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DEPARTAMENTO DE ECUACIONES FUNCTIONALES
FACULTAD DE CIENCIAS MATHEMÁTICAS
UNIVERSIDAD COMPLUTENSE, C. U. MADRID-3, SPAIN

Received November 19, 1975 (1093)

STUDIA MATHEMATICA, T. LX. (1977)

Antisymmetry of subalgebras of C^*-algebras

by

WACŁAW SZYMAŃSKI (Cracow)

Abstract. In the present paper we introduce a generalization of antisymmetric sets, known in the function algebras theory, to a noncommutative case. We prove a de Branges-type theorem and a generalization of the Bishop decomposition theorem. As applications we prove a version of the Stone–Weierstrass theorem and an approximation-type result in connection with the Bishop decomposition proved earlier.

1. Preliminaries. $L(H)$ stands for the C^*-algebra of all linear, bounded operators in a complex Hilbert space H. A *-homomorphism π of a C^*-algebra A into $L(H_\pi)$ is called a *representation of* A, the dimension of H_π is called the *dimension of* π. Characters of a C^*-algebra A are one-dimensional representations of A. A representation π of a C^*-algebra A is called *irreducible* if the algebra $\pi(A)$ has no non-trivial invariant subspace in $L(H_\pi)$. If A has the unit e, we will assume always that, for every representation π of A, $\pi(e) = I_\pi$ — the identity operator in H_π.

If $\mathscr{S}$ is a subset of $L(H)$ we denote by $C^*(\mathscr{S})$ the C^*-algebra generated by $\mathscr{S}$ and the identity. If $T \in L(H)$, we write $C^*(T)$ for $C^*(\{T\})$. By the *spectrum* $\hat{A}$ *of a C^*-algebra* A we mean the set of unitary equivalence classes of all irreducible representations of A equipped with the hull-kernel topology. For a subset K of $\hat{A}$ we write $J(K) = \bigcap\{\ker\varrho,\ \varrho\in K\}$. If J is a closed, two-sided ideal in A, then by the *hull of* J we mean the set hull(J) consisting of all $\pi\in\hat{A}$ such that $J \subset \ker\pi$. It follows from [2], 2.9.7 (ii), that $J = J\big(\mathrm{hull}(J)\big)$. The closure $\bar{K}$ of a subset K of $\hat{A}$ in that topology is equal to $\mathrm{hull}\big(J(K)\big)$, by the definition.

If two C^*-algebras are *-isomorphic, then their spectra are homeomorphic. Namely, if $\varphi\colon A_1\to A_2$ is a *-isomorphism of the C^*-algebras A_1, A_2, then the mapping $\hat{\varphi}\colon \hat{A}_1\to\hat{A}_2$ given by the formula $\hat{\varphi}\colon \varrho\to\varrho\circ\varphi^{-1}$ is the homeomorphism induced by φ. For basic facts concerning C^*-algebras we refer to [2].

2. Sets of antisymmetry. To begin with, we recall two results due to de Branges, Bishop and Glicksberg [3].

Let X be a compact Hausdorff space and let $B \subset C(X)$ be a function algebra. $B^\perp$ denotes the set of all finite, complex (regular, Borel) measures

μ on X such that $\int f d\mu = 0$ for all $f \in B$, $\operatorname{ball} B^{\perp}$ stands for the closed unit ball in $B^{\perp}$.

A subset σ of X is called *antisymmetric for* B if every function $f \in B$ real-valued on σ is constant on σ.

THEOREM A (de Branges). *Let B be a function algebra in $C(X)$. If μ is an extreme point of* $\operatorname{ball} B^{\perp}$, *then the carrier of μ is an antisymmetric set for B.*

THEOREM B (The Bishop decomposition). *Let B be a function algebra in $C(X)$. There is a family $\mathscr{K}$ of subsets of X such that $X = \bigcup \mathscr{K}$ is a partition of X and:*

(1) *Every $K \in \mathscr{K}$ is a maximal antisymmetric set for B;*

(2) *If $f \in C(X)$ and if for every $K \in \mathscr{K}$ $f|_K \in B|_K$, then $f \in B$.*

Our main purpose is to prove generalizations of these theorems to a noncommutative case.

Let A be a C^*-algebra with the unit e. Let $B \subset A$ be a subalgebra of A containing e, Z denotes the center of A.

For a subset K of $\hat{A}$ we define:

$$D_K(B) = \{b \in B\colon\ b - b^* \in J(K),\ \forall\, a \in A\ \ ab - ba \in J(K)\}.$$

Observe that for $\pi \in K$, $b \in D_K(B)$ $\pi(b)$ is a self-adjoint element in the center of $\pi(A)$. Since π is irreducible, $\pi(b) = \alpha_\pi I_\pi$ for some $\alpha_\pi \in \boldsymbol{R}$ (reals). Hence

$$D_K(B) = \{b \in B\colon\ \forall \pi \in K,\ \exists\, \alpha_\pi \in \boldsymbol{R}\colon\ \pi(b) = \alpha_\pi I_\pi\}.$$

DEFINITION. A subset K of $\hat{A}$ is *antisymmetric* (a *set of antisymmetry*) *for* B if for every $b \in D_K(B)$ there is $r \in \boldsymbol{R}$ such that, for all $\pi \in K$, $\pi(b) = rI_\pi$; in other words, all the α_π's above are equal. A subalgebra B is called *antisymmetric* if $\hat{A}$ is an antisymmetric set for B.

From the definition follows immediately that one-point subsets of $\hat{A}$ are antisymmetric sets for any subalgebra B of A. The definition implies also that all self-adjoint elements of $Z \cap B$ belong to $D_k(B)$ for every $K \subset \hat{A}$. If A is a simple C^*-algebra (i.e. A has no proper ideals), then for every $\pi \in \hat{A}$ $\ker \pi = \{0\}$. Then, for every subalgebra B of a simple C^*-algebra A and for every subset K of $\hat{A}$, $D_K(B)$ consists only of self-adjoint elements of $Z \cap B$. Simple C^*-algebras exist. For example, every UHF algebra is simple (see [6], p. 88).

Compare now our definition of the antisymmetry with the classical one (as at the beginning of this section). If $A = C(X)$ then $\hat{A} = X$ and if B is a function algebra in A then $D_K(B)$ is precisely the set of all $f \in B$ which have real values on $K \subset X$. Now it is plain that the above definition of antisymmetric sets is a natural generalization of the classical one.

First we give some immediate properties of antisymmetric sets Assume that A is a C^*-algebra with the unit e and with the center Z, B is a subalgebra of A such that $e \in B$.

Remark 1. *If $K \subset \hat{A}$ is an antisymmetric set for B, then so is its closure $\overline{K}$.*

Proof. Since $J(\overline{K}) = J(K)$, we have $D_{\overline{K}}(B) = D_K(B)$. Take $b \in D_{\overline{K}}(B)$ $\pi \in \overline{K}$. If $\varrho(b) = rI_\varrho$ for all $\varrho \in K$, then $b - re \in J(K)$, hence $\pi(b - re) = 0$ and the proof is complete.

Remark 2. *If K_1, K_2 are two antisymmetric sets for B such that $K_1 \cap K_2 \neq \emptyset$, then $K_1 \cup K_2$ is an antisymmetric set for B.*

Proof. Define $K = K_1 \cup K_2$. It is clear that $J(K) = J(K_1) \cap J(K_2)$. Hence $D_K(B) = D_{K_1}(B) \cap D_{k_2}(B)$. Take $b \in D_K(B)$. There are $r_1, r_2 \in \boldsymbol{R}$ such that for every $\pi_1 \in K_1$ $\pi_1(b) = r_1 I_{\pi_1}$ and for every $\pi_2 \in K_2$ $\pi_2(b) = r_2 I_{\pi_2}$. Since $K_1 \cap K_2 \neq \emptyset$, we must have $r_1 = r_2$ and Remark 2 is proved.

Remark 3. *Every antisymmetric set for B is contained in a maximal antisymmetric set for B. Maximal antisymmetric sets are closed.*

It follows from Remarks 1, 2 and from the Kuratowski–Zorn lemma.

Remark 4. *Suppose that all self-adjoint elements of $Z \cap B$ are scalar multiples of e. Then B is an antisymmetric algebra.*

Proof. It is known that $J(\hat{A}) = \bigcap \{\ker \varrho\colon\ \varrho \in \hat{A}\} = \{0\}$. Hence $D_{\hat{A}}(B) = \{b \in B\colon\ b \in Z,\ b = b^*\}$ and the proof is complete.

This Remark implies in particular that if the center Z is trivial, then every subalgebra B of A is antisymmetric.

The next Remark shows that the property of the antisymmetry is algebraic.

Remark 5. *Suppose that A_1, A_2 are two C^*-algebras with units e_1, e_2, respectively. Let $\varphi\colon A_1 \to A_2$ be a $*$-isomorphism. Let $B \subset A_1$ be a subalgebra of A_1 containing e_1. If K is an antisymmetric set for B, then $\hat{\varphi}(K)$ (defined in Preliminaries) is an antisymmetric set for $\varphi(B)$.*

Proof. It is easy to check that $D_{\hat{\varphi}(K)}(\varphi(B)) = \varphi(D_K(B))$. Recall that $\hat{\varphi}(\varrho) = \varrho \circ \varphi^{-1}$ for $\varrho \in \hat{A}_1$. Let $c \in D_{\hat{\varphi}(K)}(\varphi(B))$. Then there is $b \in D_K(B)$ such that $\varphi(b) = c$. If $\pi \in \hat{\varphi}(K)$ then $\pi = \varrho \circ \varphi^{-1}$ with some $\varrho \in K$. Now $\pi(c) = (\varrho \circ \varphi)(\varphi^{-1}(b)) = \varrho(b)$ and, by the assumption, K is antisymmetric for B. Hence $\pi(c) = rI_\pi$ for all $\pi \in \hat{\varphi}(K)$ and the proof is concluded.

Let A be an irreducible C^* operator algebra in $L(H)$. Remark 4 implies that A is antisymmetric. It follows from Remark 5 that an antisymmetric operator algebra need not be irreducible. As an example consider the unilateral shift U_+ of the multiplicity one. The algebra $C^*(U_+)$ is irreducible. But the algebra $C^*(U_+^2)$ is not irreducible and it is $*$-isomorphic with $C^*(U_+)$, hence, it is antisymmetric.

Now we will give an example of antisymmetric sets.

EXAMPLE 1. Denote by $J(n)$ the usual Jordan block in C^n:

$$J(n) = \begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}.$$

We define $H = \bigoplus_{n=2}^{\infty} C^n$, $S = \bigoplus_{n=2}^{\infty} J(n)$, $A = C^*(S)$. S is a power partial isometry (in the terminology of [4]). The description of $\hat{A}$ is given in [1], [4]. Namely, all irreducible representations of A have the forms:

(a) $\varphi(S) = e^{it}$ with some $t \in [0, 2\pi]$,

(b) $\pi_1(S) = U_+$,

(c) $\pi_2(S) = U_+^*$,

(d) $\varrho_n(S) = J(n)$, $n = 2, 3, \ldots$

up to the unitary equivalence. Denote by Γ the set of all characters of A, $L = \{\varrho_n, n = 2, 3, \ldots\}$. The values of all $\varphi \in \Gamma$ at S cover the whole unit circle. As for the topology of $\hat{A}$; Γ is closed, $\overline{\{\pi_i\}} = \{\pi_i\} \cup \Gamma$, $i = 1, 2$, L is dense in $\hat{A}$ (we omit the rather simple proof).

One can check that the center of A consists only of scalar multiples of the identity I in H. Hence, by Remark 4, $\hat{A}$ is an antisymmetric set for every subalgebra of A. Now we want to consider a subalgebra B of A to show some, other than $\hat{A}$, antisymmetric sets for it. Let $E = S^*S$ be the initial and $F = SS^*$ — the final projection of S. Let B be the closed in norm subalgebra (not symmetric) of A generated by S, E, F and I. It is clear that B is not commutative ($SE \neq ES$). Moreover, $B \neq A$, because $S^* \notin B$. For the distance in norm of S^* to the algebra of all lower-triangular matrices is equal to one and all elements of B are lower-triangular. We claim that Γ is an antisymmetric set for B.

For the proof let us determine the set $D_\Gamma(B)$. The algebra B is the closure in norm of all operators $p(E, F, S)$, where $p(x, y, z)$ is a polynomial in three variables x, y, z such that x and y commute and z does not commute with x, y. Let $\deg_z p$ denote the degree of p with respect to z. The elements of $D_\Gamma(B)$ are those operators $p(E, F, S)$ (and their norm-limits) for which $\varphi(p(E, F, S))$ are real for all $\varphi \in \Gamma$. Observe that for an arbitrary $\varphi \in \Gamma$ we have $\varphi(E) = \varphi(F) = 1$. Take $p(E, F, S)$ from B. For $\varphi \in \Gamma$ we have

$$\varphi(p(E, F, S)) = p(\varphi(E), \varphi(F), \varphi(S)) = p(1, 1, e^{it}),$$

where $\varphi(S) = e^{it}$, $t \in [0, 2\pi]$. This equality proves that $p(E, F, S)$ is in $D_\Gamma(B)$ if and only if $p(1, 1, e^{it})$ is real for every $t \in [0, 2\pi]$. The polynomial

$z \to p(1, 1, z)$ has real values on the unit circle if and only if $\deg_z p = 0$. But if $\deg_z p = 0$, then $p(1, 1, z)$ equals to the sum of all coefficients of p. It follows that $p(E, F, S)$ belongs to $D_\Gamma(B)$ if and only if $\deg_z p = 0$ and the sum of all coefficients of p is real. Hence $D_\Gamma(B)$ consists of norm-limits of such $p(E, F, S)$. Finally, it follows that the value of a character $\varphi \in \Gamma$ at an element of $D_\Gamma(B)$ is real and it does not depend on φ. It proves that Γ is an antisymmetric set for B.

Since one-point sets are antisymmetric, $\{\pi_1\}$, $\{\pi_2\}$ are antisymmetric for B. Since the closures of antisymmetric sets are again antisymmetric, $\{\pi_1\} \cup \Gamma$ and $\{\pi_2\} \cup \Gamma$ are closed antisymmetric sets for B. The set $D_L(B)$ consists only of scalar multiples of I, hence L is also antisymmetric for B. Its closure is equal to $\hat{A}$, hence $\hat{A}$ is antisymmetric for B, as we have remarked above in a slightly different way.

Finally, we would like to present a subset of $\hat{A}$ which is not antisymmetric for B. Choose two characters $\varphi_1, \varphi_2 \in \Gamma$ such that $\varphi_1(S) = 1$, $\varphi_2(S) = -1$. Put $K = \{\varphi_1, \varphi_2\}$. It is clear that $S \in D_K(B)$ but K is not antisymmetric, because $\varphi_1(S) \neq \varphi_2(S)$.

3. Main theorems. Before proving main results of this paper we will prove a proposition in connection with function algebras and measures in order to explain the genesis of out further considerations. Let us recall some definitions. Consider a compact Hausdorff space X and a complex, finite (regular, Borel) measure μ on X. $|\mu|$ denotes the positive total variation measure of μ. Let h be the Radon–Nikodym derivative $d\mu/d|\mu|$. Since μ is finite, h is $|\mu|$-integrable. Moreover $h \neq 0$ $|\mu|$-a.e., because μ and $|\mu|$ are mutually absolutely continuous. A point $x \in X$ belongs to the carrier of μ if and only if for every open neighbourhood U of x $|\mu|(U) > 0$. Denote by π the representation of $C(X)$ into $L(L^2(|\mu|))$ given by the formula $\pi(f)u = fu$ for $f \in C(X)$, $u \in L^2(|\mu|)$. It $x \in X$, then we write φ_x for the point-evaluation character $\varphi_x(f) = f(x)$ for $f \in C(X)$. Define the following functional on $C(X)$:

$$\varphi(f) = \int f d\mu, \; f \in C(X).$$

PROPOSITION 1. *$\ker\pi$ is the largest ideal of $C(X)$ contained in $\ker\varphi$. Moreover,*

the carrier of $\mu = \{x \in X: \ker\pi \subset \ker\varphi_x\} = \mathrm{hull}(\ker\pi)$.

Proof. It $f \in C(X)$ and $\pi(f) = 0$, then $f = 0$ $|\mu|$-a.e., because μ is finite. It follows that $\varphi(f) = 0$, hence $\ker\pi \subset \ker\varphi$. It J is an ideal of $C(X)$ such that $J \subset \ker\varphi$ and if $f \in J$, then for every $k \in C(X)$ $fk \in J$ and $\int fk\, d\mu = 0$ for all $k \in C(X)$. Using the Radon–Nikodym theorem, the continuity of the inner product in $L^2(|\mu|)$ and the density of $C(X)$ in $L^2(|\mu|)$, we obtain $\int fgh\, d|\mu| = 0$ for all $g \in L^2(|\mu|)$. It follows that $fh = 0$ $|\mu|$-a.e. and hence $f = 0$ $|\mu|$-a.e. Hence $\pi(f) = 0$ and $J \subset \ker\pi$.

Now we prove the second part of the proposition. Suppose that $x \in X$ and x does not belong to the carrier of μ. Then there is an open neighbourhood U of x such that $|\mu|(U) = 0$. Now we can find a function $f \in C(X)$ such that $f(x) \neq 0$ and $f = 0$ off U. It follows that $\varphi_x(f) \neq 0$ and for all $g \in L^2(|\mu|)$

$$\int_X fg\,d|\mu| = \int_U fg\,d|\mu| + \int_{X\setminus U} fg\,d|\mu| = 0,$$

hence $\pi(f) = 0$, thus $x \notin \mathrm{hull}(\ker\pi)$. Conversely, if $x \in X$ is not in $\mathrm{hull}(\ker\pi)$. then there is $f \in C(X)$ such that $f(x) \neq 0$, $\pi(f) = 0$. Hence $f = 0$ $|\mu|$-a.e. But f is continuous, hence there is an open neighbourhood U of x such that $f \neq 0$ on U. Finally,

$$\int_U |f|^2 d|\mu| \leqslant \int_X |f|^2 d|\mu| = 0,$$

hence $|\mu|(U) = 0$ and the proof is complete.

In what follows A will denote a C^*-algebra with the unit e. $B \subset A$ is a fixed closed subalgebra of A containing e. $B^\perp$ stands for the set of all continuous complex functionals on A vanishing on B, $\mathrm{ball}\,A$ ($\mathrm{ball}\,B^\perp$, resp.) denotes the norm-closed unit ball in A ($B^\perp$, resp.).

The following theorem is a noncommutative generalization of Theorem A.

THEOREM 1. *Suppose that φ is an extreme point of* $\mathrm{ball}\ B^\perp$. *If J is the largest two-sided ideal of A contained in* $\ker\varphi$ *then* $\mathrm{hull}(J)$ *is an antisymmetric set for B.*

Proof. Clearly, $K = \mathrm{hull}(J)$ is a closed subset of $\hat{A}$. It $A_0 = A/J$ then K can be identified with $\hat{A}_0$ ([2], 3.2.1.). Let $q\colon A \to A_0$ be the quotient map. Take $b \in D_K(B)$. Then for every $\pi \in K$ $\pi(b) = \alpha_\pi I_\pi$ for some reals α_π. We may assume $0 \leqslant \alpha_\pi \leqslant 1$ for $\pi \in K$. We have to show that all the α_π's are equal. Note that $q(b)$ is in the center of A_0, because $J = J(K)$ and for all $a \in A$ $ab - ba \in J$.

Write $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1(a) = \varphi(ba)$, $\varphi_2(a) = \varphi\big((e-b)a\big)$ for all $a \in A$. It is clear that $\varphi_1, \varphi_2 \in B^\perp$. Now we will prove that $\|\varphi_1\| + \|\varphi_2\| \leqslant 1$. It is sufficient to show that $\sup\{|\varphi_1(a) + \varphi_2(c)|,\ a, c \in \mathrm{ball}\,A\} \leqslant 1$. For take $a, c \in \mathrm{ball}\,A$. Then we have

$$\begin{aligned} |\varphi_1(a) + \varphi_2(c)| &= \big|\varphi\big(ba + (e-b)c\big)\big| \leqslant \big\|q\big(ba + (e-b)c\big)\big\| \\ &= \sup_{\pi\in K} \big\|\pi\big(ba + (e-b)c\big)\big\| \\ &\leqslant \sup_{\pi\in K} \big(\alpha_\pi\|\pi(a)\| + (1-\alpha_\pi)\|\pi(c)\|\big) \leqslant 1, \end{aligned}$$

because for every $a \in A$ $\|q(a)\| = \sup_{\pi\in K}\|\pi(a)\|$ ([2], 2.7.1.), by the identification of K with $\hat{A}_0$.

Since φ is an extreme point of $\mathrm{ball}\,B^\perp$, we conclude that $\varphi_1 = r\varphi$ for some $r \in [0, 1]$. It follows that $\varphi\big((b - re)a\big) = 0$ for all $a \in A$. Since for all $a \in A$ $ab - ba \in J \subset \ker\varphi$, we have also $\varphi\big(c(b - re)\big) = 0$ for all $c \in A$. It follows that φ annihilates the two-sided ideal generated by $b - re$. But, by the definition of J, $b - re \in J$ and hence $\pi(b) = rI_\pi$ for all $\pi \in K$. The proof is completed.

I would like to express my thanks to the Referee for the essential simplification of my initial proof of Theorem 1. The enclosed proof is just this simplified version.

Now we consider an ideal of B instead of the whole algebra B. Then the following theorem holds:

THEOREM 2. *Let G be an ideal of B. Suppose that φ is an extreme point is* $\mathrm{ball}\,G^\perp$. *If J is the largest two-sided ideal contained in* $\ker\varphi$, *then* $\mathrm{hull}(J)$ *of antisymmetric for B.*

To prove this theorem it is sufficient to repeat step by step the proof of Theorem 1, observing additionally that if $\varphi \in G^\perp$ and $b \in B$ then the functional $\varphi_b(a) = \varphi(ba)$ for $a \in A$ is also an element of $G^\perp$.

Let us point out that by Proposition 1 one can consider Theorem A and a part of Theorem 2.5. in [3] as special cases of our Theorems 1 and 2, respectively. As a consequence of the previous two theorems we are able to prove the following theorem which generalizes Theorem B and a result of Glicksberg ([3], Theorem 2.5).

THEOREM 3. *Suppose that A, B are as above and that G is an ideal in B. Then there is a family $\mathscr{K}$ of subsets of $\hat{A}$ such that $\hat{A} = \bigcup\mathscr{K}$ is a partition of $\hat{A}$ and:*

(1) *every $K \in \mathscr{K}$ is a maximal (closed) antisymmetric set for B;*

(2) *if $a \in A$ and if for every $K \in \mathscr{K}$ there is $b \in B$ such that $a - b \in J(K)$, then $a \in B$;*

(3) *if $a \in A$ and if for every $K \in \mathscr{K}$ there is $g \in G$ such that $a - g \in J(K)$, then $a \in G$.*

Proof. Since all one-point subsets of $\hat{A}$ are antisymmetric for B, (1) follows from Remark 3. To prove (2), we will apply the technique used by Glicksberg in [3]. Let $a \in A$ be as in (2). Suppose that $a \notin B$. By the Hahn–Banach theorem, we find a continuous functional φ on A such that $\varphi \in \mathrm{ball}\,B^\perp$, $\varphi(a) \neq 0$. By the Krein–Milman theorem we may choose φ as an extreme point of $\mathrm{ball}\,B^\perp$. Let J be the largest two-sided ideal contained in $\ker\varphi$. By Theorem 1, $\mathrm{hull}(J)$ is an antisymmetric set for B. By (1), there is $K \in \mathscr{K}$ such that $\mathrm{hull}(J) \subset K$. It follows that $J = J\big(\mathrm{hull}(J)\big) \supset J(K)$. Our assumptions imply now that there is $b \in B$ such that $a - b \in J(K)$. But $J(K) \subset J \subset \ker\varphi$, hence $\varphi(a) = \varphi(b) = 0$ which is a contradiction. The proof of (2) is finished.

To prove (3), we repeat the proof of (2) using Theorem 2 instead of Theorem 1. Now our theorem is completely proved.

We call the family $\mathscr{K}$ of Theorem 3 the *Bishop decomposition for* B.

4. Applications. In the function algebras theory the Bishop decomposition appears as a generalization of the Stone–Weierstrass theorem. In the noncommutative situation considered in this paper we are also able to prove a verison of the Stone–Weierstrass theorem. First we introduce some terminology. Let us fix a C^*-algebra A. Let π_1, π_2 be two irreducible representations of A. We will write $\pi_1 \simeq \pi_2$ if π_1 and π_2 are unitarily equivalent. Similarly, if S is a subset of A, we write $\pi_1|_S \simeq \pi_2|_S$ if there is a unitary operator $U\colon H_{\pi_1} \to H_{\pi_2}$ such that, for all $a \in S$, $U\pi_1(a) = \pi_2(a)U$. We say that a subset S of A *separates* $\hat{A}$ if for any two irreducible representations π_1, π_2 of A $\pi_1 \not\simeq \pi_2$ implies $\pi_1|_S \not\simeq \pi_2|_S$. The following proposition is a generalization of the Stone–Weierstrass theorem.

PROPOSITION 2. *Suppose we are given a C^*-algebra A with the unit e and with the center Z. Suppose that dimensions of all irreducible representations of A are equal (finite or not). If a C^*-subalgebra B of A satisfies the following conditions:*

(a) $e \in B$,

(b) $B \cap Z$ *separates* $\hat{A}$,

(c) *for all irreducible representations π of A $\pi(B) = \pi(A)$,*

then $B = A$.

Proof. Let $\mathscr{K}$ be the Bishop decomposition for B. Take $K \in \mathscr{K}$ and two points $\tilde{\pi}_1, \tilde{\pi}_2 \in K$. Suppose that $\tilde{\pi}_1 \neq \tilde{\pi}_2$. It means that there are two irreducible representations π_1, π_2 of A such that $\pi_1 \not\simeq \pi_2$ and the unitary equivalence class of π_i equals to $\tilde{\pi}_i$, $i = 1, 2$. By our assumption (b), $\pi_1|_{B\cap Z} \not\simeq \pi_2|_{B\cap Z}$. Hence, for every unitary operator $U\colon H_{\pi_1} \to H_{\pi_2}$, there is $b \in B\cap Z$ such that $U\pi_1(b) \neq \pi_2(b)U$. Since $B \cap Z$ is a C^*-algebra, we may choose b as a self-adjoint element of $B\cap Z$. But self-adjoint elements of $B \cap Z$ lie in $D_K(B)$ and K is antisymmetric. Hence there is $r \in \boldsymbol{R}$ such that $\pi_1(b) = rI_{\pi_1}$, $\pi_1(b) = rI_{\pi_2}$. This is a contradiction which proves that every maximal, antisymmetric set for B, must contain strictly one point of $\hat{A}$. Now, if $a \in A$ then, by (c), there is $b \in B$ such that $\pi(a) = \pi(b)$ and applying Theorem 3 we finish the proof.

Some remarks are now in order. There are several noncommutative generalizations of the Stone–Weierstrass theorem. Our generalization has a connection with one of them due to Fell (see [2], 11.1). He called a subalgebra B of a C^*-algebra A *rich* if B separates $\hat{A}$ and for every irreducible representation π of $A\pi|_B$ is irreducible. He also proved ([2], 11.1.6) that every rich subalgebra of a GCR algebra is equal to the whole algebra. If B satisfies the conditions of Proposition 2 then, obviously, it is rich. Hence we see that Proposition 2 says something new in case where A is not GCR.

Now we would like to present an example in which assumptions of our Proposition 2 are satisfied non-trivially. We will also show that those assumptions are essential.

EXAMPLE 2. Consider a commutative von Neumann algebra $\mathscr{C} \subset L(H)$ with the identity $I \in L(H)$. By $M_n(\mathscr{C})$ we will denote the C^*-algebra of all $n \times n$ matrices over $\mathscr{C}$ (i.e whose entires are elements of $\mathscr{C}$). Takesaki in [5] proved that all irreducible representations of $A = M_n(\mathscr{C})$ are exactly n-dimensional. Moreover, to every irreducible representation π of A there corresponds a non-zero character φ of $\mathscr{C}$ such that π is unitarily equivalent to the representation $\tilde{\varphi}$ of A of the form $\tilde{\varphi}([T_{ij}]) = [\varphi(T_{ij})]$, $T_{ij} \in \mathscr{C}$ for $i, j = 1, \ldots, n$. Conversely, every non-zero character φ of $\mathscr{C}$ induces the representation $\tilde{\varphi}$ (as above) of A which is irreducible.

We can consider $\mathscr{C}$ as a subalgebra of A if we embed it into A as ollows: for $T \in \mathscr{C}$

$$T \to T_n = \begin{bmatrix} T & & 0 \\ & T & \\ 0 & & \ddots \\ & & & T \end{bmatrix}.$$

This embedding is a *-monomorphism. One can check easily that $\mathscr{C}$ is equal to the center of A.

Let π_1, π_2 be two irreducible representations of A and let φ_1, φ_2 be two characters of $\mathscr{C}$ corresponding to π_1, π_2, respectively. Observe that the following three conditions are equivalent:

(i) $\pi_1 \simeq \pi_2$,

(ii) $\varphi_1 = \varphi_2$,

(iii) $\pi_1|_{\mathscr{C}} \simeq \pi_2|_{\mathscr{C}}$.

Indeed, implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are trivial. The only thing to prove is (iii) $\Rightarrow$ (ii). If $\pi_1|_e \simeq \pi_2|_e$, then, for $T \in \mathscr{C}$, $\varphi_1(T)I_n \simeq \varphi_2(T)I_n$. Hence $\varphi_1(T) = \varphi_2(T)$.

This equivalence implies that $\mathscr{C}$ separates $\hat{A}$. Let B be a C^*-subalgebra of A containing $\mathscr{C}$ and such that for all $\pi \in \hat{A}$ $\pi(B)$ is irreducible (in this case it is equivalent to say that $\pi(B) = \pi(A) = L(C^n)$). Proposition 2 implies $B = A$. For instance, take B as a C^*-subalgebra of A containing $\mathscr{C}$ and an operator

$$\tilde{S} = \begin{bmatrix} 0 & & 0 \\ T_1 & 0 & \\ & \ddots & \ddots \\ 0 & & T_{n-1} & 0 \end{bmatrix}_i,$$

where T_i, $i = 1, \ldots, n-1$ are invertible. Since for all $\pi \in \hat{A}$ $\pi(\tilde{S})$ is an irreducible operator, we have $B = A$. The C^*-subalgebra $\mathscr{C}$ of A satisfies (a) and (b) of Proposition 2, it does not satisfy (c) and $\mathscr{C} \neq A$, hence (c) is essential.

Let now $B = C^*(\tilde{U})$, where

$$\tilde{U} = \begin{bmatrix} 0 & & & 0 \\ I & 0 & & \\ & \ddots & \ddots & \\ 0 & & I & 0 \end{bmatrix}.$$

One can check that the center of B is trivial, hence $B \cap \mathscr{C}$ consists of scalar multiples of the identity. It follows that $B \neq A$ (and, by Remark 4, $\hat{A}$ is the only maximal, antisymmetric set for B). If φ is a character of $\mathscr{C}$, then $\tilde{\varphi}(\tilde{U}) = J(n)$ (as in Example 1). Hence for any two irreducible representations π_1, π_2 of A $\pi_1|_B \simeq \pi_2|_B$. It is clear now that B does not separate $\hat{A}$ and (b) of Proposition 2 is essential.

As the second application of Theorem 3 we will prove now an approximation-type property for subalgebras of C^*-algebras. Consider a C^*-algebra A with unit e. Let K be a closed subset of $\hat{A}$. In this case $K = \operatorname{hull}(J(K))$. Let $\tau\colon A \to A/J(K)$ denote the quotient map. As we have observed in the proof of Theorem 1, we may identify K with the spectrum $\big(A/J(K)\big)^{\wedge}$ of $A/J(K)$. If $a, b \in A$, we define $\|a-b\|_K = \sup\{\|\pi(a)-\pi(b)\|, \pi \in K\}$. Now we have $\|a-b\|_K = \|\tau(a-b)\|$.

Proposition 3. *Let A be a C^*-algebra with the unit e, let B be its closed subalgebra containing e and let $\mathscr{K}$ be the Bishop decomposition for B. Then if $a \in A$, we have:*

$$\|a-B\| = \sup\{\|a-B\|_K,\ K \in \mathscr{K}\},$$

where $\|a-B\| = \inf\{\|a-b\|, b \in B\}$ and $\|a-B\|_K = \inf\{\|a-b\|_K, b \in B\}$.

Proof. Let us recall that all elements K of $\mathscr{K}$ are closed in $\hat{A}$. Since for every $x \in A$ $\|x\| = \sup\{\|\pi(x)\|, \pi \in \hat{A}\}$, we have $\|a-B\| \geqslant \sup\{\|a-B\|_K, K \in \mathscr{K}\}$. To prove the converse inequality observe that the dual space $(A/B)'$ of A/B and $B^{\perp}$ are isometrically isomorphic as normed linear spaces. Hence

$$\|a-B\| = \sup\{|f(a)|\colon f \in \operatorname{ball} B^{\perp}\}.$$

It follows from the proof of the Krein–Milman theorem that the last supremum is assumed at an extreme point φ of $\operatorname{ball} B^{\perp}$.

Let J be the largest two-sided ideal contained in $\ker\varphi$. By Theorem 1, $\operatorname{hull}(J)$ is an antisymmetric set for B, hence there is $K \in \mathscr{K}$ such that $\operatorname{hull}(J) \subset K$. It follows that $J(K) \subset J(\operatorname{hull}(J)) = J \subset \ker\varphi$. Let $\tau\colon A \to A/J(K)$

be the quotient map. Now we have for every $b \in B$:

$$|\varphi(a)| = |\varphi(a-b)| \leqslant \|\tau(a-b)\| = \|a-b\|_K$$

and the proof is completed.

This proposition implies a result proved by Glicksberg ([3], p. 419) in connection with the commutative Bishop decomposition.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES, CRACOW

Received March 25, 1975
New version November 25, 1975 (979)