\lambda) \) is a homogeneous polynomial of degree \( n \), and one can show readily that \( \bar{\phi}(\lambda) \exp(-2\pi i \langle \lambda, \xi \rangle) \) is square integrable on \( \Omega \). Therefore, by Remark 3,

\[
\int_{\Omega} e^{2\pi i \langle \lambda, \xi \rangle} \hat{f}(\lambda, \cdot) d\lambda = \int_{\Omega} e^{2\pi i \langle \lambda, \xi \rangle} e^{-2\pi i \langle \lambda, \delta \rangle} \delta^\lambda \hat{f}(\lambda, \cdot) d\lambda
\]

is absolutely convergent for every \( \xi \in V \). Choosing coordinates in \( W \times \Omega \), and applying Morera's theorem in combination with Fubini's theorem one shows that \( U F \) is holomorphic in each coordinate of \( (z, \xi) \), and hence by Hartogs's theorem holomorphic in \( D \). Since \( \bar{\phi}(\lambda) \exp(-2\pi i \langle \lambda, \delta \rangle) \) is the bounded function of \( \lambda \) on \( \Omega \), by Remark 3 one concludes that the quantity multiplying \( \exp(2\pi i \langle \lambda, \xi \rangle) \) in (10) is square integrable on \( \Omega \). Therefore, by Fubini's theorem for every \( \xi \in V \)

\[
\int_{\Omega} \left[ U \hat{f}(z + it + i\Phi(\xi, \cdot), \lambda) \right]^2 d\lambda = \int_{\Omega} e^{-4\pi i \langle \lambda, \delta \rangle} \delta^\lambda \hat{f}(\lambda, \cdot) d\lambda
\]

Integrating this equality on \( V \) we get

\[
\| U \hat{f} \|_{L^2(\Omega)}^2 = \int_{\Omega} e^{-4\pi i \langle \lambda, \delta \rangle} \delta^\lambda \hat{f}(\lambda, \cdot) d\lambda \leq \| \hat{f} \|_{L^2(\Omega)}^2.
\]

From (11) we conclude that \( \hat{f} \in H^1 \). If \( t_n \in \Omega \) is a sequence tending to \( 0 \), then by the dominated convergence theorem we have that \( \| U \hat{f} \|_{L^2(\Omega)} \) converges to \( \| \hat{f} \|_{L^2(\Omega)} \), and that therefore}

\[
\| U \hat{f} \|_{L^2(\Omega)}^2 = \| \hat{f} \|_{L^2(\Omega)}^2.
\]

If \( \hat{G} \in H^1 \) and \( \hat{F} \sim \hat{G} \), then (12) implies that \( \| U \hat{F} - U \hat{G} \|_{L^2(\Omega)} = \| \hat{F} - \hat{G} \|_{L^2(\Omega)} = 0 \), i.e. that equivalent \( \hat{F} \) give rise to the same \( U \hat{F} \). Therefore \( U \) defines a linear map from \( H^1 \) to \( H^1 \) which we continue to write \( U \). The equation (12) shows that \( U \) maps \( H^1 \) isometrically into \( H^1 \). Now let \( t_n \in \Omega \) be a sequence converging to \( 0 \), then (11) (with \( \hat{U} \hat{G}_{t_n} - \hat{U} \hat{G}_{t} \) instead of \( \hat{f} \)) and the dominated convergence theorem show that \( \| \hat{U} \hat{G}_{t_n} \|_{L^2(\Omega)} \) is a Cauchy sequence in \( L^2(\Omega) \). Therefore \( U \hat{G}_{t_n} \) converges in \( L^2(\Omega) \) norm to an element of \( L^2(\Omega) \). We omit the proof that the sequential limit can be replaced by \( t \in \Omega \) tending to \( 0 \). We therefore have

**Remark 5.** Assertion (iii) of the theorem holds for every \( F \in H^1 \) which admits the representation (1).

Let now \( F \in H^2 \cap H^1 \) and let \( \xi \in V \) be arbitrary but fixed. (By Lemma 3, \( H^2 \cap H^1 = \emptyset \). Set \( \Omega^1 = \{ \delta \in \Omega: \delta - \Phi(\xi, \cdot) \in D \} \). For \( \delta \in \Omega \) \( z \mapsto \hat{F}(z, \xi) \) belongs to \( H^2 \cap H^1 \) by Lemma 2. By the theory of \( H^2 \) spaces on tube domains \( \Omega=\{ \delta \in \Omega: \delta - \Phi(\xi, \cdot) \in D \} \), \( U \hat{F}(\xi, \cdot) \) belongs to \( U \hat{F}(\xi, \cdot) \). We can therefore define a function \( \hat{F}(\lambda, \cdot) \) by

\[
\hat{F}(\lambda, \cdot) = \int_{\Omega} e^{2\pi i \langle \lambda, \xi \rangle} \hat{F}(z, \xi) dz.
\]

**Remark 6.** By the \( H^2 \) theory for tube domains \( \hat{F}(\lambda, \xi) \) is supported in \( D \). Since \( \hat{F}(\lambda, \cdot) = \hat{F}(U(z), \xi) \) is continuous and integrable. Therefore Fourier inversion can be applied to (13) everywhere.

If \( \delta \) is another element of \( \Omega \), then for \( \lambda \in \Omega \), by the \( H^2 \) theory for tube domains,

\[
\hat{F}(\lambda, \xi) = \exp(-2\pi i \langle \lambda, \delta \rangle) \hat{F}(\lambda, \xi) = \exp(-2\pi i \langle \lambda, \delta' \rangle) \hat{F}(\lambda, \xi).
\]

Therefore for \( \delta \in \Omega \), \( \exp(2\pi i \langle \lambda, \delta \rangle) \hat{F}(\lambda, \xi) \) is independent of \( \delta \). Denote this function by \( \hat{F}(\lambda, \cdot) \).

Now let \( x = x + iy \in W \) be such that \( (x, \xi) \in D \), i.e. \( y \in \Omega \). Note that \( F(x, \cdot) = F(x, \xi) \). By Remark 6, we can apply Fourier inversion
to (13). If we now express \( \hat{F} \) in terms of \( F \) in the Fourier inversion formula, we get
\[
F(x, \zeta) = \int e^{i(x, \zeta) \cdot \xi} \hat{F}(\xi) d\xi = U \hat{F}(x, \zeta).
\]
Since \( \zeta \) was arbitrary, (14) holds for every \((x, \zeta) \in D\).

We now prove that \( \hat{F} \in \mathcal{H}^1 \). Again fix \( \zeta \in V \), and also \( \lambda \in \mathcal{O} \). If \( \delta \in \Omega \), then there is a polydisc \( \Delta = V \times \mathcal{O} \) centered at \( \zeta \) such that \( \delta \subseteq \Omega \) for \( \zeta \in \Delta \). Now by (15) and by the definition of \( \hat{F} \) we have for \( \zeta \in \delta \) that
\[
\hat{F}(\lambda, \zeta) = e^{i(\lambda, \zeta) \cdot \zeta} \int e^{-i(\lambda, \zeta) \cdot \xi} F(\xi, \zeta) d\xi.
\]
Exactly as before, by combining the theorems of Fubini, Morera and Hartogs, we can show that \( \hat{F}(\lambda, \cdot) \) is holomorphic in \( \lambda \). Since \( \zeta \) was arbitrary in \( V \), it follows that \( \hat{F}(\lambda, \cdot) \) is an entire function. By Remark 6, we know that \( \hat{F}(\cdot, \zeta) \) is continuous for every \( \zeta \in V \), therefore by Remark 1, \( \hat{F} \) is measurable on \( \mathcal{O} \times V \).

Now let \( t \in \Omega \), then Plancherel's theorem applied to (13) gives for every \( \zeta \in V \)
\[
\int |F(x + it \zeta, \zeta)|^2 dx = \int e^{-i(\lambda, \zeta) \cdot \zeta} |\hat{F}(\lambda, \zeta)|^2 d\lambda.
\]
Integrating this equality on \( V \) we have
\[
\|F\|^2_{H^1(B)} = \int_{\Omega} e^{-i(\lambda, \zeta) \cdot \zeta} |\hat{F}(\lambda, \zeta)|^2 d\lambda dt \leq \|\hat{F}\|^2_{L^2}. \tag{16}
\]
By Fatou's lemma it follows that \( \hat{F} \in \mathcal{H}^1 \). Taking suprema over \( \Omega \) we see that \( \|F\|^2_{H^1} = \|\hat{F}\|^2_{L^2} \). We conclude that the map \( F \to \hat{F} \) maps the subspace \( \mathcal{H}^\alpha \cap \mathcal{H}^\beta \) of \( \mathcal{H} \) isometrically into \( \mathcal{H} \). By Lemma 1, the range of this map is contained in a complete space, and therefore, if we denote by \( \mathcal{M} \) the closure in \( \mathcal{H} \) of \( \mathcal{H} \cap \mathcal{H}^\alpha \), it extends uniquely to an isometry \( T \) of \( \mathcal{M} \) into \( \mathcal{H} \). Now for \( \mathcal{F} \in \mathcal{H}^\alpha \cap \mathcal{H}^\beta \) (14) holds, and therefore for such \( \mathcal{F} \), \( \mathcal{U} \mathcal{F} \mathcal{V} = \mathcal{F} \), i.e., \( \mathcal{U} \) and \( \mathcal{V} \) are isometries of \( \mathcal{H}^\alpha \cap \mathcal{H}^\beta \). By continuity it follows that \( \mathcal{U} \) and \( \mathcal{V} \) are isometries of all of \( \mathcal{M} \), and hence if \( \mathcal{F} \in \mathcal{M} \), then \( \mathcal{U} \mathcal{F} \mathcal{V} = \mathcal{F} \), i.e., \( \mathcal{U} \) and \( \mathcal{V} \) are isometries of \( \mathcal{M} \). Therefore the unitary maps \( U_{\mathcal{M} \mathcal{F}} \) and \( V_{\mathcal{M} \mathcal{F}} \) are inverses of each other. By Remark 5 it follows that assertion (iii) holds for every \( \mathcal{F} \in \mathcal{M} \).

We now prove that \( \mathcal{M} = \mathcal{H} \). Let \( \mathcal{F} \in \mathcal{H} \), and let \( \mathcal{G} \) be the function introduced in Lemma 3. By that lemma and Schwartz's inequality \( \mathcal{G} \mathcal{F} \in \mathcal{H}^\alpha \cap \mathcal{H}^\beta \). Since assertion (iii) of the theorem holds in \( \mathcal{M} \), there exists an element \( (\mathcal{G} \mathcal{F})^\alpha \) of \( \mathcal{H}^\alpha \) such that \( (\mathcal{G} \mathcal{F})^\alpha \) tends to \( (\mathcal{G} \mathcal{F})^\alpha \) as \( t \to 0 \) tends to zero. Consider first the case \( \delta = 1 \). For every sequence \( t_k \in \Omega \), \( t_k \to 0 \) (fixed once and for all in this proof) \( (\mathcal{G} \mathcal{F})^\alpha \) almost everywhere on \( B \). Since \( \mathcal{G} \mathcal{F} \) is a Cauchy sequence in the complete space \( \mathcal{M} \), and therefore tends in \( \mathcal{H} \) to an element \( H \) of \( \mathcal{M} \). Now let \( t \in \Omega \) be arbitrary but fixed, then
\[
\|H\|^2_{\mathcal{H}^1(B)} = \|F\|^2_{\mathcal{H}^1(B)} \leq \int |(\mathcal{G} \mathcal{F})^\alpha - (\mathcal{G} \mathcal{F})^\alpha|^2 dx dt
\]
because (iii) holds in \( \mathcal{M} \). Since \( \mathcal{G} \mathcal{F} \to \mathcal{G} \mathcal{F} \) tends to zero boundedly, we have that \( \mathcal{G} \mathcal{F} \) is a Cauchy sequence in the complete space \( \mathcal{M} \), and therefore tends in \( \mathcal{H} \) to an element \( H \) of \( \mathcal{M} \). Now let \( t \in \Omega \) be arbitrary but fixed, then
\[
\|H\|^2_{H^1(B)} \leq \|\mathcal{G} \mathcal{F} - \mathcal{F}\|^2_{L^2(B)}
\]
and therefore \( (\mathcal{G} \mathcal{F})^\alpha \to H \) in \( L^2(B) \) norm. On the other hand, \( (\mathcal{G} \mathcal{F})^\alpha \to H \) everywhere on \( B \). Consequently \( \|H\|^2_{\mathcal{H}^1(B)} = \|H\|^2_{\mathcal{H}^1(B)} \) because both functions are continuous. Therefore \( \mathcal{M} = \mathcal{H} \). Therefore \( \mathcal{M} = \mathcal{H} \).

To prove (iv) let \( (w, a) \in D \) and \( F \in \mathcal{H} \). By the assertions of the theorem already proved, we have
\[
F(w, a) = \int_{\mathcal{O}} e^{i(x, \zeta) \cdot \zeta} \mathcal{F}(\lambda, \zeta) d\lambda dt \leq \|\mathcal{F}\|^2_{L^2},
\]
where \( \mathcal{F} = U^{-1}F \in \mathcal{H} \). Taking \( \mathcal{F}(\lambda, \cdot) \in \mathcal{H}^\alpha \) for almost every \( \lambda \in \mathcal{O} \), we have \( \mathcal{F}(\lambda, \cdot) = \langle \mathcal{F}(\lambda, \cdot) \rangle K_{\mathcal{O}} \) for almost every \( \lambda \in \mathcal{O} \). Introducing this into (15) and rewriting the integral formally as a double integral we have (only formally, so far)
\[
F(w, a) = \int_{\mathcal{O}} e^{i(x, \zeta) \cdot \zeta} \mathcal{F}(\lambda, \zeta) d\lambda dt \leq \|\mathcal{F}\|^2_{L^2},
\]
Denote the quantity in curly brackets by \( T_{\mathcal{M} \mathcal{F}} (w, \zeta) \). A straightforward check verifies that \( (\mathcal{F}(\lambda, \cdot) \to T_{\mathcal{M} \mathcal{F}} (w, \zeta) \) belongs to \( \mathcal{H} \). Therefore the double integral in (16) is absolutely convergent (this justifies the passage from (15) to (16)) and equal to \( \langle \mathcal{F} \rangle T_{\mathcal{M} \mathcal{F}} \) in \( \mathcal{H} \). Consequently,
\[
F(w, a) = \langle \mathcal{F} \rangle T_{\mathcal{M} \mathcal{F}}.
\]
Now a simple calculation shows that \( U T_{\mathcal{M} \mathcal{F}} \) is the function \( T_{\mathcal{M} \mathcal{F}} \) defined by (2), hence in view of the fact that \( U \) is unitary, (17) yields
\[
F(w, a) = \langle \mathcal{F} \rangle T_{\mathcal{M} \mathcal{F}}.
\]
But this is equation (3) in assertion (iv) of the theorem whose proof is now complete.
STUDIA MATHEMATICA, T. LXI. (1977)

Integrability of seminorms, the 0-1 law and the affine kernel for product measures

by

J. HOFFMANN-JØRGENSEN (Aarhus, Denmark)

Abstract. Let $(X_n)$ be a sequence of random variables taking values in a measurable linear space $E$, and let $\varphi$ be a quasi-convex subadditive function on $\mathbb{R}^m$. The first part of the paper deals with the problem of finding conditions, which assures that $E(e^{\varphi\mathcal{M}})$ is finite for some positive $\varepsilon$, where $\mathcal{M} = \sup\{X_1, \ldots, X_n, 0, 1, \ldots\}$. In the second and third part of the paper we take $E = \mathbb{R}$, and we show that $\mathcal{M}$ has no mass points, then every linear subspace of $\mathbb{R}^m$ has probability 0 or 1. Finally, we study the affine kernel of $(X_n)$, i.e. the intersection of all affine subspaces of probability 1, and we give an analytic expression for this.

1. Introduction. If $\mu$ is a Gaussian measure on a locally convex space, there are three main results which have proved to be useful.

The first is the result of Fernique stating that, if $\varphi$ is a measurable a.e. finite seminorm, then $E[\exp(\varphi)] < \infty$ where $E$ denotes expectation with respect to $\mu$. A similar result has been proved by C. Borel (11) for certain other classes of measures. In Section 2 of this paper we shall prove some results in this direction when $\mu$ is a product measure on $(\mathbb{R}^n_+, \mathcal{B}_{\mathbb{R}^n})$ and $(\mathbb{R}^n_+, \mathcal{B}_{\mathbb{R}^n})$ is a measurable linear space. Here we define a measurable linear space, $(E, \mathcal{B})$, to be a linear space $E$ equipped with a $\sigma$-algebra $\mathcal{B}$ satisfying

\[
(x, y) \mapsto x + y \text{ is measurable: } (E \times E, \mathcal{B} \otimes \mathcal{B}) \to (E, \mathcal{B}),
\]

\[
(x, y) \mapsto xy \text{ is measurable: } (E \times R, \mathcal{B} \otimes \mathcal{B}(\mathbb{R})) \to (E, \mathcal{B}).
\]

The methods and the results of that section are closely related to the results of Marcus and Jain in [9] and to the results in [2] and [3].

The second result is the 0-1 law by Kallianpur in [5]. C. Borel has in [1] shown that the same result holds for certain other classes of measure. In Section 3 we show that, if $\mu$ is a product measure on $\mathbb{R}^m$ with nonatomic factors, then $\mu(\mathcal{A}) = 0$ or 1 for all $\mu$-measurable affine subsets.

The third result on Gaussian measures, which has proved to be a very powerful tool, is the reproducing kernel Hilbert space which, in case $\mu$...