pour toutes les décompositions $h = \sum_{i=1}^{n} h_i$, $h_i \in H$ et si l'on remarque que

$$|h| \geqslant |h|' \geqslant \frac{1}{\gamma} \varrho(h).$$

Additif. Pour plus de détails concernant les résultats de cet article, voir notre livre à paraître *Propriétés spectrales des algèbres de Banach*. A propos de la remarque 4, C. Apostol nous a signalé un exemple d'algèbre de Banach où le rayon spectral est continu mais le spectre discontinu.

Bibliographie

[1] B. Aupetit, *Continuité du spectre dans les algèbres de Banach avec involution*, Pacific J. Math. 56 (1975), p. 321–324.
[2] — *Uniforme continuité du spectre dans les algèbres de Banach avec involution*, C. R. Acad. Sci. Paris 284 (1977), p. 1125–1127.
[3] — *Caractérisation spectrale des algèbres de Banach commutatives*, Pacific J. Math. 63 (1976), p. 23–35.
[4] — *Caracterisation spectrale des algèbres de Banach de dimension finie*, J. Functional Analysis 25 (1977), à paraître.
[5] A. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, New York 1973.
[6] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton 1967.
[7] — *Capacity in Banach algebras*, Indiana Univ. J. 20 (1971), p. 855–863.
[8] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York 1966.
[9] J. D. Newburgh, *The variation of spectra*, Duke Math. J. 18 (1951), p. 165–176.
[10] C. E. Rickart, *General theory of Banach algebras*, Van Nostrand, Princeton 1960.
[11] E. Vesentini, *On the subharmonicity of the spectral radius*, Boll. Un. Mat. Ital. 4 (1968), p. 427–429.
[12] — *Maximum theorems for spectra. Essays on topology and related topics dedicated to Georges de Rham*, Springer-Verlag, New York 1970.
[13] V. S. Vladimirov, *Methods of the theory of functions of many complex variables*, M. I. T Press, Cambridge, Mass. 1966.
[14] B. Yood, *On axioms for B*-algebras*, Bull. Amer. Math. Soc. 76 (1970), p. 80–82.

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Decompositions of set functions with values in a topological semigroup

by

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Abstract. This paper contains a generalization of Theorem 3.11 of L. Drewnowski [1], concerning generalized Hewitt–Yosida and Lebesgue decompositions, to the case of Hausdorff topological semigroups.

0. Preliminaries. Throughout this paper,
S is an abstract space,
$\mathcal{R}$ is a $\mathfrak{S}$-ring of subsets of S,
H is comutative, Hausdorff, completely regular topological semigroup with identity $\boldsymbol{O}$ under the operation $+$ and topology τ such that the families $\{x+U\}$, where x runs through all elements of H and U runs through all elements of $\mathcal{U}$ ($\mathcal{U}$ an open basis of $\boldsymbol{O}$) are open basis for H.

0.1. Definitions.
(1) Let I be any index set;

$$f(I) = \{j\colon j \subset I \text{ and } j \text{ is finite}\}.$$

(2) For any J directed by $<$ and $x\colon J \to H$, $y \in H$,

$$\lim_j x_j = x$$

iff for every neighborhood U_y of y there exists $j_0 \in J$ such that for every $j \in J$ with $j > j_0$ we have $x_j \in U_y$.

(3) Let I be any index set and $x\colon I \to H$, $y \in H$; then

$$\sum_{i\in I} x_i = y \quad \text{iff} \quad \lim_{j\in f(I)} S_j = y \quad \text{where } S_j = \sum_{k\in j} x_k$$
and $J = f(I)$ directed by $\subset$.

(4) Let $x\colon I \to H$. The family $(x_i\colon i \in I)$ is *summable* in H iff there exists $y \in H$ such that $\sum_{i\in I} x_i = y$.

For any $\mu\colon \mathcal{R}\to H$,

(5) μ is *finitely additive* on H iff for every non-empty disjoint $\mathcal{A}\in f(\mathcal{R})$ with $\bigcup_{A\in\mathcal{A}} A\in\mathcal{R}$ we have

$$\mu\Big(\bigcup_{A\in\mathcal{A}} A\Big) = \sum_{A\in\mathcal{A}}\mu(A).$$

(6) μ is *σ-additive* on H iff for every non-empty countable, disjoint $\mathcal{A}\in f(\mathcal{R})$ with $\bigcup_{A\in\mathcal{A}} A\in\mathcal{R}$ we have

$$\mu\Big(\bigcup_{A\in\mathcal{A}} A\Big) = \sum_{A\in\mathcal{A}}\mu(A).$$

(7) $a(\mathcal{R}, H) = \{\mu\colon \mu\colon \mathcal{R}\to H$ and μ is finitely additive$\}$.

(8) μ is *exhaustive* (or *s-bounded*) iff for every disjoint sequence $(E_n)\subset\mathcal{R}$,

$$\lim_n \mu(E_n) = \boldsymbol{O}.$$

(9) $\mathrm{ea}(\mathcal{R}, H) = \{\mu\colon \mu\in a(\mathcal{R}, H)$ and μ is exhaustive$\}$.

(10) $\check{\mu}(E) = \{\mu(F)\colon F\subset E$ and $F\in\mathcal{R}\}$.

1. *s*-Cauchy net and Cauchy condition.

1.1. Definitions.

(1) For any J directed by $<$ and $x\colon J\to H$ x is an *s-Cauchy* net iff for every neighborhood U of $\boldsymbol{O}$ there exists $j_0\in J$ such that, for every $j, k\in J$ with $j, k > j_0$, we have (see [4])

(sC) $$(x_j+U)\cap(x_k+U)\neq\emptyset.$$

(2) Let I be any index set and $x\colon I\to H$. Then x *satisfies the Cauchy condition* iff

(Cc) *for every neighborhood U of $\boldsymbol{O}$ there exists $j_0\in f(I)$ such that for every $j'\in f(I\setminus j_0)$ we have*

$$S_{j'}\in U,\quad \textit{where}\quad S_{j'} = \sum_{k\in j'} x_k.$$

(3) For $A\subset H$, A is *s-complete* iff every *s*-Cauchy net in A converges to some point in A.

(4) A is *s-precomplete* iff the closure of A is *s*-complete.

(5) For $\mu\colon \mathcal{R}\to H$, μ is *s*-precomplete iff the range of μ (e.a. $\check{\mu}(S)$) is *s*-precomplete.

1.2. Lemma. *If J is directed by $<$ and $x\colon J\to H$ is convergent, then x is an s-Cauchy net.*

Proof. There exists $y\in H$ such that $\lim_j x_j = y$. Then for every neighborhood of $\boldsymbol{O}$, U, there exists $j_0\in J$ such that for $j, k\in J$ with $j, k > j_0$ we have

$$x_j\in y+U,\qquad x_k\in y+U.$$

Hence

$$x_j = y+u_1,\quad x_k = y+u_2,\quad \text{where } u_i\in U\ (i=1,2).$$

Thus

$$x_j+u_2 = y+u_2+u_1 = x_k+u_1,$$

so

$$(x_j+U)\cap(x_k+U)\neq\emptyset.$$

1.3. Corollary. *If $x\colon I\to H$ and the family $(x_i\colon i\in I)$ is summable in H, then $\big(S_j\colon j\in f(I)\big)$ is an s-Cauchy net, where $f(I)$ is directed by $\subset$.*

1.4. Lemma. *Definition* 1.1 (1) *of an s-Cauchy net is equivalent to the following: for every $j\in J$ with $j > j_0$ we have*

$$(x_j+U)\cap(x_{j_0}+U)\neq\emptyset.$$

1.5. Lemma. *If $x\colon I\to H$ and if x satisfies the Cauchy condition, then $\big(S_j\colon j\in f(I)\big)$ is an s-Cauchy net, where $f(I)$ is directed by $\subset$.*

Proof. By Definition (Cc), for every neighborhood U of $\boldsymbol{O}$, there exists $j_0\in f(I)$ such that, for every $j'\in f(I\setminus j_0)$, we have $S_{j'}\in U$.

Now, let $j\in f(I)$ and $j > j_0$. Then $S_j = S_{j\setminus j_0}+S_{j_0}$, but $j\setminus j_0\in f(I\setminus j_0)$. Thus $S_j\in S_{j_0}+U$, so

$$(S_j+U)\cap(S_{j_0}+U)\neq\emptyset$$

for every $j\in f(I)$, $j > j_0$. In view of 1.4., this completes the proof.

2. Fréchet-Nikodym topology and $\mathfrak{S}$-additivity.

2.1. Definitions (see [1], [2]).

(1) A topology Γ on $\mathcal{R}$ is called a *Fréchet-Nikodym topology* (shortly: FN-*topology*) iff $\mathcal{R}$ (with the symmetric difference $E\triangle F = (E\setminus F)\cup(F\setminus E)$ as addition) is a topological group under Γ and if, moreover, the operation of intersection $(E, F)\mapsto E\cap F$ is uniformly continuous on $\mathcal{R}$.

(2) $\eta\colon \mathcal{R}\to[0, \infty[$ is a *submeasure* on $\mathcal{R}$ iff $\eta(\emptyset) = 0$, $A\subset B\Rightarrow\eta(A)\leqslant\eta(B)$ and $\eta(A\cup B)\leqslant\eta(A)+\eta(B)$.

(3) η is a submeasure on $\mathcal{R}$,

$\Gamma(\eta)$ is the FN-topology on $\mathcal{R}$ determined by η, that is, by the *Fréchet–Nikodym ecart* $(A, B)\mapsto\eta(A\triangle B)$.

(4) $\mu\in a(\mathcal{R}, H)$, Γ is the FN-topology on $\mathcal{R}$, $\mu\ll\Gamma$ iff μ is Γ-continuous.

(5) For $\mu\in a(\mathcal{R}, H)$, there exists the coarsest FN-topology, $\Gamma(\mu)$, with respect to which μ is continuous. If $\mathcal{U}$ is a base of neighborhood of $\boldsymbol{O}$ in H, then the classes

(6) $\mathscr{U}_U = \{E \in \mathscr{R}:\ \mu(F) \in U \text{ for each } F \subset E,\ F \in \mathscr{R}\},\ U \in \mathscr{U}$, constitute a base of neighborhoods of $\emptyset$ in $(\mathscr{R}, \Gamma(\mu))$.

(7) For $\mu, \nu \in a(\mathscr{R}, H)$, $\mu \ll \nu$ iff $\Gamma(\mu) \subset \Gamma(\nu)$.

(8) Classes $\mathscr{U}_\varepsilon = \{E:\ E \in \mathscr{R}:\ \eta^{\vee}(E) \subset]0, \varepsilon[\}$, $\varepsilon > 0$, form a base of neighborhoods of $\emptyset$ in $(\mathscr{R}, \Gamma(\eta))$, where η is a submeasure on $\mathscr{R}$.

2.2. Lemma. *Let $\mu \in a(\mathscr{R}, H)$ and η be a submeasure on $\mathscr{R}$, then $\eta \ll \mu$ iff for every $\varepsilon > 0$ there exists a neighborhood of $\boldsymbol{O}$, $U \in \mathscr{U}$ such that, for every $E \in \mathscr{R}$ for which $\mu^{\vee}(E) \subset U$, we have $\eta^{\vee}(E) \subset]0, \varepsilon[$.*

2.3. Definitions.

(1) For $\mathscr{A}, \mathscr{B} \subset \mathscr{R}$

$$\mathscr{A} \mathbin{\mathring{\cap}} \mathscr{B} = \{A \cap B:\ A \in \mathscr{A} \text{ and } B \in \mathscr{B}\}.$$

(2) $\mathscr{D}$ is a class of pairwise disjoint sets from $\mathscr{R}$.

(3) $\Delta = \Delta(\mathscr{R})$ is the set all classes $\mathscr{D}$.

(4) $\Delta_f = \{\mathscr{D}:\ \mathscr{D} \in \Delta \text{ and } \mathscr{D} \text{ is a finite class}\}$.

(5) $\Delta_c = \{\mathscr{D}:\ \mathscr{D} \in \Delta \text{ and } \mathscr{D} \text{ is a countable class}\}$.

(6) For $\mathscr{D}_1, \mathscr{D}_2 \in \Delta$,

$\mathscr{D}_1 \leqslant \mathscr{D}_2$ iff for every $D_2 \in \mathscr{D}_2$ there exists $D_1 \in \mathscr{D}_1$ such that $D_2 \subset D_1$.

(7) $\leqslant$ is a partial order in Δ.

(8) Given a set $\mathfrak{S} \subset \mathscr{R} \times \Delta(\mathscr{R})$, let us write

$$\mathfrak{S}[E] = \{\mathscr{D} \in \Delta:\ (E, \mathscr{D}) \in \mathfrak{S},\ E \in \mathscr{R}\},$$

$$\Delta_{\mathfrak{S}} = \bigcup_{E \in \mathscr{R}} \mathfrak{S}[E].$$

(9) $\mathfrak{S} \subset \mathscr{R} \times \Delta(\mathscr{R})$ is *additive* on $\mathscr{R}$ iff the following conditions are satisfied:

(a1) $\Delta_f \subset \Delta_{\mathfrak{S}}$ and $\bigcup_{E \in \mathscr{R}} \{E\} \times \mathfrak{S}[E] = \mathfrak{S}$.

(a2) If $E \in \mathscr{R}$ and $\mathscr{D}_1, \mathscr{D}_2 \in \mathfrak{S}[E]$, then $\mathscr{D}_1 \mathbin{\mathring{\cap}} \mathscr{D}_2 \in \mathfrak{S}[E]$.

(a3) If $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}[E]$, then $\bigcup \mathscr{D} \subset E$.

(a4) If $E, F \in \mathscr{R}$, $F \subset E$ and $\mathscr{D} \in \mathfrak{S}[E]$, then $\mathscr{D} \mathbin{\mathring{\cap}} F \in \mathfrak{S}[F]$.

(a5) If $E_1, E_2 \in \mathscr{R}$, $E_1 \cap E_2 = \emptyset$ and $\mathscr{D}_i \in \mathfrak{S}[E_i]$ $(i = 1, 2)$, then

$$\mathscr{D}_1 \cup \mathscr{D}_2 \in \mathfrak{S}[E_1 \cup E_2].$$

(a6) If $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}[E]$, and each $D \in \mathscr{D}$ is the union of two disjoint sets $D_1, D_2 \in \mathscr{R}$, then

$$\mathscr{D}^* = \{D_i:\ D_i \in \mathscr{D},\ i = 1, 2\} \in \mathfrak{S}[E].$$

(10) $\mu: \mathscr{R} \to H$ is $\mathfrak{S}$-*additive* iff for every $E \in \mathscr{R}$ and $\mathscr{D} \in \mathfrak{S}[E]$ the family $(\mu(D):\ D \in \mathscr{D})$ is summable in H and $\sum_{D \in \mathscr{D}} \mu(D) = \mu(E)$, or, equivalently,

$$\lim_{\mathscr{D}' \in f(\mathscr{D})} \mu(E \setminus \bigcup \mathscr{D}') = \boldsymbol{O}.$$

(11) FN-topology Γ on $\mathscr{R}$ is $\mathfrak{S}$-*continuous* iff

$$(\Gamma) \lim_{\mathscr{D}' \in f(\mathscr{D})} (E \setminus \bigcup \mathscr{D}') = \emptyset$$

for each $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}[E]$.

(12) η is submeasure on $\mathscr{R}$. η is $\mathfrak{S}$-*continuous* iff

$$\lim_{\mathscr{D}' \in f(\mathscr{D})} \eta(E \setminus \bigcup \mathscr{D}') = \boldsymbol{O}, \quad E \in \mathscr{R},\ \mathscr{D} \in \mathfrak{S}[E].$$

(13) $\mu: \mathscr{R} \to H$ is $\mathfrak{S}$-singular iff $\Gamma(\mu)$ is $\mathfrak{S}$-singular.

2.4. Lemma. *An FN-topology on $\mathscr{R}$ is $\mathfrak{S}$-singular iff each submeasure both $\mathfrak{S}$-continuous and Γ-continuous, vanishes on $\mathscr{R}$.*

3. Existence of μ', μ'' and their properties.

3.1. Definitions. For any $\mathscr{D} \in \Delta$, $f(\mathscr{D})$ is directed by $\subset$.

(1) $\mu(\mathscr{D}) = \lim_{\mathscr{D}' \in f(\mathscr{D})} \mu(\bigcup \mathscr{D}')$. For $\mathfrak{S}$-additivity on $\mathscr{R}$, $E \in \mathscr{R}$ and $f(\mathscr{D})$ directed by $\subset$, $\mathfrak{S}[E]$ is directed by $\leqslant$.

(2) $\mu'(E) = \lim_{\mathscr{D} \in \mathfrak{S}[E]} \mu(\mathscr{D})$.

(3) $\mu(E, \mathscr{D}) = \lim_{\mathscr{D}' \in f(\mathscr{D})} \mu(E \setminus \bigcup D')$.

(4) $\mu''(E) = \lim_{\mathscr{D} \in \mathfrak{S}[E]} \mu(E, \mathscr{D})$.

3.2. Lemma. *$\mu \in \mathrm{ea}(\mathscr{R}, H)$ is exhaustive iff for each $\mathscr{D} \in \Delta$ the family $(\mu(D):\ D \in \mathscr{D})$ satisfies the Cauchy condition* (see [1]).

3.3. Proposition. *Let $\mu \in \mathrm{ea}(\mathscr{R}, H)$; then the family $(\mu(D):\ D \in \mathscr{D})$ is an s-Cauchy net, for every $\mathscr{D} \in \Delta$.*

Proof. This follows from Lemmas 3.2 and 1.5.

3.4. Corollary. *If $\mu \in \mathrm{ea}(\mathscr{R}, H)$ and μ is s-precomplete, then for every $\mathscr{D} \in \Delta$ $\mu(\mathscr{D})$ exists.*

3.5. Lemma. *If $\mu \in \mathrm{ea}(\mathscr{R}, H)$ and μ is s-precomplete, $\mathscr{D} \in \Delta$, then for each closed neigborhood U of $\boldsymbol{O}$ in H there exists $\mathscr{D}' \in f(\mathscr{D})$ such that if $\mathscr{D}'' \in f(\mathscr{D} \setminus \mathscr{D}')$ and for each $D \in \mathscr{D}''$, $\mathscr{D}_D \in \Delta$ and $\bigcup \mathscr{D}_D \subset D$, then $\sum_{D \in \mathscr{D}''} \mu(\mathscr{D}_D) \in U$.*

Proof. Otherwise, there is a neighborhood U of $\boldsymbol{O}$ such that for each $\mathscr{D}' \in f(\mathscr{D})$ there exists $\mathscr{D}'' \in f(\mathscr{D} \setminus \mathscr{D}')$ and a family $(\mathscr{D}_D)_{D \in \mathscr{D}''}$, where $\mathscr{D}_D \in \Delta$ and $\bigcup \mathscr{D}_D \subset D$, for which we have $\sum_{D \in \mathscr{D}''} \mu(\mathscr{D}_D) \notin U$. Then there exists $\mathscr{D}''' \in f(\mathscr{D}'')$ such that for each $(\mathscr{D}_D)_{D \in \mathscr{D}'''}$, $\bigcup \mathscr{D}_D \subset D$, $\mathscr{D}_D \in \Delta$, and we have $\sum_{D \in \mathscr{D}'''} \mu(\mathscr{D}_D) = \mu\big(\bigcup_{D \in \mathscr{D}'''} \bigcup \mathscr{D}_D\big) \notin U$. Hence we find a disjoint sequence $(A_n) \subset \mathscr{R}$ such that $u(A_n)$ non$\to \boldsymbol{O}$, but $\mu \in \mathrm{ea}(\mathscr{R}, H)$.

3.6. Lemma. *If $\mu \in \mathrm{ea}(\mathscr{R}, H)$ and μ is s-precomplete $\bigcup \mathscr{D} \subset E$ and*

$\mathscr{D} \in \Delta$, $E \in \mathscr{D}$, *then for each closed neighborhood* U *of* $\boldsymbol{O}$ *we have* $\mu(\mathscr{D}) \in \check{\mu}(E)+U$.

Proof. By Proposition 3.3, $\mu(\mathscr{D})$ exists. Then for each $U \in \mathscr{U}$ there exists $\mathscr{D}' \in f(\mathscr{D})$ such that

$$\mu(\mathscr{D}\setminus\mathscr{D}') \in U.$$

Hence

$$\mu(\mathscr{D}) = \mu(\mathscr{D}')+\mu(\mathscr{D}\setminus\mathscr{D}') \in \mu(\mathscr{D}')+U,$$

but

$$\mu(\mathscr{D}') = \sum_{D\in\mathscr{D}'}\mu(D) = \mu\big(\bigcup\mathscr{D}'\big) \subset \check{\mu}(E)$$

so

$$\mu(\mathscr{D}) \in \check{\mu}(E)+U.$$

3.7. Proposition. *If* $\mu \in \mathrm{ea}(\mathscr{R}, H)$ *and* μ *is s-precomplete,* $\emptyset \neq \Delta_0 \subset \Delta$ *and* Δ_0 *is directed by* $\leqslant$, *then* $\lim\limits_{\mathscr{D}\in\Delta_0}\mu(\mathscr{D})$ *exists.*

Proof. By Proposition 3.3, $\mu(\mathscr{D})$ exists for every $\mathscr{D} \in \Delta$. Suppose that a family $\big(\mu(\mathscr{D})\colon \mathscr{D}\in\Delta_0\big)$ does not satisfy (sC). Then by Lemma 1.4, there exist a sequence $(\mathscr{D}_n) \subset \Delta_0$ and a neighborhood U of $\boldsymbol{O}$ such that

$$\mathscr{D}_1 \leqslant \mathscr{D}_2 \leqslant \ldots$$

and

$$\big(\mu(\mathscr{D}_n)+U\big)\cap\big(\mu(\mathscr{D}_{n+1})+U\big) = \emptyset, \quad n = 1, 2, \ldots$$

Given a neighborhood U_1 of $\boldsymbol{O}$ such that $U_1+U_1 \subset U$. Now let V_n be a closed neighborhood of $\boldsymbol{O}$, such that

$$V_0+V_0+V_0 \subset U_1 \quad\text{and}\quad V_{n+1}+V_{n+1} \subset V_n.$$

Applying Lemma 3.5 to the $\mathscr{D}_n$, V_n, there exists $\mathscr{D}_n' \in f(\mathscr{D}_n)$ such that, if $\mathscr{D}_n'' \in f(\mathscr{D}_n\setminus\mathscr{D}_n')$ and for each $D \in \mathscr{D}''$, $\mathscr{D}_D \in \Delta$ and $\bigcup\mathscr{D}_D \subset D$, then

$$(*)\qquad \sum_{D\in\mathscr{D}_n''}\mu(\mathscr{D}_D) \in V_n.$$

Write

$$E_n = \bigcup\mathscr{D}_n' = \bigcup_{D\in\mathscr{D}_n'} D, \quad F_n = \bigcap_{k=0}^{n} E_k, \quad \text{for } n = 1, 2, \ldots$$

Then $F_{n+1} \subset F_n$ and $\lim\limits_n \mu(F_n\setminus F_{n+1}) = \boldsymbol{O}$. Hence for some $N \in \boldsymbol{N}$,

$$\mu(F_n\setminus F_{n+1}) \in V_0,$$

so

$$\mu(F_n) = \mu(F_{n+1})+\mu(F_n\setminus F_{n+1}) \in \mu(F_{n+1})+V_0.$$

But

$$E_n\setminus F_n = E_n\setminus\bigcap_{k=0}^{n}E_k = \bigcup_{k=0}^{n}(E_n\setminus E_k)$$
$$= (E_n\setminus E_0)\cup(E_n\cap E_0\setminus E_1)\cup\ldots\cup(E_n\cap E_0\cap\ldots\cap E_{n-1}\setminus E_n).$$

Therefore

$$E_n\setminus E_k = \bigcup\mathscr{D}_n'\setminus\bigcup\mathscr{D}_k' \subset \bigcup\mathscr{D}_n',$$

where $\mathscr{D}_n' \in f(\mathscr{D}_n)$ but $\mathscr{D}_k \leqslant \mathscr{D}_n$. Hence for every $D_n \in \mathscr{D}_n'$ there exists $D_k^n \in \mathscr{D}_k$, such that $D_n \subset D_k^n$, so there exists $\mathscr{D}_k'' \in f(\mathscr{D}_k\setminus\mathscr{D}_k')$ such that

$$E_n\setminus E_k \subset \mathscr{D}_k'',$$

and

$$G_k^n = E_n\cap E_0\cap\ldots\cap E_{k-1}\setminus E_k \subset \bigcup\mathscr{D}_k'',$$
$$G_k^n = \bigcup_{D\in\mathscr{D}_k''} G_k^n\cap D.$$

Write

$$\mathscr{D}_D = \{G_k^n\cap D, \emptyset\},$$

by $(*)$ we have

$$\sum_{D\in\mathscr{D}_k''}\mu(G_k^n\cap D) = \mu(G_k^n) \in V_k,$$

so

$$\mu(E_n\setminus F_n) \in \sum_{k=0}^{n-1} V_k.$$

But

$$\mu(E_n) \in \mu(\mathscr{D}_n)+V_n,$$

and

$$\mu(E_n) = \mu(F_n)+\mu(E_n\setminus F_n) \subset \mu(F_n)+V_0+\sum_{k=1}^{n-1}V_k.$$

Now, by the assumption,

$$V_0 \supset V_1+V_1 \supset V_1+V_2+V_2 \supset \ldots$$
$$\ldots \supset V_1+V_2+\ldots+V_{n-1}+V_n+V_n \supset V_1+\ldots+V_{n-1},$$

hence

$$\mu(E_n) \subset \mu(F_n)+V_0+V_0 \subset \mu(F_{n+1})+V_0+V_0+V_0 \subset \mu(F_{n+1})+U_1,$$

for $n \geqslant N$. Now

$$\mu(E_{n+1}) \in \mu(\mathscr{D}_{n+1})+V_{n+1},$$

and similarly,

$$\mu(E_{n+1}) \subset \mu(F_{n+1}) + V_0 + V_0 \subset \mu(F_{n+1}) + U_1,$$

for $n \geqslant N$. Thus

$$\mu(E_N) \in \mu(\mathscr{D}_N) + V_N, \qquad \mu(E_{N+1}) \in \mu(\mathscr{D}_{N+1}) + V_{N+1},$$

and

$$\mu(E_N) \in \mu(F_{N+1}) + U_1, \qquad \mu(E_{N+1}) \in \mu(F_{N+1}) + U_1,$$

hence

$$\begin{aligned} \mu(E_N) &= \mu(\mathscr{D}_N) + v_N, & v_N &\in V_N, \\ \mu(E_{N+1}) &= \mu(\mathscr{D}_{N+1}) + v_{N+1}, & v_{N+1} &\in V_{N+1}, \\ \mu(E_N) &= \mu(F_{N+1}) + u_1^1, & u_1^1 &\in U_1, \\ \mu(E_{N+1}) &= \mu(F_{N+1}) + u_2^1, & u_2^1 &\in U_1, \end{aligned}$$

so

$$\begin{aligned} \mu(F_{N+1}) + u_1^1 &= \mu(\mathscr{D}_N) + v_N, \\ \mu(F_{N+1}) + u_2^1 &= \mu(\mathscr{D}_{N+1}) + v_{N+1}, \end{aligned}$$

and

$$\mu(F_{N+1}) + u_1 + u_2 = \mu(\mathscr{D}_N) + v_N + u_2 = \mu(\mathscr{D}_{N+1}) + v_{N+1} + u_1.$$

Write

$$\begin{aligned} u_1 &= v_N + u_2^1 \subset U_1 + U_1 \subset U, \\ u_2 &= v_{N+1} + u_1^1 \subset U_1 + U_1 \subset U. \end{aligned}$$

Then

$$\mu(\mathscr{D}_N) + u_1 = \mu(\mathscr{D}_{N+1}) + u_2, \quad \text{where} \quad u_i \in U.$$

This contradicts the assumption.

3.8. PROPOSITION. *If $\mu \in \mathrm{ea}(\mathscr{R}, H)$ and μ is s-precomplete, then $\mu'(\cdot) = \lim\limits_{\mathscr{D}\in\mathfrak{S}(\cdot)} \lim\limits_{\mathscr{D}'\in f(\mathscr{D})} \mu\big((\cdot) \setminus \bigcup \mathscr{D}'\big)$ is exhaustive.*

Proof. Let (E_n) be a disjoint sequence in $\mathscr{R}$. By Lemma 3.6, for each $\mathscr{D}_n \in \mathfrak{S}[E_n]$, the closed neighborhood U and closed $V \in \mathscr{U}$ such that $V + V \subset U$, we have

$$\mu(\mathscr{D}_n) \in \mu^{\check{}}(E_n) + V,$$

hence

$$\mu'(E_n) \in \overline{\mu^{\check{}}(E_n) + V}.$$

But exhaustivity of μ implies exhaustivity of $\mu^{\check{}}$, then for $n \geqslant N$ we have $\mu^{\check{}}(E_n) \subset V$, so $\mu'(E_n) \in \overline{V+V} \subset U$, and $\mu'(E_n) \in U$ for $n \geqslant N$.

3.9. PROPOSITION. *If $\mu \in \mathrm{ea}(\mathscr{R}, H)$ and μ is s-precomplete, then μ' is $\mathfrak{S}$-additive.*

Proof. Let $\tilde{\mathscr{U}}$ be a uniformity induced by the topology τ (see [5]). Given any $U \in \tilde{\mathscr{U}}$, there exists $V \in \tilde{\mathscr{U}}$ such that $V \circ V \subset U$. Applying Lemma 3.5 for $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}[E]$, there exists $\mathscr{D}_0 \in f(\mathscr{D})$ such that if $\mathscr{D}_D \in \mathfrak{S}[D]$, $D \in \mathscr{D}$, then

$$\Big(O, \sum_{D\in\mathscr{D}'} \mu(\mathscr{D}_D)\Big) \in V,$$

for every $\mathscr{D}' \subset \mathscr{D} \setminus \mathscr{D}_0$.

Let $\mathscr{D}^* \in \mathfrak{S}[E]$, $\mathscr{D} \leqslant \mathscr{D}^*$ be such that

$$\big(\mu(\mathscr{D}^*), \mu'(E)\big) \in V.$$

Since $\mu(\mathscr{D}^*) = \sum\limits_{D\in\mathscr{D}} \mu(\mathscr{D}^* \mathring{\cap} D)$, for each $\mathscr{D}_1 \in f(\mathscr{D})$, $\mathscr{D}_0^1 \subset \mathscr{D}_1$, $\mathscr{D}_0$, $\mathscr{D}_1 \in f(\mathscr{D})$, we have

$$\Big(\mu(\mathscr{D}^*), \sum_{D\in\mathscr{D}_1} \mu(\mathscr{D}^* \mathring{\cap} D)\Big) \in V, \quad \text{for } \mathscr{D}_1 \supset \mathscr{D}_0.$$

Hence for $\mathscr{D}_1 \supset \mathscr{D}_0 \cup \mathscr{D}_0^1$, $\big(\mu'(E), \sum\limits_{D\in\mathscr{D}_1} \mu(\mathscr{D}^* \mathring{\cap} D)\big) \in V \circ V \subset U$.

3.10. PROPOSITION. *If $\mu \in \mathrm{ea}(\mathscr{R}, H)$ and μ is s-precomplete, $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}[E]$, then $\mu(E, \mathscr{D})$ exists.*

Proof. Suppose that a family $\big(\mu(E \setminus \bigcup \mathscr{D}') \colon \mathscr{D}' \in f(\mathscr{D})\big)$ does not satisfy (sC). Then there exist a sequence $(\mathscr{D}_n) \subset f(\mathscr{D})$ and a neighborhood U of O such that

$$\mathscr{D}_n \subset \mathscr{D}_{n+1} \subset \ldots,$$

and

$$\big(\mu(E \setminus \bigcup \mathscr{D}_n) + U\big) \cap \big(\mu(E \setminus \bigcup \mathscr{D}_{n+1}) + U\big) = \varnothing.$$

Write $F_n = E \setminus \bigcup \mathscr{D}_n$, for $n = 1, 2, \ldots$ Then $F_{n+1} \subset F_n$, so $\lim\limits_n \mu(F_n \setminus F_{n+1}) = O$. Hence for some $N \in \mathbf{N}$,

$$\mu(F_n \setminus F_{n+1}) \in U,$$

for $n \geqslant N$. But

$$\mu(F_n) = \mu(F_{n+1}) + \mu(F_n \setminus F_{n+1}) \subset \mu(F_{n+1}) + U,$$

so for $n = N$

$$\mu(E \setminus \bigcup \mathscr{D}_N) \in \mu(E \setminus \bigcup \mathscr{D}_{N+1}) + U.$$

This contradicts the assumption.

3.11. Remark. Analogously, $\mu(\bigcup \mathscr{D}, \mathscr{D}) = \lim\limits_{\mathscr{D}', \mathscr{D}'\in f(\mathscr{D})} \mu(\bigcup \mathscr{D} \setminus \bigcup \mathscr{D}')$ exists. But $\mu(E \setminus \bigcup \mathscr{D}') = \mu(E \setminus \bigcup \mathscr{D}) + \mu(\bigcup \mathscr{D} \setminus \bigcup \mathscr{D}')$, for every $\mathscr{D}' \in f(\mathscr{D})$. Then $\mu(E, \mathscr{D}) = \mu(E \setminus \bigcup \mathscr{D}) + \mu(\bigcup \mathscr{D}, \mathscr{D})$.

3.12. PROPOSITION. *If $\mu \in \mathrm{ea}(\mathscr{R}, H)$, μ is s-precomplete and H satisfies the cancellation laws ($x + y = x + z \Rightarrow y = z$), then $\mu''(E)$ exists for every $E \in \mathscr{R}$.*

Proof. By 3.10, $\mu(E, \mathscr{D})$ exists for every $\mathscr{D} \in \mathfrak{S}[E]$. Since for every $\mathscr{D}' \in f(\mathscr{D})$, we have

$$\mu(E) = \mu\big(E \setminus \bigcup \mathscr{D}'\big) + \mu\big(\bigcup \mathscr{D}'\big),$$

then

$$(*) \qquad \mu(E) = \lim_{\mathscr{D}' \in f(\mathscr{D})} \mu\big(E \setminus \bigcup \mathscr{D}'\big) + \lim_{\mathscr{D}' \in f(\mathscr{D})} \mu\big(\bigcup \mathscr{D}'\big) = \mu(E, \mathscr{D}) + \mu(\mathscr{D}).$$

Now by Proposition 3.7, the family $\{\mu(\mathscr{D}) \colon \mathscr{D} \in \mathfrak{S}[E]\}$, satisfies $(s\mathrm{C})$. Then for every $U \in \mathscr{U}$ there exists $\mathscr{D}_0 \in \mathfrak{S}[E]$ such that, for every $\mathscr{D}_j, \mathscr{D}_k \geqslant \mathscr{D}_0$, $\mathscr{D}_j, \mathscr{D}_k \in \mathfrak{S}[E]$ we have

$$\big(\mu(\mathscr{D}_j) + U\big) \cap \big(\mu(\mathscr{D}_k) + U\big) \neq \emptyset.$$

Hence, there exist $v_j, v_k \in U$ such that

$$\mu(\mathscr{D}_j) + v_j = \mu(\mathscr{D}_k) + v_k,$$

and by $(*)$

$$\mu(E, \mathscr{D}_j) + \mu(\mathscr{D}_j) = \mu(E, \mathscr{D}_k) + \mu(\mathscr{D}_k).$$

So

$$\mu(E, \mathscr{D}_j) + \mu(\mathscr{D}_j) + v_j = \mu(E, \mathscr{D}_k) + \mu(\mathscr{D}_k) + v_j,$$

$$\mu(E, \mathscr{D}_j) + \mu(\mathscr{D}_k) + v_k = \mu(E, \mathscr{D}_k) + \mu(\mathscr{D}_k) + v_j,$$

and by the cancellation laws,

$$\mu(E, \mathscr{D}_j) + v_k = \mu(E, \mathscr{D}_k) + v_j.$$

Hence a family $\{\mu(\mathscr{D}) \colon \mathscr{D} \in \mathfrak{S}[E]\}$ satisfies $(s\mathrm{C})$.

3.13. Proposition. *If $\mu \in \mathrm{ea}(\mathscr{R}, H)$ μ is s-precomplete and μ'' exists, then μ'' is an $\mathfrak{S}$-singular.*

Proof. Let η be $\mathfrak{S}$-continuous and $\eta \ll \mu''$. Suppose that for some $E \in \mathscr{R}$ we have $\eta(E) > \varepsilon > 0$. Let $U \in \mathscr{U}$ be such that $\mu''^{\vee}(F) \subset U$ implies $\eta(G) < \varepsilon$ for every $G \subset F$, $G \in \mathscr{R}$. Thus $\mu''^{\vee}(E) \not\subset U$, hence there exists $E_1 \subset E$, $E_1 \in \mathscr{R}$ such that $\mu''(E_1) \notin U$. So for each $\mathscr{D} \in \mathfrak{S}[E_1]$ there exists $\mathscr{D}_1 \in \mathfrak{S}[E_1]$ such that $\mu(E, \mathscr{D}_1) \notin U$ and $\mathscr{D}_1 \geqslant \mathscr{D}$, hence for every $\mathscr{D}_1' \in f(\mathscr{D}_1)$ there exists $\mathscr{D}_1'' \supset \mathscr{D}_1'$, $\mathscr{D}_1'' \in f(\mathscr{D}_1)$ such that $\mu\big(E \setminus \bigcup \mathscr{D}_1''\big) \notin U$. In other words,

$$(\cdot) \qquad \forall \mathscr{D} \in \mathfrak{S}[E_1]\ \exists \mathscr{D}_1 \geqslant \mathscr{D},\ \mathscr{D}_1 \in \mathfrak{S}[E_1]\ \forall \mathscr{D}_1' \in f(\mathscr{D}_1)\ \exists \mathscr{D}_1'' \supset \mathscr{D}_1,$$
$$\mathscr{D}_1'' \in f(\mathscr{D}_1) \text{ such that } \mu\big(E_1 \setminus \bigcup \mathscr{D}_1''\big) \notin U.$$

Now let $\mathscr{D}_1 \in \mathfrak{S}[E_1]$ have the property $(\cdot)$. Given any $\mathscr{D}_2 \in \mathfrak{S}[E \setminus E_1]$, denote $\mathscr{D}_0 = \mathscr{D}_1 \cup \mathscr{D}_2$, it is clear that $\mathscr{D}_0 \in \mathfrak{S}[E]$. Since η is $\mathfrak{S}$-continuous and $\eta(E) > \varepsilon$, we can find $\mathscr{D}_0' \in f(\mathscr{D}_0)$ with $\eta\big(\bigcup \mathscr{D}_0'\big) > \varepsilon$. Let $\mu''^{\vee} = (\mu'')^{\vee}$. Hence $\mu''^{\vee}\big(\bigcup \mathscr{D}_0'\big) \not\subset U$. But $\mathscr{D}_0 = \mathscr{D}_1 \cup \mathscr{D}_2$ and $\mathscr{D}_1 \in \mathfrak{S}[E_1]$ and $\mathscr{D}_2 \in \mathfrak{S}[E \setminus E_1]$, then $\mathscr{D}_0' = \mathscr{D}_1' \cup \mathscr{D}_2'$, where $\mathscr{D}_1' \in f(\mathscr{D}_1)$, $\mathscr{D}_2' \in f(\mathscr{D}_2)$ and $\bigcup \mathscr{D}_1 \subset E$, $\bigcup \mathscr{D}_2' \subset E \setminus E_1$. Hence by $(\cdot)$, there exists $\mathscr{D}_1'' \supset \mathscr{D}_1$, $\mathscr{D}_1'' \in f(\mathscr{D}_1)$ such that $\mu\big(E_1 \setminus \bigcup \mathscr{D}_1''\big) \not\subset U$, this implies $\mu''^{\vee}\big(E_1 \setminus \bigcup \mathscr{D}_1''\big) \not\subset U$.

Let us denote $F_1 = \bigcup \mathscr{D}_0'$, $G_1 = E_1 \setminus \bigcup \mathscr{D}_1''$. We have $F_1, G_1 \subset E$, $F_1 \cap G_1 = \emptyset$, and $\eta(F_1) > \varepsilon$, $\mu''^{\vee}(F_1) \not\subset U$, $\mu''^{\vee}(G_1) \not\subset U$. Applying the same argument to the set F_1, we shall find a set $F_2, G_2 \subset F_1$ such that $F_2 \cap G_2 = \emptyset$, and $\eta(F_2) > \varepsilon$, $\mu''^{\vee}(F_2) \not\subset U$, $\mu''^{\vee}(G_2) \not\subset U$, etc. Thus there exists a disjoint sequence of sets $G_n \in \mathscr{R}$ such that $\mu''^{\vee}(G_n) \not\subset U$ for $n = 1, 2, \ldots$ This contradicts to the exhaustivity of μ.

Now by Lemma 2.4, we have the following theorem.

3.14. Theorem. *Let $\mu \in \mathrm{ea}(\mathscr{R}, H)$, if μ is s-precomplete and H satisfies the cancellation laws, then μ can be written in the form*

$$\mu = \mu' + \mu'',$$

where $\mu', \mu'' \in \mathrm{ea}(\mathscr{R}, H)$, μ' is $\mathfrak{S}$-additive and μ'' is $\mathfrak{S}$-singular.

Proof. For every $E \in \mathscr{R}$, $\mathscr{D} \in \mathfrak{S}[E]$ and $\mathscr{D}' \in f(\mathscr{D})$ we have

$$\mu(E) = \mu\big(\bigcup \mathscr{D}'\big) + \mu\big(E \setminus \bigcup \mathscr{D}'\big),$$

but $\mu'(E)$, $\mu''(E)$ exist by Propositions 3.7, 3.12 and have the above properties by Propositions 3.8, 3.9 and 3.13. Hence

$$\mu(E) = \mu'(E) + \mu''(E), \qquad \text{for every} \quad E \in \mathscr{R}.$$

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References

[1] L. Drewnowski, *Decompositions of set functions*, Studia Math. 48 (1973), pp. 23–48.
[2] — *Topological rings of sets, continuous set functions, integration I, II, III*, Bull. Acad. Polon. Sci. Sér. sci. math., astr. et phys. 20 (1972), pp. 269–276, 277–286, 439–445.
[3] N. Dunford and J. T. Schwartz, *Linear operators, Part I*, New York 1958.
[4] M. Sion, *A theory of semigroup valued measures*, Springer-Verlag, Berlin–Heidelberg–New York 1973.
[5] H. Schaefer, *Topological vector spaces*, New York 1966.
[6] T. Traynor, *A general Hewitt–Yosida decomposition*, Canad. J. Math. 24 (1972), pp. 1164–1169.

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