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On the structure of separable $\mathscr{L}_p$ spaces $(1 < p < \infty)$

by

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Abstract. It is shown that L_p, $(\sum X_p)_{l_p}$, B_p and $(\sum l_2)_{l_p}$ $(1 < p < \infty)$ are primary. The proof for L_p is then extended to a class of rearrangement invariant function spaces. Also, if X is a subspace of $(\sum l_2)_{l_p} = Z_p$ $(1 < p < \infty)$ which contains a subspace Y isomorphic to Z_p and $\varepsilon > 0$, then there is a subspace $Z \subseteq Y$ with $d(Z, Z_p) \leqslant 1+\varepsilon$ and a projection P of Z_p onto Z with $\|P\| \leqslant 1+\varepsilon$.

Introduction. A Banach space X is said to be *primary* if whenever $X = Y \oplus Z$ then either Y or Z is isomorphic to X. It is known that c_0, l_p $(1 \leqslant p \leqslant \infty)$ and $C[0,1]$ are primary (see [13] and [9]). In the first part of Section 1 of this paper we show that L_p $(1 < p < \infty)$ is primary. The main technique in the proof is a result of Casazza and Lin (Lemma 1.1 of Section 1). In the latter part of Section 1, we employ a similar argument to show that certain other $\mathscr{L}_p$ spaces (namely, $(\sum X_p)_{l_p}$, $(\sum l_2)_{l_p}$ and B_p (see [14] for the definitions)) are primary.

In Section 2 we turn to the study of the isomorphic structure of subspaces of $(\sum l_2)_{l_p} = Z_p$. In particular, we show that if X is a subspace of Z_p which contains an isomorph of Z_p, then for all $\varepsilon > 0$ there is a subspace Y of X with $d(Y, Z_p) \leqslant 1+\varepsilon$ and such that there is a projection P of Z_p onto Y with $\|P\| \leqslant 1+\varepsilon$.

We use standard Banach space notation throughout as may be found, for example, in the book of Lindenstrauss and Tzafriri [10]. By *subspace* we mean closed linear subspace. If $A \subseteq X$, by $[A]$ we mean the smallest subspace containing A. $X \sim Y$ means that X is isomorphic to Y.

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1. In this section we will show that L_p, $(\sum X_p)_{l_p}$, $(\sum l_2)_{l_p}$ and B_p are primary. The basic technique we use is essentially due to Casazza and Lin [4]. We wish to thank Professor W. B. Johnson for pointing out to us that the following lemma follows easily from the arguments in [4].

LEMMA 1.1. *Let (x_n) be a bounded unconditional basis for X with biorthogonal functionals (x_n^*). Assume that T is an operator on X such that (Tx_{n_i}) is a block basis of (x_n) for some subsequence (x_{n_i}) and $|x_{n_i}^*(Tx_{n_i})| \geqslant \varepsilon$ for all i and some fixed $\varepsilon > 0$. Then the basic sequence $(Tx_{n_i})_{i=1}^{\infty}$ is equivalent to $(x_{n_i})_{i=1}^{\infty}$ and $[(Tx_{n_i})_{i=1}^{\infty}]$ is complemented in X.*

Our next lemma follows immediately from a theorem of Gamlen and Gaudet [6].

LEMMA 1.2. *If (h_i) is the Haar basis for L_p $(1 < p < \infty)$ and $\{h_i\}_{i=1}^{\infty} = \{h_{i_n}\}_{n=1}^{\infty} \cup \{h_{i_m}\}_{m=1}^{\infty}$, then either $[(h_{i_n})] \sim L_p$ or $[(h_{i_m})] \sim L_p$.*

We also wish to recall that if (x_n) is an unconditional basic sequence in L_p, then there is a constant $k < \infty$ such that for all finitely non-zero sequences of scalars (a_i),

$$(1)\quad k^{-1}\Big(\int\Big(\sum |a_i x_i(t)|^2\Big)^{p/2} dt\Big)^{1/p} \leqslant \Big\|\sum a_i x_i\Big\|_p \leqslant k\Big(\int\Big(\sum |a_i x_i(t)|^2\Big)^{p/2} dt\Big)^{1/p}$$

(see [7]).

THEOREM 1.3. *L_p $(1 < p < \infty)$ is primary.*

Proof. It is well known that $L_p \sim L_p(l_2)$, where

$$(2)\quad L_p(l_2) = \Big\{(f_i)_{i=1}^{\infty}\colon f_i \in L_p \text{ and } \|(f_i)\| = \Big(\int\Big(\sum_{i=1}^{\infty} |f_i|^2\Big)^{p/2}\Big)^{1/p} < \infty\Big\}.$$

Let (h_i) be the Haar basis for L_p. Then $(h_{ij})_{i,j=1}^{\infty}$ is an unconditional basis for $L_p(l_2)$, where

$$h_{ij} = (0, 0, \ldots, 0, h_i, 0, \ldots)$$

(h_i stands in the jth place). For these facts and some related results see [15].

Assume $L_p(l_2) = X \oplus Y$ and let P_X (respectively, P_Y) denote the projection of $L_p(l_2)$ onto X with kernel Y (respectively, the projection of $L_p(l_2)$ onto Y with kernel X). We shall show that either X or Y contains a complemented isomorph of L_p. The fact that L_p is primary then follows from the well-known decomposition technique of Pełczyński [13].

Since $h_{ij} = P_X h_{ij} + P_Y h_{ij}$ either $h_{ij}^*(P_X h_{ij}) \geqslant \frac{1}{2}$ or $h_{ij}^*(P_Y h_{ij}) \geqslant \frac{1}{2}$ for each i and j (here (h_{ij}^*) are the functionals biorthogonal to (h_{ij})). Let

$$I = \{i\colon h_{ij}^*(P_X h_{ij}) \geqslant \tfrac{1}{2} \text{ for an infinite number of } j\},$$

and

$$J = \{i\colon h_{ij}^*(P_Y h_{ij}) \geqslant \tfrac{1}{2} \text{ for an infinite number of } j\}.$$

By Lemma 1.2, either $[(h_i)_{i\in I}] \sim L_p$ or $[(h_i)_{i\in J}] \sim L_p$. Without loss of generality we assume that $[(h_i)]_{i\in I} \sim L_p$ and enumerate I as $\{i_n\}_{n=1}^{\infty}$.

Since $(P_X h_{ij})_{i,j=1}^{\infty}$ converges weakly to 0, we may assume (by standard perturbation arguments) that there are integers j_n such that $(P_X h_{i_n,j_n})_{n=1}^{\infty}$ is a block of the basis (h_{ij}) and $h_{i_n,j_n}^*(P_X h_{i_n,j_n}) \geqslant \frac{1}{2}$. By Lemma 1.1, $(P_X h_{i_n,j_n})_{n=1}^{\infty}$ is equivalent to $(h_{i_n,j_n})_{n=1}^{\infty}$ and $[(P_X h_{i_n,j_n})_{n=1}^{\infty}]$ is complemented in $L_p(l_2)$. But by (1) and (2), $(h_{i_n,j_n})_{n=1}^{\infty}$ is equivalent to $(h_{i_n})_{n=1}^{\infty}$ and thus we have shown that X contains a complemented isomorph of L_p. ■

The second named author presented a different proof of Theorem 1.3 at the conference "The Geometry of Banach Spaces" at Oberwolfach, 1973. A proof similar to that and an extension to the case $p = 1$ has been given by Maurey [11].

Our next theorem shows that certain other $\mathcal{L}_p$ spaces with a "nice matrix form" are primary. In what follows X_p and B_p $(1 < p < \infty, p \neq 2)$ are the $\mathcal{L}_p$ spaces of Rosenthal [14].

THEOREM 1.4. *$(\sum X_p)_{l_p}$, B_p and $(\sum l_2)_{l_p}$ are primary $(1 < p < \infty$, $p \neq 2)$.*

(J. Lindenstrauss has independently obtained this result for $(\sum l_2)_{l_p}$.)

Proof. It suffices by duality to prove the theorem for $p > 2$ and we shall consider only the case of $(\sum X_p)_{l_p}$ (the proofs for the other spaces are similar and simpler).

We regard X_p as $[(x_{ij})_{i,j=1}^{\infty}]$, where $(x_{ij})_{i,j=1}^{\infty}$ is a sequence of independent symmetric 3-valued random variables in $L_p[0,1]$ such that

$$(3)\quad \frac{\|x_{ij}\|_{L_2}}{\|x_{ij}\|_{L_p}} = w_i \quad \text{for all } i \text{ and } j,$$

$\sum w_i^{2p/p-2} = \infty$ and $w_i \downarrow 0$ (cf. [14]). Thus an unconditional basis for $(\sum X_p)_{l_p}$ is given by $(x_{n,i,j})_{n,i,j=1}^{\infty}$, where for each n $(x_{n,i,j})_{i,j=1}^{\infty}$ is a sequence as in (3) above, and

$$\Big\|\sum_n \sum_i \sum_j a_{nij} x_{nij}\Big\| = \Big(\sum_n \Big\|\sum_i \sum_j a_{nij} x_{nij}\Big\|^p\Big)^{1/p}.$$

Let $(\sum X_p)_{l_p} = Y \oplus Z$ and let P_Y be the projection onto Y with kernel Z and define P_Z similarly. As in the proof of the previous theorem, we need only show that either Y or Z contains a complemented isomorph of $(\sum X_p)_{l_p}$.

Let $(x_{n,i,j}^*)$ be the functionals biorthogonal to $(x_{n,i,j})$ and for each n set

$$(4)\quad \begin{aligned} A_n &= \{i\colon x_{n,i,j}^*(P_Y x_{n,i,j}) \geqslant \tfrac{1}{2} \text{ for an infinite number of } j\},\\ B_n &= \{i\colon x_{n,i,j}^*(P_Z x_{n,i,j}) \geqslant \tfrac{1}{2} \text{ for an infinite number of } j\}. \end{aligned}$$

Then for each n either

$$\sum_{i\in A_n} w_i^{2p/p-2} = \infty \quad \text{or} \quad \sum_{i\in B_n} w_i^{2p/p-2} = \infty.$$

Thus, without loss of generality, we may assume that

(5) $\sum_{i\in A_n} w_i^{2p/(p-2)} = \infty$ for all $n\in I$ for some infinite set of integers I.

Let $a\colon N\to\{(n,i)\colon n\in I,\ i\in A_n\}$ be a bijection. We claim that $(P_Y x_{a(k),j(k)})_{k=1}^\infty$ is a small perturbation of a block of the basis $(x_{n,i,j})_{n,i,j=1}^\infty$ for some choice of the $j(k)$'s. To see this, let $m\in N$ and set

$$Q_m\Big(\sum_n\sum_i\sum_j a_{nij}x_{nij}\Big) = \sum_{n=1}^m\sum_{i=1}^m\sum_{j=1}^m a_{nij}x_{nij}.$$

Let ε_k be an arbitrary sequence of positive numbers decreasing to 0. Now for each n and i, $P_Y x_{nij}$ converges weakly to 0 as $j\to\infty$, thus, if we let $x_{a(1),j(1)} = x_{1,1,1}$, then there is an integer m, such that $\|(I-Q_{m_1})P_Y x_{a(1),j(1)}\| < \varepsilon_1$ and an $x_{a(2),j(2)}$ such that $\|Q_{m_1}P_Y x_{a(2),j(2)}\| < \varepsilon_2$. Suppose $x_{a(k),j(k)}$ has been chosen. Then there is an integer m_k and a $j(k+1)$ such that $\|(I-Q_{m_k})P_Y x_{a(k),j(k)}\| < \varepsilon_k$ and $\|Q_{m_k}P_Y x_{a(k+1),j(k+1)}\| < \varepsilon_{k+1}$. Since by (4), $P_Y x_{a(k),j(k)}$ is bounded away from 0 in norm, a sufficiently small choice of the ε_k's yields the claim.

The theorem follows by Lemma 1.1 once we observe that $[(x_{a(k),j(k)})_{k=1}^\infty] \sim (\sum X_p)_{l_p}$. This in turn follows from (5), the definition of a and the following result of Rosenthal [14]: There is a $K<\infty$ such that if (x_n) is a sequence of 3-valued symmetric independent random variables with

$$\frac{\|x_n\|_2}{\|x_n\|_p} = w_n,\ w_n\downarrow 0 \quad\text{and}\quad \sum w_n^{2p/p-2} = \infty,\quad\text{then}$$

$$d([(x_n)], X_p) \leqslant K. \ \blacksquare$$

Remarks. 1. By a similar argument it can be shown that $(\sum l_r)_{l_p}$ is primary, $1\leqslant r, p<\infty$.

2. We do not know if X_p itself is primary. A simpler version of the above proof yields that, if $X_p = Y\oplus Z$, then either Y or Z contains a complemented isomorph of X_p.

3. The proof of Theorem 1.3 can be extended to show that, if X is a reflexive rearrangement invariant function space on $[0,1]$ with indices a, β, $0<\beta\leqslant a<1$, then X is primary (see [2] or [3] for definitions). Thus, in particular, every reflexive Orlicz space on $[0,1]$ is primary. This extension was the result of a conversation with A. Pełczyński, to whom we are grateful. We sketch the proof below.

From results of Boyd [2] one can easily obtain the following theorem:

THEOREM 1.5. *Let T be a bounded linear operator on L_p for all p, $1<p<\infty$. If X is a rearrangement invariant function space with indices a, β, $0<\beta\leqslant a<1$, then T is continuous on X (i.e., $TX\subset X$ and $\|T\|_X<\infty$).*

By arguments of Mitjagin ([12], pp. 85–91), it can be shown that Theorem 1.5 also holds with L_p replaced by $L_p(l_2)$ and X replaced by $X(l_2)$. Moreover, X can be shown to be isomorphic to $X(l_2)$.

If we examine the proof of Theorem 1.6, we see that the following results are needed:

(i) The Haar system, $\{h_i\}_{i=1}^\infty$, is an unconditional basis for X and the corresponding system $\{h_{ij}\}_{i,j=1}^\infty$ is an unconditional basis for $X(l_2)$;

(ii) If $\{1,2,3,\ldots\} = I\cup J$, then either $[h_i]_{i\in I}$ or $[h_i]_{i\in J}$ is isomorphic to X;

(iii) If $[h_i]_{i\in I}\sim X$, then $[h_{ij_i}]_{i\in I}\sim X$;

(iv) If $X\sim Y\oplus Z$ and $Y\sim A\oplus B$ with $X\sim A$, then $X\sim Y$.

The first three of these can be obtained from the corresponding results for L_p and Theorem 1.5. Indeed, consider (ii). An examination of the proof of the result for L_p [6] shows that one can construct an operator T from $L_p[0,1]$ to $[h_i]_{i\in I}$ (say) which is an isomorphism for all p, $1<p<\infty$. Thus, by Theorem 1.5, T is continuous on X. Let P be the basis projection from $L_p[0,1]$ onto $[h_i]_{i\in I}$. Then $T^{-1}P$ is continuous on X, by Theorem 1.5, and hence T is an isomorphism from X onto $[h_i]_{i\in I}$.

Finally, (iv) follows from arguments of Mitjagin ([12], p. 95).

The techniques used here would have wider application if the following problem has an affirmative solution.

PROBLEM. If Y is isomorphic to a complemented subspace of X and X is isomorphic to a complemented subspace of Y is Y isomorphic to X?

2. Let $Z_p = (\sum l_2)_{l_p}$ $(1<p<\infty)$. Our aim in this section is to prove the following theorem.

THEOREM 2.1. *Let X be a subspace of Z_p which contains a subspace Y isomorphic to Z_p. Then for any $\delta>0$ there is a subspace $Z\subseteq Y$ with $d(Z,Z_p)\leqslant 1+\delta$ and a projection P of Z_p onto Z with $\|P\|\leqslant 1+\delta$.*

We note that a theorem of Pełczyński shows that l_p possesses a similar property [13].

We first introduce the basic notation we shall be using. Let the natural basis of Z_p be given by $(e_{ij})_{i,j=1}^\infty$, where

$$\Big\|\sum_i\sum_j a_{ij}e_{ij}\Big\| = \Big(\sum_i\Big(\sum_j a_{ij}^2\Big)^{p/2}\Big)^{1/p}.$$

Let Q_n be the natural projection onto the first n Hilbert spaces and for $E\subseteq N$ (finite or infinite) let P_E be the projection onto those Hilbert spaces indexed by E. Thus

$$Q_n\Big(\sum_{i,j} a_{ij}e_{ij}\Big) = \sum_{i=1}^n\sum_{j=1}^\infty a_{ij}e_{ij}$$

and

$$P_E\Big(\sum_{i,j} a_{ij}e_{ij}\Big) = \sum_{i\in E}\sum_{j=1}^{\infty} a_{ij}e_{ij}.$$

Also let $Q^n = I - Q_n$ be the natural projection onto those Hilbert spaces past the first n.

The idea of the proof of Theorem 2.1 is to construct a sequence of almost disjoint Hilbert subspaces in Y. The following lemma will be very useful. We omit the proof which is quite standard.

LEMMA 2.2. *Let (y_i) be a block basis of $(e_{ij})_{i,j=1}^{\infty}$. Then, if (a_i) is a finitely non-zero sequence of scalars, we have*

(1) *if* $p \geqslant 2$, $\big(\sum |a_i|^p \|y_i\|^p\big)^{1/p} \leqslant \big\|\sum a_i y_i\big\| \leqslant \big(\sum |a_i|^2 \|y_i\|^2\big)^{1/2}$;

(2) *if* $p < 2$, $\big(\sum |a_i|^2 \|y_i\|^2\big)^{1/2} \leqslant \big\|\sum a_i y_i\big\| \leqslant \big(\sum |a_i|^p \|y_i\|\big)^{1/p}$.

Lemma 2.3 provides a sufficient condition for a subspace of Z_p to be isometric to l_2 and well complemented.

LEMMA 2.3. *Let (y_i) be a normalized block basis of (e_{ij}) such that $\|P_{\{k\}}y_i\| = \lambda_k$ for all k and i. Then $\big\|\sum a_i y_i\big\| = \big(\sum a_i^2\big)^{1/2}$ for all scalars (a_i) and $[(y_i)]$ is norm-1 complemented in Z_p.*

Proof. The first assertion was observed by Rosenthal (cf. p. 292 of [14]) so we shall confine ourselves to producing the desired projection P.

Let $P_{\{k\}}y_j = y_{kj}$ so that $\|y_{kj}\| = \lambda_k$ for all j and k and $\sum \lambda_k^p = 1$. For each j let $f_j = \sum_{k=1}^{\infty}\lambda_k^{p-2}y_{kj}$. Then $f_j \in Z_q$ $(1/p+1/q = 1)$ and $\|f_j\|_q = 1$. For $x \in Z_p$ define

$$Px = \sum_{j=1}^{\infty} f_j(x)y_j.$$

By definition, $f_j(y_l) = 0$ if $j \neq l$ while

$$f_j(y_j) = \sum_{k=1}^{\infty}\lambda_k^{p-2}\langle y_{kj}, y_j\rangle = \sum_{k=1}^{\infty}\lambda_k^{p-2}\lambda_k^2 = 1.$$

Thus $P(y_i) = y_i$ for all i and it remains only to check that $\|P\| = 1$.

Let $x \in Z_p$ with $\|x\| = 1$ and let c_{kj} denote the vector x restricted to the support (with respect to the e_{ij}'s) of y_{kj}. Thus $\big(\sum_k\big(\sum_j \|c_{kj}\|^2\big)^{p/2}\big)^{1/p} \leqslant 1$. We shall show $\|Px\| \leqslant 1$.

By the definition of P and the first part of the lemma,

$$\|Px\|^2 = \sum_j |f_j(x)|^2 = \sum_j\Big(\sum_k \lambda_k^{p-2}\langle y_{kj}, c_{kj}\rangle\Big)^2$$
$$\leqslant \sum_j\Big(\sum_k \lambda_k^{p-2}\|y_{kj}\|\,\|c_{kj}\|\Big)^2 = \sum_j\Big(\sum_k \lambda_k^{p-1}\|c_{kj}\|\Big)^2.$$

Thus, by Minkowski's inequality and Hölder's inequality,

$$\|Px\| \leqslant \sum_k\Big(\sum_j\big(\lambda_k^{p-1}\|c_{kj}\|\big)^2\Big)^{1/2} = \sum_k \lambda_k^{p-1}\Big(\sum_j \|c_{kj}\|^2\Big)^{1/2}$$
$$\leqslant \Big(\sum_k\big(\lambda_k^{p-1}\big)^q\Big)^{1/q}\Big(\sum_k\Big(\sum_j \|c_{kj}\|^2\Big)^{p/2}\Big)^{1/p} = \Big(\sum_k\Big(\sum_j \|c_{kj}\|^2\Big)^{p/2}\Big)^{1/p} \leqslant 1. \ \blacksquare$$

Our next lemma will allow us to replace subspaces isomorphic to l_2 by subspaces nearly isometric to l_2. It follows easily from an argument given in [5] and Lemma 2.2.

LEMMA 2.4. *Let X be a subspace of Z_p which is isomorphic to l_2. Then for all $\varepsilon > 0$, X contains a subspace Y with $d(Y, l_2) \leqslant 1+\varepsilon$.*

Proof. This was proved in [14] for $1 < p \leqslant 2$ so we assume $p > 2$. Let $X = [(x_i)]$, where (x_i) is equivalent to the unit vector basis of l_2. By the generalization of Rosenthal and the second named author of an argument of James [5], there is a normalized block basis (y_i) of (x_i) such that for all finitely non-zero sequences of scalars (a_i),

$$(1-\varepsilon)\Big(\sum |a_i|^2\Big)^{1/2} \leqslant \Big\|\sum a_i y_i\Big\|.$$

Since (y_i) converges weakly to 0, by passing to a subsequence, we may assume that (y_i) is a block of (e_{ij}). But then by (1) of Lemma 2.2, we are done. Of course, the case $p < 2$ could be proved similarly. $\blacksquare$

Remark. It is possible using a slightly different argument to take the y_i's as blocks of constant coefficient and constant length. This can be accomplished by taking long averages in order to "kill the l_p part" of the x_i's.

Our next lemma asserts that every Hilbert subspace of Z_p must contain a subspace which "dies off uniformly". We wish to thank L. E. Dor for correcting an error in the proof of this lemma. If $Q: X \to Z$ and Y is a subspace of X by $Q|_Y$ we mean the operator obtained by restricting Q to Y.

LEMMA 2.5. *If X is a subspace of Z_p $(1 < p < \infty, p \neq 2)$ which is isomorphic to l_2, then there is a subspace $Y \subseteq X$ for which $\lim_{n\to\infty}\|Q^n|_Y\| = 0$.*

Proof. If $1 < p < 2$, then we may take $Y = X$ (cf. [14]) so we restrict ourselves to the case $p > 2$.

CLAIM. *For every $\delta > 0$ there is an $\varepsilon > 0$ such that, if $Y \subseteq Z_p$ and $d(Y, l_2) \leqslant 1+\varepsilon$, then for some integer n, $\|Q^n|_Y\| \leqslant \delta$.*

If not, then for some fixed $\delta > 0$ and any $\varepsilon > 0$ we can find a normalized block basis (y_i) of (e_{ij}) with $\|Q^n y_n\| > \delta$ for all n and such that (y_i) is $(1+\varepsilon)$-equivalent to the unit vector basis of l_2. By passing to a subsequence of (y_n), we obtain disjoint finite subsets $E_{n_i} \subseteq N$ so that $\|P_{E_{n_i}}y_{n_i}\| > \delta$ and $\|P_{E_{n_i}}y_l\| = 0$ for $i > l$. Given $\alpha > 0$, since $\delta < \|P_{E_{n_i}}y_{n_i}\| \leqslant 1$, we may assume (by passing to a subsequence) that $|\|P_{E_{n_l}}y_{n_l}\|-\eta|<\alpha$ for all i and some $\eta \geqslant \delta$. Also for some $k<l$, $\|P_{E_{n_k}}y_{n_l}\|<\alpha$. Indeed, the set $\{k\colon \|P_{E_{n_k}}y_{n_l}\|\geqslant \alpha$ for all $l>k\}$ is finite by the disjointness of the E_i's and the fact that $\|y_n\|=1$ for all n.

For simplicity we thus assume that we have y_n, y_m and disjoint finite sets $E, F\subseteq N$ so that

$$\|P_E y_n\| = \|P_F y_m\| = \eta \geqslant \delta$$

and

$$\|P_F y_n\| = \|P_E y_m\| = 0.$$

Then, by Lemma 2.2,

$$\begin{aligned}\|y_n+y_m\|^p &\leqslant \|P_E y_n + P_F y_m\|^p + \|(I-P_E)y_n+(I-P_F)y_m\|^p\\ &\leqslant (\eta^p+\eta^p)+[(1-\eta^p)^{2/p}+(1-\eta^p)^{2/p}]^{p/2}\\ &= 2^{p/2}-(2^{p/2}-2)\eta^p \leqslant 2^{p/2}-(2^{p/2}-2)\delta^p.\end{aligned}$$

But this contradicts the fact that (y_n) is $(1+\varepsilon)$-equivalent to the unit vector basis of l_2 (provided ε is taken sufficiently small depending upon δ) and the claim is proved.

Using the claim and Lemma 2.4 repeatedly, we can find vectors $(y_i)\subseteq X$ and integers $n_i\uparrow\infty$ so that (y_i) is 2-equivalent to the unit vector basis of l_2 and $\|Q^{n_i}y_j\|<2^{-i}$ for all j and i. Assuming without loss of generality that (y_i) is a block basis of (e_{ij}), we see that if $y=\sum a_i y_i$ then by Lemma 2.2

$$\|Q^{n_i}y\| = \Big\|\sum_j a_i Q^{n_i} y_i\Big\| \leqslant \Big(\sum |a_i|^2\Big)^{1/2} 2^{-i} \leqslant 2^{-i}\cdot 2\,\|y\|. \ \blacksquare$$

Proof of Theorem 2.1. We shall construct a sequence of "almost disjoint" Hilbert subspaces of X. First assume $p>2$ and let $\delta>0$. By the hypothesis on X, there are $K<\infty$ and subspaces $Y_n\subseteq X$ such that $d(Y_n, l_2)\leqslant K$ for all n and, if $y_n\in Y_n$, then

$$K^{-1}\Big(\sum \|y_n\|^p\Big)^{1/p} \leqslant \Big\|\sum y_n\Big\| \leqslant K\Big(\sum \|y_n\|^p\Big)^{1/p}. \tag{3}$$

Also we have

(4) *For all integers N and $\varepsilon>0$ there is an integer n_0 such that, if $n\geqslant n_0$, then $\|Q_N y\|\leqslant \varepsilon\|y\|$ for all $y\in Y_n$.*

Indeed, if (4) is false, there are $y_m\in Y_{n_m}$, $n_m\uparrow\infty$ with $\|y_m\|=1$ and $\|Q_N y_m\|\geqslant\varepsilon$ for all m and some fixed N. By (3), (y_m) is equivalent to the unit vector basis of l_p but $\|Q_N y_m\|\geqslant\varepsilon$ implies that (y_m) is equivalent to the unit vector basis of l_2, a contradiction.

Let $\varepsilon_i\downarrow 0$ be arbitrary. Using (4) and Lemma 2.5, we can inductively

construct integers $m_i\uparrow\infty$, subspaces $X_i\subseteq Y_{m_i}$ and disjoint finite subsets E_i of N with ($\sim E$ denotes the complement of E)

$$\|P_{\sim E_i}x\|\leqslant \varepsilon_i\|x\| \quad \text{for} \quad x\in X_i. \tag{5}$$

For each i choose unit vectors $(x_{ij})_{j=1}^\infty\subseteq X_i$ so that

$$\|P_{E_i}x_{ij}-y_{ij}\|\leqslant \varepsilon_i 2^{-j}, \tag{6}$$

where $(y_{ij})_{j=1}^\infty\subseteq[(e_{k,l})_{l=1,\,k\in E_i}^\infty]$ is a block basis of $(e_{k,l})$ satisfying

$$\|P_{\{k\}}y_{ij}\| = \lambda_{ik} \quad \text{for} \quad k\in E_i$$

(λ_{ik} is independent of j).

By Lemma 2.3,

$$\Big\|\sum a_i y_i\Big\| = \Big(\sum |a_i|^2\|y_i\|^2\Big)^{1/2}$$

and $[(y_{ij})_{j=1}^\infty]$ is norm-1 complemented in $[(e_{k,l})_{l=1,\,k\in E_i}^\infty]$. Thus $[(y_{ij})_{i,j=1}^\infty]$ is isometric to Z_p and norm-1 complemented in Z_p.

By standard perturbation arguments, the proof will be completed if we show that the operator $T\colon[(x_{ij})_{i,j=1}^\infty]\to[(y_{ij})_{i,j=1}^\infty]$ given by $Tx_{ij}=y_{ij}$ satisfies $\|T\|\,\|T^{-1}\|\leqslant 1+\delta$ (provided the ε_i's are taken sufficiently small).

By (6), it suffices to show that the operator $S\colon[(x_{ij})_{i,j=1}^\infty]\to[(P_{E_i}x_{ij})_{i,j=1}^\infty]$ defined by $Sx_{ij}=P_{E_i}x_{ij}$ satisfies

(7) $\|S\|\,\|S^{-1}\|\leqslant(1+\delta)^{1/2}$ if the ε_i's are chosen sufficiently small.

Let $(a_{i,j})_{i,j=1}$ be a finitely non-zero sequence of scalars. Then, if $x=\sum_i\sum_j a_{ij}x_{ij}$,

$$\begin{aligned}\|Sx\| &= \Big\|\sum_i P_{E_i}\Big(\sum_j a_{ij}x_{ij}\Big)\Big\|\\ &\leqslant \Big\|\sum_i\sum_j a_{ij}x_{ij}\Big\| + \Big\|\sum_i P_{\sim E_i}\Big(\sum_j a_{ij}x_{ij}\Big)\Big\|\\ &\leqslant \|x\| + \sum_i \varepsilon_i\Big\|\sum_j a_{ij}x_{ij}\Big\|\\ &\leqslant \|x\| + \sum_i \varepsilon_i K\|x\| = \Big(1+K\sum_i\varepsilon_i\Big)\|x\|.\end{aligned}$$

Here we have used (3) and (5).

Similarly,

$$\begin{aligned}\|x\| &= \Big\|\sum_i\sum_j a_{ij}x_{ij}\Big\|\\ &\leqslant \Big\|\sum_i P_{E_i}\Big(\sum_j a_{ij}x_{ij}\Big)\Big\| + \Big\|\sum_i P_{\sim E_i}\Big(\sum_j a_{ij}x_{ij}\Big)\Big\|\\ &\leqslant \|Sx\| + \sum_i \varepsilon_i K\|x\|,\end{aligned}$$

or

$$\Big(1-K\sum_i \varepsilon_i\Big)\|x\|\leqslant \|Sx\|.$$

(7) follows by taking (ε_i) small enough to insure that

$$\Big[1-K\Big(\sum\varepsilon_i\Big)\Big]^{-1}\Big[1+K\Big(\sum\varepsilon_i\Big)\Big] < (1+\delta)^{1/2},$$

and this completes the case $p>2$.

The case $p<2$ may be proved in a similar fashion once we have established the following

LEMMA 2.6. *Let Y be a subspace of Z_p $(1<p<2)$ which is isomorphic to Z_p. Then for every n and $\varepsilon>0$ there is a subspace $W\subseteq Y$, $W\sim l_2$ such that*

(8) $$\|Q_n w\|\leqslant \varepsilon\|w\| \quad \text{for all } w\in W.$$

Proof. Let $Y=[(y_{ij})_{i,j=1}^\infty]$, where $(y_{ij})_{j=1}^\infty$ is K-equivalent to the unit vector basis of l_2 for each i and, if $y_i\in Y_i=[(y_{ij})_{j=1}^\infty]$, then

(9) $$K^{-1}\Big(\sum\|y_i\|^p\Big)^{1/p}\leqslant\Big\|\sum y_i\Big\|\leqslant K\Big(\sum\|y_i\|^p\Big)^{1/p}.$$

By passing to subsequences (using a diagonal process), we may assume that $(y_{ij})_{i,j=1}^\infty$ is a block basis of (e_{ij}).

To prove the lemma we need only show that for all integers n and $\delta>0$ there is a normalized block basis (w_i) of (y_{ij}) which is equivalent to the unit vector basis of l_2 and such that

$$\|Q_n w_i\|\leqslant\delta \quad \text{for all } i.$$

Indeed, if this is true, then by passing to a subsequence we may assume $(Q_n w_i\|Q_n w_i\|^{-1})$ is 2-equivalent to the unit vector basis of l_2 (here we are using that $p<2$ (cf. [14])). Then by Lemma 2.2,

$$\begin{aligned}\Big\|Q_n\Big(\sum a_i w_i\Big)\Big\| &= \Big\|\sum a_i Q_n(w_i)\Big\|\\ &\leqslant 2\Big(\sum|a_i|^2\|Q_n w_i\|^2\Big)^{1/2}\leqslant 2\delta\Big(\sum|a_i|^2\Big)^{1/2}\\ &\leqslant 2\delta\Big\|\sum a_i w_i\Big\|,\end{aligned}$$

which proves (8).

Thus let n and $\delta>0$ be arbitrary and assume that $\|Q_n y_{ij}\|\geqslant\delta$ for all i and $j\geqslant N_i$. We next observe that there is an $\eta>0$ such that for all m there is an i with

$$\|Q^m y_{ij}\|\geqslant\eta \quad \text{for an infinite number of } j.$$

If not, then for all $\eta>0$ there is an m such that for all j

$$\|Q^m y_{ij}\|\leqslant\eta \quad \text{for all but a finite number of } j.$$

Thus there are (y_{i,j_i}) and $m_i\uparrow\infty$ so that for all l,

$$\|Q^{m_l}y_{i,j_i}\|<2^{-l} \quad \text{for all } j.$$

But then, by a result of Arazy and Lindenstrauss (proof of Theorem 1 of [1]), a subsequence of (y_{i,j_i}) is equivalent to the unit vector basis of l_2, contradicting (9). Thus by relabeling the y_{ij}'s necessary, we may assume that we have disjoint sets $E_i\subseteq N$ and an $\eta>0$ such that

$$\|P_{E_i}y_{ij}\|\geqslant\eta \quad \text{for all } i \text{ and } j.$$

We now employ an averaging argument to produce the desired sequence (w_j). Let n and $\delta>0$ be given. For a fixed integer k (to be choosen below) and arbitrary j let $x_j=\sum_{i=1}^k y_{ij}$ and $w_j=\|x_j\|^{-1}x_j$. Since $w_j\in[(y_{ij})_{i=1,j=1}^{k,\infty}]$, (w_j) is equivalent to the unit vector basis of l_2. We shall show that, if k is taken sufficiently large, then $\|Q_n w_j\|<\delta$. For any j,

$$\|Q_n x_j\|=\Big\|\sum_{i=1}^k Q_n y_{ij}\Big\|\leqslant c\Big(\sum_{i=1}^k\|Q_n y_{ij}\|^2\Big)^{1/2}\leqslant ck^{1/2}.$$

(Here c is a constant depending only on $d(Q_nZ_p, l_2)$.) Since the E_i's are disjoint, Lemma 2 of [8] yields

$$\|x_j\|\geqslant\Big(\sum_{i=1}^k\|P_{E_i}y_{ij}\|^p\Big)^{1/p}\geqslant\eta k^{1/p}.$$

Thus $\|Q_n w_j\|\leqslant\eta^{-1}k^{-1/p}ck^{1/2}$, which is turn smaller than δ if k is sufficiently large. This completes the proof of Lemma 2.6 and Theorem 2.1. ∎

Remarks and Questions. 1. The third named author has recently shown that, if X is a complemented subspace of Z_p, then X is isomorphic to one of the four spaces: l_p, l_2, $l_p\oplus l_2$, or Z_p. This result was obtained in [16] under the assumption that X has an unconditional basis.

2. G. Schechtman has proved that there are an infinite number of distinct isomorphic types of L_p subspaces of Z_p $(p>2)$ [17].

3. If X is a subspace of Z_p $(p>2)$ which does not contain an isomorph of Z_p, is X isomorphic to a subspace of $l_p\oplus l_2$? If the answer is yes, does the same result hold for subspaces of L_p?

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A general result on the equivalence between derivation of integrals and weak inequalities for the Hardy–Littlewood maximal operator

by

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Abstract. In this paper we consider integrals of functions belonging to $\varphi(L)$ classes, and their differentiation properties with respect to a translation invariant (B–F) differentiation basis. We prove that the differentiation of certain integrals is equivalent to a certain property of weak type for the maximal function of Hardy–Littlewood, which is associated to the basis. In a sense, this is a sharp result (see Peral [3]).

Introduction. We consider for each $x \in \boldsymbol{R}^n$, a family of open bounded sets $\mathscr{B}(x)$ such that each $B \in \mathscr{B}(x)$ verifies:

(i) $x \in B$;

(ii) there is a sequence $\{B_k\}_{k\in N} \subset \mathscr{B}(x)$ such that $\delta(B_k) \to 0$ as $k \to \infty$ ($\delta(B_k)$ stands for the diameter of B_k).

If these conditions are satisfied, we say that $\{B^k\}$ *contracts to* x, and that $\mathscr{B} = \bigcup_{x\in\boldsymbol{R}^n} \mathscr{B}(x)$ is a *differentiation basis* in $\boldsymbol{R}^n$.

$\mathscr{B}$ is a Busemann–Feller (B–F) basis, if for each $B \in \mathscr{B}$ with $y \in B$, we have $B \in \mathscr{B}(y)$.

A differentiation basis $\mathscr{B}$ is *translation invariant*, if each translation of $B \in \mathscr{B}$ belongs also to $\mathscr{B}$.

We denote by $\mathscr{B}_r$ and $\mathscr{B}_r(x)$ all the elements in $\mathscr{B}$ and $\mathscr{B}(x)$ with a diameter less than r.

If B is a measurable set, then $|B|$ will be its measure.

Let f be a locally integrable function on $\boldsymbol{R}^n$, i.e. $f \in L^1_{\text{loc}}(\boldsymbol{R}^n)$; we define the *upper* and *lower derivatives* of the integral of f with respect to $\mathscr{B}$ by:

$$\bar{D}\left(\int f; x\right) = \sup\left\{\limsup_{k\to\infty} \frac{1}{|B_k|}\int_{B_k} f(y)\,dy:\ B_k \to x;\ \{B_k\} \subset \mathscr{B}(x)\right\},$$

$$\underline{D}\left(\int f; x\right) = \inf\left\{\liminf_{k\to\infty} \frac{1}{|B_k|}\int_{B_k} f(y)\,dy:\ B_k \to x;\ \{B_k\} \subset \mathscr{B}(x)\right\}.$$