STUDIA MATHEMATICA, T. LXI. (1977)

On the heredity of weak compactness in biprojective tensor product spaces

by

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Abstract. For a fairly large class of locally convex spaces, it is therein verified that the usual variations of the notion of weak compactness are hereditary on biprojective tensor product vector spaces.

1. Introduction. Let E and F be locally convex spaces, $G = E \otimes F$ their biprojective tensor product vector space and let A and B be subsets of E and F, respectively. In this article we are interested in a verification of the various usual typs of weak compactness on $A \otimes B$, which are derived by heredity from those of the factors A and B. Thus, it is first proved, without restrictions to E and F, that (relative) weak sequential compactness and weak conditional compactness are hereditary (cf. Proposition 4.1 and Theorem 4.4 below). On the other hand, if E (or) F is a Šmulian space (Definition 2.1) and A and B are (relatively) weakly countably compact (resp. relatively weakly compact), then $A \otimes B$ is (relatively) weakly countably compact (resp. relatively (weakly) $\sigma(\hat{G}, G')$-compact, where $\hat{G}$ is the completion of G) (cf. Theorems 4.2 and 4.3 below). In particular, Lemma 3.1 and Theorem 4.3 below constitute, for the case under consideration, extended forms of the results of Lewis ([4]; Lemma 2.1) and Junghenn ([1]; Lemma 3.1), respectively, which also motivated the present work. Moreover, the techniques applied therein, being interesting in themselves, are essentially different from those of [4] and [1] indicated above.

2. Preliminaries. All vector spaces considered in the following are over the field C of complex numbers. The topological spaces involved are assumed to be Hausdorff unless otherwise indicated. If E is a locally convex space, in the sequel we denote by E^* and E' the algebraic and the topological dual of E, respectively. On the other hand, we use the leter N for the set of natural numbers.

Now, let E be a locally convex space. Then, a subset B of E is said to be (*relatively*) *countably compact* if every sequence in B has a cluster point (in E) in B. B is said to be (*relatively*) *sequentially compact* if each sequence in B possesses a subsequence converging to a point (of E) of B.

B is called *relatively compact* if its closed hull $\bar{B}$ in E is compact. It is immediate that (relative) compactness of B and (relative) sequential compactness of B both imply (relative) countable compactness of B. Furthermore, if B is relatively weakly countably compact, then it is also bounded (cf. [3], 24.1 (1)). On the other hand, B is called *conditionally compact* if every sequence in B has a Cauchy subsequence. In this case, following the arguments of Theorem 7.7 of [2], it is not difficult to show that B is precompact and hence bounded.

We now state the following

DEFINITION 2.1. By a *Šmulian space* we mean a locally convex space E with the property that each relatively weakly countably compact subset of E is relatively weakly sequentially compact.

A locally convex space (E, τ) such that there exists a metrizable locally convex topology on E, which is coarser than τ, is a Šmulian space (cf. [3], 24.1 (3)). Thus, in particular, if E is a metrizable locally convex space, or a strict (LF)-space, or E has weakly separable topological dual E', then E is a Šmulian space (cf. also [3], 24.1 (4) and 24.1 (2)).

3. On convergence of sequences in biprojective tensor product spaces. In this section we obtain certain interesting results, which also will be essentially applied in the next section.

Thus, let $\langle E, F\rangle$ be a dual pair of vector spaces and let $\mathcal{S}$ be a (directed) family of $\sigma(F, E)$-bounded subsets of F, which covers F. We consider the (Hausdorff) locally convex topology τ of $\mathcal{S}$-convergence on E and the topological dual E' of the locally convex space (E, τ). By the definition of τ, the (absolute) polars $A^\circ \subseteq E$ of the members A of $\mathcal{S}$ constitute a local base for τ and hence F is a vector subspace of E'. On the other hand, as is known, a subset M of E' is τ-equicontinuous if and only if it is contained in the bipolar $A^{\circ\circ}$ (with respect to the duality $\langle E, E'\rangle$) of some $A \in \mathcal{S}$. Thus, in particular, each $A \in \mathcal{S}$ is τ-equicontinuous so that, by [5] (III, 4.3), any $A \in \mathcal{S}$ is $\sigma(E', E)$-closed if and only if it is $\sigma(E^*, E)$-closed.

First, we get the following result which extends the corresponding part of [4], Lemma 1.1:

LEMMA 3.1. *Let $\langle E, F\rangle$ be a dual pair of vector spaces and let $\mathcal{S}$ be a (directed) cover of F, consisting of $\sigma(F, E)$-bounded and $\sigma(E^*, E)$-closed subsets of F. We consider the locally convex topology τ of $\mathcal{S}$-convergence on E and the topological dual E' of the locally convex space (E, τ). Then the topologies $\sigma(E, F)$ and $\sigma(E, E')$ have the same τ-bounded null sequences in E.*

Proof. Let (x_n) be a τ-bounded, $\sigma(E, F)$-null sequence in E. Then the sequence (x_n) is clearly uniformly bounded on each $A \in \mathcal{S}$, because A is τ-equicontinuous. On the other hand, by hypothesis, the preceding comments, and the theorem of Alaoglu–Bourbaki, each $B \in \mathcal{S}$ is $\sigma(E', E)$-compact. Moreover, if $z' \in E'$, then, by the foregoing comments and the theorem of (absolute) bipolars, there exists $B \in \mathcal{S}$ such that $z' \in B^{\circ\circ} = \overline{\Gamma(B)}$, the absolutely convex $\sigma(E', E)$-closed hull of B. Now, by the hypothesis, one has: $\lim_n \langle x_n, x'\rangle = 0$ for every $x' \in B$ so that, by [5] (IV, 11.3), we finally have $\lim_n \langle x_n, z'\rangle = 0$, so (x_n) is a $\sigma(E, E')$-null sequence which clearly yields the assertion.

On the other hand, by Proposition 1.4 of [6] and the preceding lemma, we also obviously have

LEMMA 3.2. *Under the assumptions of Lemma 3.1, the topologies $\sigma(E, F)$ and $\sigma(E, E')$ have the same τ-bounded Cauchy sequences in E.*

Let E and F be locally convex spaces and let $E \otimes F$ be the corresponding tensor product vector space. Then the respective *biprojective (tensorial) locally convex topology* is the topology ε of $\mathcal{S}$-convergence on $E \otimes F$, where $\mathcal{S}$ is the family of the sets $A' \otimes B'$ with A' and B' weakly closed equicontinuous subsets of E' and F', respectively. The locally convex space $G := E \otimes_\varepsilon F$ thus obtained is referred to as the (respective) *biprojective tensor product space*. Now, by the comments preceding Lemma 3.1, each set $A' \otimes B'$ is ε-equicontinuous. On the other hand, by the fact that the canonical bilinear map $\otimes \colon E' \times F' \mapsto (E \otimes_\varepsilon F)'$ is clearly weakly continuous, it is easily verified that $A' \otimes B'$ is (weakly) $\sigma(G', G)$-compact and hence (weakly) $\sigma(G', G)$-closed. Moreover, if A and B are subsets of E and F, respectively, both containing non-zero elements, then, by the continuity of the canonical bilinear map $\otimes \colon E \times F \mapsto E \otimes_\varepsilon F$, it easily follows that $A \otimes B$ is ε-bounded if and only if both A and B are bounded.

Now we are in a position to prove the following interesting result which allows an extention of [1], Lemma 2.1 (cf. also Theorem 4.3 below) and will be essentially applied in the next section. That is, we have:

PROPOSITION 3.3. *Let E and F be locally convex space, $G := E \otimes_\varepsilon F$ the respective biprojective tensor product space and let G' be the topological dual of G. Moreover, let (x_n) be a weakly null sequence in E and let (y_n) be a bounded sequence in F, having a weak cluster point y in F. Then $0 = 0 \otimes y$ in G is a (weak) $\sigma(G, G')$-cluster point of the sequence $(x_n \otimes y_n)$.*

Proof. First, let $(z_a)_{a \in A}$ be a subnet of the sequence $(y_n)_{n \in \boldsymbol{N}}$, defined by the map $q\colon A \mapsto \boldsymbol{N}$, which net converges weakly to y. On the other hand, we consider the subnets $(u_\beta)_{\beta \in B}$ and $(v_\beta)_{\beta \in B}$, where $B = \boldsymbol{N} \times A$, of $(x_n)_{n \in \boldsymbol{N}}$ and $(z_a)_{a \in A}$, respectively, with $u_\beta = x_{\mathrm{pr}_1(\beta)}$ and $v_\beta = y_{q(\mathrm{pr}_2(\beta))}$, which converge weakly to 0 and y, respectively. Now the net $(u_\beta \otimes v_\beta)_{\beta \in B}$ converges weakly to $0 = 0 \otimes y$ in G. In fact, suppose this is not the case; that means, there exist $\varepsilon > 0$, an element φ of the topological dual G'

of G and a cofinal subset D of B such that

$$\varphi(u_\delta \otimes v_\delta) \geqslant \varepsilon \quad \text{for every } \delta \in D. \tag{3.1}$$

Now, by the definition of the topology ε on $E \otimes F$ (cf. also the related comments before Lemma 3.1), there exist weakly closed equicontinuous subsets A' and B' of the topological duals E' and F', respectively, such that $\varphi \in (A' \otimes B')^{\circ\circ} = \overline{\Gamma(A' \otimes B')}$, the absolutely convex $\sigma(G', G)$-closed hull of $A' \otimes B'$. Moreover, by the comments following Lemma 3.2, $A' \otimes B'$ is ε-equicontinuous and $\sigma(G', G)$-compact. Now $M := \mathrm{pr}_1(D)$ is clearly a cofinal subset of N such that for every $\mu \in M$ there exists $\delta_\mu \in A$ with $(\mu, \delta_\mu) \in D$. Thus, we may consider the sequences $\bar{x}_\mu = u_{(\mu,\delta_\mu)} = x_\mu$ and $\bar{y}_\mu = v_{(\mu,\delta_\mu)} = y_{a(\delta_\mu)}$. In this respect, we remark that $(\bar{x}_\mu)$, being a cofinal subnet of (x_n), converges weakly to 0 and, by the equicontinuity of A', B' and $A' \otimes B'$, the sequences $(\bar{y}_\mu)$ and $(\bar{x}_\mu \otimes \bar{y}_\mu)$ are clearly uniformly bounded on B' and $A' \otimes B'$, respectively, so that let $\lambda > 0$ with $|\langle \bar{y}_\mu, y' \rangle| \leqslant \lambda$ for every $y' \in B'$. Now we have

$$|\langle \bar{x}_\mu \otimes \bar{y}_\mu, x' \otimes y' \rangle| = |\langle \bar{x}_\mu, x' \rangle| \cdot |\langle \bar{y}_\mu, y' \rangle| \leqslant \lambda |\langle \bar{x}_\mu, x' \rangle| \to 0$$

for every $(x', y') \in A' \times B'$ so that, by [5] (IV, 11.3) and the foregoing, we get $\lim\limits_n \varphi(\bar{x}_\mu \otimes \bar{y}_\mu) = 0$, a contradiction to (3.1) above.

Finally, as it is easily verified by the definition of the net $(u_\beta \otimes v_\beta)_{\beta \in B}$, $0 = 0 \otimes y$ is a (weak) $\sigma(G, G')$-cluster point of the sequence $(x_n \otimes y_n)$ and the proof is completed.

4. On weak compactness in biprojective tensor product spaces. Now, by applying the results of Section 3, we are in position to state and prove the main results of this paper, concerning hereditary variations of the notion of weak compactness on biprojective tensor product spaces.

Thus, we first have

PROPOSITION 4.1. *Let E and F be locally convex spaces, $G := E \underset{\varepsilon}{\otimes} F$ the respective biprojective tensor product space and let A and B be relatively weakly sequentially compact subsets of E and F, respectively. Then $A \otimes B$ is relatively (weakly) $\sigma(G, G')$-sequentially compact.*

Proof. Let (x_n) and (y_n) be sequences in A and B converging weakly to x and y in E and F, respectively. Then for every $x' \in E'$ and $y' \in F'$ it obviously follows

$$\langle x_n \otimes y_n - x \otimes y, x' \otimes y' \rangle = \langle x_n - x, x' \rangle \langle y_n, y' \rangle + \langle x, x' \rangle \langle y_n - y, y' \rangle,$$

that is, $(x_n \otimes y_n)$ converges to $x \otimes y$ in the topology $\sigma(E \otimes F, E' \otimes F')$, so that by Lemma 3.1 above $(x_n \otimes y_n)$ converges to $x \otimes y$ in the (weak) $\sigma(G, G')$-topology, which clearly yields the assertion.

THEOREM 4.2. *Let E and F be locally convex spaces such that E is a Šmulian space, $G := E \underset{\varepsilon}{\otimes} F$ the respective biprojective tensor product space and let A and B be relatively weakly countably compact subsets of E and F, respectively. Then $A \otimes B$ is relatively (weakly) $\sigma(G, G')$-countably compact.*

Proof. Let (x_n) and (y_n) be sequences in A and B having the elements x and y of E and F as weak cluster points, respectively. By the assumption that E is a Šmulian space, we may suppose that $x_n \to x$ weakly. Then, by the relation

$$x_n \otimes y_n - x \otimes y = (x_n - x) \otimes y_n + x \otimes (y_n - y),$$

the (separate) weak continuity of the tensors and the Proposition 3.3 above, it obviously follows that $x \otimes y$ is a (weak) $\sigma(G, G')$-cluster point of the sequence $(x_n \otimes y_n)$ which proves the assertion.

On the other hand, we also get the following result which, by the comments following Definition 2.1, extends it ([1], Lemma 3.1). That is, we have

THEOREM 4.3. *Let E and F be locally convex spaces such that E is a Šmulian space, $G := E \underset{\varepsilon}{\otimes} F$ the corresponding biprojective tensor product space and let A and B be relatively weakly compact subsets of E and F, respectively. Then $A \otimes B$ is relatively (weakly) $\sigma(\hat{G}, G')$-compact, where $\hat{G}$ is the completion of G.*

Proof. By the preceding theorem, $A \otimes B$ is relatively (weakly) $\sigma(G, G')$-countably compact and hence, clearly, relatively (weakly) $\sigma(\hat{G}, G')$-countably compact, so that the assertion is now obtained by [5], IV, Theorem 11.2, and the proof is finished.

Moreover, we finally prove

THEOREM 4.4. *Let E and F be locally convex spaces, $G := E \underset{\varepsilon}{\otimes} F$ the respective biprojective tensor product space and let A and B be subsets of E and F, respectively, both containing non-zero elements. Then, the following assertions are equivalent:*

(1) *Each of A and B is conditionally weakly compact.*

(2) *$A \otimes B$ is conditionally (weakly) $\sigma(G, G')$-compact.*

Proof. (1) *implies* (2). Let $(x_n \otimes y_n)_{n \in N}$ be a sequence in $A \otimes B$, where (x_n) and (y_n) are weakly Cauchy sequences in A and B, respectively and let $x' \in E'$ and $y' \in F'$. Then, by the fact that each of (x_n) and (y_n) is bounded, there are $\lambda > 0$ and $\mu > 0$ with $|\langle x_n, x' \rangle| \leqslant \lambda$ and $|\langle y_n, y' \rangle| \leqslant \mu$ for every $n \in N$. Thus, for every $n, m \in N$ we obviously have

$$\begin{aligned} &|\langle x_n \otimes y_n - x_m \otimes y_m, x' \otimes y' \rangle| \\ &\qquad \leqslant |\langle (x_n - x_m) \otimes y_n, x' \otimes y' \rangle| + |\langle x_m \otimes (y_n - y_m), x' \otimes y' \rangle| \\ &\qquad \leqslant \mu |\langle x_n - x_m, x' \rangle| + \lambda |\langle y_n - y_m, y' \rangle|, \end{aligned}$$

that is, $(x_n \otimes y_n)$ is clearly a (weak) $\sigma(E \otimes F, E' \otimes F')$-Cauchy sequence. On the other hand, by hypothesis (cf. also Section 2), it follows that $A \otimes B$ is ε-bounded, so that, by Lemma 3.2, $(x_n \otimes y_n)$ is a $\sigma(G, G')$-Cauchy sequence in $A \otimes B$, that is, $A \otimes B$ is conditionally (weakly) $\sigma(G, G')$-compact.

(2) *implies* (1). Let $(z_n)_{n \in N}$ be a sequence in A and let $y \in B$ with $y \neq 0$. Then there exists by hypothesis a subsequence (x_n) of (z_n) such that $(x_n \otimes y)$ is a (weakly) $\sigma(G, G')$-Cauchy sequence. Now if $x' \in E'$ and $y' \in F'$ with $|\langle y, y'\rangle| = 1$, then for every $n, m \in N$ obviously follows

$$|\langle x_n \otimes y - x_m \otimes y, x' \otimes y'\rangle| = |\langle (x_n - x_m) \otimes y, x' \otimes y'\rangle| = |\langle x_n - x_m, x'\rangle| \cdot |\langle y, y'\rangle| = |\langle x_n - x_m, x'\rangle|.$$

Thus, (x_n) is clearly a weak Cauchy sequence in A and hence A is conditionally weakly compact and the proof is completed.

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Received January 28, 1975 (941)

STUDIA MATHEMATICA, T. LXI. (1977)

The algebra of finitely additive measures on a partially ordered semigroup

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Abstract. The algebra of all finitely additive measures on a discrete semigroup which is a product of totally ordered sets provided with the multiplication max is studied. It is found that all proper maximal left ideals are the kernels of complex homomorphisms, and that the quotient of the algebra by its radical is isomorphic with the usual measure algebra on the almost periodic compactification of the original semigroup.

The algebra of all finitely additive measures on a discrete semigroup is in general very difficult to study. This is in part because any other algebra of measures (finite or countably additive) on the same semigroup provided with any topology can be obtained as a quotient of this one, and so we are in a sense asking to study all these algebras at once. Success therefore depends on severely restricting the class of semigroups under consideration. This policy was followed in [5] where we treated the case of a totally ordered semigroup (that is, a totally ordered set given the multiplication max). In the present paper, we offer similar results for finite direct products of such semigroups.

In Section 2, we show that every maximal left ideal of the algebra (which is, of course, non-commutative) is the kernel of a complex-valued homomorphism, and thus is two-sided. The exact form of the complex homomorphisms is in fact given in Theorem 2.9. The quotient of the algebra by its radical is therefore commutative; it turns out to be the algebra of countably additive measures on a certain compact semigroup. This semigroup is the almost periodic compactification of the original semigroup (and in fact coincides with the weakly almost periodic compactification, which we found in [5]). Moreover, it is a finite product of compact totally ordered semigroups.

Our justifications for presenting these results are first, that the appearance of the almost periodic (rather than the weakly almost periodic) compactification should be recorded. Secondly, the proofs we gave in [5] do not extend to the present case. Moreover, although the greater generality gives an appearance of greater complexity, the methods of this