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THE OHIO STATE UNIVERSITY, COLUMBUS, OHIO
and
STANFORD UNIVERSITY, STANFORD, CALIFORNIA

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A general result on the equivalence between derivation of integrals and weak inequalities for the Hardy–Littlewood maximal operator

by

IRENEO PERAL ALONSO (Madrid)

Abstract. In this paper we consider integrals of functions belonging to $\varphi(L)$ classes, and their differentiation properties with respect to a translation invariant (B–F) differentiation basis. We prove that the differentiation of certain integrals is equivalent to a certain property of weak type for the maximal function of Hardy–Littlewood, which is associated to the basis. In a sense, this is a sharp result (see Peral [3]).

Introduction. We consider for each $x \in \mathbf{R}^n$, a family of open bounded sets $\mathcal{B}(x)$ such that each $B \in \mathcal{B}(x)$ verifies:

(i) $x \in B$;

(ii) there is a sequence $\{B_k\}_{k\in N} \subset \mathcal{B}(x)$ such that $\delta(B_k) \to 0$ as $k \to \infty$ ($\delta(B_k)$ stands for the diameter of B_k).

If these conditions are satisfied, we say that $\{B^k\}$ *contracts to* x, and that $\mathcal{B} = \bigcup_{x\in\mathbf{R}^n} \mathcal{B}(x)$ is a *differentiation basis* in $\mathbf{R}^n$.

$\mathcal{B}$ is a Busemann–Feller (B–F) basis, if for each $B \in \mathcal{B}$ with $y \in B$, we have $B \in \mathcal{B}(y)$.

A differentiation basis $\mathcal{B}$ is *translation invariant*, if each translation of $B \in \mathcal{B}$ belongs also to $\mathcal{B}$.

We denote by $\mathcal{B}_r$ and $\mathcal{B}_r(x)$ all the elements in $\mathcal{B}$ and $\mathcal{B}(x)$ with a diameter less than r.

If B is a measurable set, then $|B|$ will be its measure.

Let f be a locally integrable function on $\mathbf{R}^n$, i.e. $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$; we define the *upper* and *lower derivatives* of the integral of f with respect to $\mathcal{B}$ by:

$$\bar{D}\Big(\int f; x\Big) = \sup\Big\{\limsup_{k\to\infty} \frac{1}{|B_k|}\int_{B_k} f(y)\,dy:\ B_k \to x;\ \{B_k\} \subset \mathcal{B}(x)\Big\},$$

$$\underline{D}\Big(\int f; x\Big) = \inf\Big\{\liminf_{k\to\infty} \frac{1}{|B_k|}\int_{B_k} f(y)\,dy:\ B_k \to x;\ \{B_k\} \subset \mathcal{B}(x)\Big\}.$$

We say that $\mathscr{B}$ *differentiates* the integral of f, if

$$\bar{D}\left(\int f;\, x\right) = \underline{D}\left(\int f;\, x\right) = f(x) \text{ a.e.}$$

The maximal operators of Hardy–Littlewood associated to $\mathscr{B}$ and $\mathscr{B}_r$ are defined respectively by

$$Mf(x) = \sup_{x\in B\in\mathscr{B}} \frac{1}{|B|}\int_B |f(y)|\,dy,$$

$$M_r f(x) = \sup_{x\in B\in\mathscr{B}_r} \frac{1}{|B|}\int_B |f(y)|\,dy.$$

When $\mathscr{B}$ is a B–F basis, it is easy to see that $Mf(x)$, $M_r f(x)$, $\bar{D}(\int f; x)$ and $\underline{D}(\int f;\, x)$ are measurable functions, if $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ (see Guzmán [1]).

We also consider the $\varphi(L)$ classes:

$$\varphi(L) = \left\{f\in L^1_{\mathrm{loc}}(\mathbf{R}^n):\ \int \varphi(|f|)\,dx < \infty\right\},$$

where φ: $[0, \infty)\to[0, \infty)$ is strictly increasing and such that $\lim\limits_{x\to 0}\varphi(x) = \varphi(0) = 0$.

Note that the Orlicz spaces are included here.

Previous results and statement of the main theorem. For the proof of the main theorem we will use the following results:

LEMMA 1. *Let $\{E_k\}_{k\in \mathbf{N}}$ be a sequence of measurable sets contained in Q, where Q is a cube in $\mathbf{R}^n$ with center at the origin and side length $1/4$, such that*

$$\sum_{k=1}^{\infty} |E_k| = \infty.$$

Then there is a sequence $\{x_k\}_{k\in N}\subset Q^*$, where Q^* is the unit cube in $\mathbf{R}^n$ such that, if $A_k = x_k + E_k$, then almost every $x\in Q$ belongs to an infinite family of A_k's.

This lemma was proved by A. P. Calderón. See Zygmund [5], Vol. II, p. 165.

LEMMA 2. *Let $\mathscr{B}$ be a differentiation basis in $\mathbf{R}^n$. Then the following are equivalent:*

(1) *$\mathscr{B}$ differentiates integrals of functions $f\in\varphi(L)$;*

(2) *for each $\lambda > 0$, for each decreasing sequence of positive real numbers $\{b_k\}_{k\in N}$ such that $\lim\limits_{k\to\infty} b_k = 0$ and for each decreasing sequence of positive functions with compact support $\{f_k\}_{k\in N}\subset\varphi(L)$ such that $\lim\limits_{k\to\infty}\int\varphi(f_k)\,dx = 0$, we have*

$$\lim_{k\to\infty} |\{x\in\mathbf{R}^n:\ M_{b_k} f_k(x) > \lambda\}| = 0.$$

See Peral [3] for the proof of this lemma.

In the theorem the results of Guzmán–Welland [2] and Rubio [4], are generalized to $\varphi(L)$ classes and to translation invariant basis. The techniques used, are also different.

THEOREM. *Let $\mathscr{B}$ be a* B–F *translation invariant differentiation basis in $\mathbf{R}^n$. Then the following are equivalent:*

(I) *$\mathscr{B}$ differentiates integrals of functions $f\in\varphi(L)$;*

(II) *there are constants $c > 0$ and $r > 0$ such that, for every $\lambda > 0$ and every $f\in\varphi(L)$,* we have

$$|\{x\in\mathbf{R}^n:\ M_r f(x) > \lambda\}| \leqslant c\int\varphi\left(\frac{a_n|f|}{\lambda}\right)dx, \tag{2.1}$$

where a_n is a constant which depends only on the dimension n.

In comparison with the case of $\varphi(u) = u^p$, we say that M_r is of *weak type* φ, if (II) is satisfied.

The above theorem gives, in a sense, the best result (see Peral [3]). We have also the following corollaries.

COROLLARY 1. *Under the hypothesis of the theorem, if we also assume $\varphi(u) = u^p$, then we get the differentiation of integrals of functions in $L^p(\mathbf{R}^n)$ to be equivalent to the truncated maximal operator so that it is of weak type (p, p).*

COROLLARY 2. *If $\mathscr{B}$ is also homothecy invariant, then $\mathscr{B}$ differentiates integrals of functions $f\in\varphi(L)$ if and only if there is a $c > 0$ such that, for each $\lambda > 0$ and each $f\in\varphi(L)$,*

$$|\{x\in\mathbf{R}^n:\ Mf(x) > \lambda\}| \leqslant c\int\varphi\left(\frac{a_n|f|}{\lambda}\right).$$

This corollary is essentially Rubio's result in [4].

Proof of the main theorem. (I) $\Rightarrow$ (II).

First step. First of all we prove the theorem for functions $f\in\varphi(L)$ with support on a fixed cube $Q\subset\mathbf{R}^n$.

Assume that the inequality (2.1) is not true; then, for each $r > 0$ each $c > 0$, there exists $\lambda > 0$ and $f\in\varphi\big(L(Q)\big)$, $f\geqslant 0$, such that

$$|\{x\in\mathbf{R}^n:\ M_r f(x) > \lambda\}| > c\int\varphi\left(\frac{|f|}{\lambda}\right)dx.$$

Let $\{r_k\}_{k\in N}$ be a decreasing sequence of positive real numbers which converges to zero, and let $\{c_k\}$ be another sequence of positive real numbers such that

$$\sum_{k=1}^{\infty} c_k^{-1} < \infty.$$

Consider now the corresponding sequences $\{\lambda_k\}_{k\in N}$ and $\{f_k^*\}_{k\in N}\in\varphi(L(Q))$, $f_k^*\geqslant 0$, which verify for every fixed k:

$$|\{x\in \boldsymbol{R}^n:\ M_{r_k}f_k(x)>1\}|\geqslant c_k\int\varphi(f_k)\,dx,$$

where $f_k(x)=\lambda_k^{-1}f_k^*(x)$.

Let $E_k=\{x\in\boldsymbol{R}^n:\ M_{r_k}f_k(x)>1)$. Then, for every k, $|E_k|>0$ and

$$E_k\subset A=\{x\in\boldsymbol{R}^n:\ d(x,Q)\leqslant 2r_k\}.$$

Let a_k be a positive integer such that

$$|A|\leqslant a_k|E_k|\leqslant 2|A|.$$

Then

$$\sum_{k=1}^{\infty}a_k|E_k|=\infty.$$

Let $\{E_k^i\}_{k\in N}$ (where $E_k^i=E_k$), $i=1,2,\ldots,a_k$, which satisfies the hypothesis of Lemma 1. Then there exists the sequence $\{x_k^i\}_{k\in N}$, $i=1,\ldots,a_k$ such that almost every point of Q belongs to an infinite number of sets of the family $\{E_k(x_k^i)\}_{k\in N}$, where $E_k(x_k^i)=x_k^i+E_k^i$, $i=1,\ldots,a_k$.

Let $f_k^i(x)=f_k(x-x_k^i)$, $k\in\boldsymbol{N}$, $i=i,\ldots,a_k$; then

$$\sum_{k=1}^{\infty}\sum_{i=1}^{a_n}\int\varphi\big(f_k^i(x)\big)\,dx\leqslant\sum_{k=1}^{\infty}\frac{a_k|E_k|}{c_k}\leqslant 2|A|\sum_{k=1}^{\infty}c_k^{-1}<\infty.$$

Define

$$h_k(x)=\sup_{\substack{j>k\\ i=1,\ldots,a_j}}f_j^i(x),\qquad k\in\boldsymbol{N}.$$

Then it is obvious, from the above, that $h_k\in\varphi(L)$ and that $\{h_k\}_{k\in N}$ is a decreasing sequence which verifies $\lim\limits_{k\to\infty}\int\varphi\big(h_k(x)\big)\,dx=0$.

But, on the other hand,

$$\{x:\ M_{r_k}h_k(x)>1\}\supset\bigcup_{j>k}\Big(\bigcup_{i=1}^{a_j}\{x:\ M_{r_k}f_j^i(x)>1\}\Big)$$

$$\supset\bigcup_{j>k}\Big(\bigcup_{i=1}^{a_j}E_j(x_j^i)\Big)\supset Q-N_k,$$

where $|N_k|=0$, for all $k\in N$.

Then

$$\lim_{k\to\infty}|\{x:\ M_{r_k}h_k(x)>1\}|>|Q|\neq 0,$$

but this is a contradiction, according to Lemma 2. This proves the first step.

Second step. If we consider a cube Q of $\boldsymbol{R}^n$ and the constants r_Q and c_Q of the first step, both constants are valid for each cube of the same size as Q, because the basis is translation invariant.

Let l be the side of Q, and consider

$$r<\tfrac{1}{2}\min\{r_Q,l\}.$$

Let us fix the cube Q of side length l with one vertix on the origin and the others with nonnegative coordinates. Let $\{Q_i\}_{i=1,\ldots,2^n}$ be the cubes obtained from Q by the translations $l\vec{n}$, where n is a vector with coordinates that take only the values zero or one. Let $\boldsymbol{Z}^n$ be the integer lattice.

For every $i=1,2,\ldots,2^n$ and every $\vec{m}\in\boldsymbol{Z}^n$, let us consider

$$Q_{i\overline{m}}=2l\overline{m}+Q_i$$

and

$$A_i=\bigcup_{\overline{m}\in Z^n}Q_{i\overline{m}},\qquad i=1,2,\ldots,2^n.$$

Then

$$\bigcup_{i=1}^{n}A_i=\boldsymbol{R}^n,\qquad\text{and}\qquad |A_i\cup A_j|=0\ \text{if}\ i\neq j.$$

The operator M_r acts independently over each cube of A_i, because $2r<1$.

Let $f\in\varphi(L(\boldsymbol{R}^n))$, $f\geqslant 0$, then we define $f_{i\overline{m}}=f\cdot\chi_{Q_{i\overline{m}}}$ ($\chi_{Q_{i\overline{m}}}$ characteristic function of $Q_{i\overline{m}}$) and $f_i=\sum_{\overline{m}\in Z^n}f_{i\overline{m}}$. It is clear that

$$f=\sum_{i=1}^{2^n}f_i.$$

From the first step we have

$$|\{x:\ M_rf_{i\overline{m}}(x)>\lambda\}|\leqslant c_Q\int_{Q_{i\overline{m}}}\varphi\left(\frac{f_{i\overline{m}}}{\lambda}\right)dx$$

for $\lambda>0$; and this implies

$$|\{x:\ M_rf_i(x)>\lambda\}|=\sum_{\overline{m}\in Z^n}|\{x:\ M_rf_{i\overline{m}}>\lambda\}|\leqslant c_Q\int\varphi\left(\frac{f_i}{\lambda}\right)ds.$$

Then

$$|\{x:\ M_rf(x)>\lambda\}|\leqslant c_Q\int\varphi\left(\frac{2^nf}{\lambda}\right)dx.$$

Note that for functions in $\varphi(L)$, the constant a_n is 2^{n+1}.

(II) $\Rightarrow$ (I). It is a trivial consequence of Lemma 2.

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DEPARTAMENTO DE ECUACIONES FUNCTIONALES
FACULTAD DE CIENCIAS MATHEMÁTICAS
UNIVERSIDAD COMPLUTENSE, C. U. MADRID-3, SPAIN

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Antisymmetry of subalgebras of C^*-algebras

by

WACŁAW SZYMAŃSKI (Cracow)

Abstract. In the present paper we introduce a generalization of antisymmetric sets, known in the function algebras theory, to a noncommutative case. We prove a de Branges-type theorem and a generalization of the Bishop decomposition theorem. As applications we prove a version of the Stone–Weierstrass theorem and an approximation-type result in connection with the Bishop decomposition proved earlier.

1. Preliminaries. $L(H)$ stands for the C^*-algebra of all linear, bounded operators in a complex Hilbert space H. A*-homomorphism π of a C^*-algebra A into $L(H_\pi)$ is called a *representation of* A, the dimension of H_π is called the *dimension of* π. Characters of a C^*-algebra A are one-dimensional representations of A. A representation π of a C^*-algebra A is called *irreducible* if the algebra $\pi(A)$ has no non-trivial invariant subspace in $L(H_\pi)$. If A has the unit e, we will assume always that, for every representation π of A, $\pi(e) = I_\pi$ — the identity operator in H_π.

If $\mathscr{S}$ is a subset of $L(H)$ we denote by $C^*(\mathscr{S})$ the C^*-algebra generated by $\mathscr{S}$ and the identity. If $T \in L(H)$, we write $C^*(T)$ for $C^*(\{T\})$. By the *spectrum* $\hat{A}$ *of a* C^**-algebra* A we mean the set of unitary equivalence classes of all irreducible representations of A equipped with the hull-kernel topology. For a subset K of $\hat{A}$ we write $J(K) = \bigcap\{\ker\varrho,\ \varrho\in K\}$. If J is a closed, two-sided ideal in A, then by the *hull of* J we mean the set hull(J) consisting of all $\pi\in\hat{A}$ such that $J \subset \ker\pi$. It follows from [2], 2.9.7 (ii), that $J = J\big(\mathrm{hull}(J)\big)$. The closure $\bar{K}$ of a subset K of $\hat{A}$ in that topology is equal to $\mathrm{hull}\big(J(K)\big)$, by the definition.

If two C^*-algebras are *-isomorphic, then their spectra are homeomorphic. Namely, if $\varphi\colon A_1 \to A_2$ is a *-isomorphism of the C^*-algebras A_1, A_2, then the mapping $\hat{\varphi}\colon \hat{A}_1 \to \hat{A}_2$ given by the formula $\hat{\varphi}\colon \varrho \to \varrho\circ\varphi^{-1}$ is the homeomorphism induced by φ. For basic facts concerning C^*-algebras we refer to [2].

2. Sets of antisymmetry. To begin with, we recall two results due to de Branges, Bishop and Glicksberg [3].

Let X be a compact Hausdorff space and let $B \subset C(X)$ be a function algebra. $B^\perp$ denotes the set of all finite, complex (regular, Borel) measures