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On Ω-stability and structural stability of endomorphisms satisfying Axiom A

by

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Abstract. The main result of this paper is the following theorem:

If $f\colon M\to M$ is a C^r-map ($r\geqslant 1$) on a smooth, compact connected, boundaryless manifold M which satisfies Axiom A, then f is C^r-stable iff f satisfies the no-cycle condition and for every i $f|_{\Omega_i}$ (Ω_i — a component of Spectral Decomposition) is either a one-one map or a quasi-expanding map (i.e. $\dim(E^u_x) = \dim(M)$ for $x\in\Omega_i$).

We give some simple examples of Axiom A endomorphisms which satisfy the properties under consideration.

0. Introduction. In paper [6] we proved that if an Anosov endomorphism (a weak Anosov endomorphism in the terminology introduced in [3]) is structurally stable, then it is either an Anosov diffeomorphism or an expanding map. The same result for ε-stability is proved in [3]. In the present paper we develop the ideas from [3] and [6] and prove the following:

THEOREM A. *If $f\colon M\to M$ is a C^r-map ($r\geqslant 1$) called a C^r-endomorphism* (1) *on a smooth, compact, connected, boundaryless manifold M which satisfies Axiom* A, *then the following two conditions are equivalent*:

1 *f is C^r-stable,*

2 *f satisfies the no-cycle condition and for every i, $1\leqslant i\leqslant I$, $f|_{\Omega_i}$ is either a one-one map or a quasi-expanding map.*

We recall some definitions and notations. For a topological space X and a map $f\in C(X, X)$ a point x is said to be *non-wandering* if for each neighbourhood U of x there is a positive integer n such that $f^n(U)\cap U\neq\emptyset$. The set of all non-wandering points will be denoted by $\Omega(f)$.

One says that an endomorphism g on M is *topologically conjugate* (*Ω-conjugate*) to f if there is a homeomorphism $h\colon M\to M$ ($h\colon \Omega(f)\to\Omega(g)$) satisfying $g\circ h = h\circ g$. The map f is called C^r *structurally stable* (C^r *Ω-stable*) if there is a C^r neighbourhood N of f such that any $g\in N$ is topologically conjugate (Ω-conjugate) to f.

(1) This terminology is not consistent with [6]; there we assume that an endomorphism is a regular map.

Recall from [6] that $f \in C^r(M, M)$ satisfies Axiom A iff

(a) The periodic points of f are dense in $\Omega(f)$;

(b) $\Omega(f)$ is a hyperbolic set, i.e. $\mathrm{Sing}(f) \cap \Omega(f) = \emptyset$ ($\mathrm{Sing}(f)$ denotes the set of all singular points of f) and there exist constants $C > 0$, $0 < \mu < 1$ and a Riemannian metric $\langle\cdot,\cdot\rangle$ on TM such that for every f-trajectory (x_n) contained in $\Omega(f)$ there is a splitting of

$$\bigcup_{n=-\infty}^{+\infty} T_{x_n}M = E^s \oplus E^u = \bigcup_{n=-\infty}^{+\infty} E^s_{x_n} \oplus \bigcup_{n=-\infty}^{+\infty} E^u_{x_n}$$

which is preserved by the derivative Df and the following conditions are satisfied for $n = 0, 1, \ldots$:

$$\|Df^n(v)\| \leqslant C\mu^n \|v\| \quad \text{for} \quad v \in E^s,$$
$$\|Df^n(v)\| \geqslant C^{-1}\mu^{-n} \|v\| \quad \text{for} \quad v \in E^u$$

(for the properties of hyperbolic sets of endomorphisms see [6]).

Recall that (a) implies $\Omega(f)$ is an f-invariant set $(f(\Omega(f)) = \Omega(f))$. If $\Lambda \subset M$ and $f(\Lambda) \subset \Lambda$, then by $\tilde{\Lambda}(f)$ we denote an inverse limit of the system $\ldots \leftarrow \Lambda \xleftarrow{f|\Lambda} \Lambda \xleftarrow{f|\Lambda} \Lambda \leftarrow \ldots$ and by $\tilde{f}$ the shift operator $(x_n) \to (fx_n)$. Recall that $\Omega(\tilde{f}) = \widetilde{\Omega(f)}(f)$ (to simplify the notation we denote $\widetilde{\Omega(f)}(f)$ by $\tilde{\Omega}(f)$).

For $(x_n) \in \tilde{\Omega}(f)$ we define

$$W^u_{f,x_0} = \{y \in M\colon \text{there exists a } (y_n) \in \tilde{M}(f) \text{ such that } y = y_0 \text{ and } \varrho(x_n, y_n) \xrightarrow[n\to-\infty]{} 0\}.$$

(Notice that W^u_{f,x_0} can depend on the whole f-trajectory (x_n), see [6]),

$$W^s_{f,x_0} = \{y \in M\colon \varrho(f^n(y), x_n) \xrightarrow[n\to+\infty]{} 0\}.$$

If x is periodic, then $W^u_{f,x}$ denotes the unstable manifold of the periodic trajectory of x; the same notation will be used in the local case.

Denote by $W^{s(u)}_{f,x_0,a}$ (or $W^{s(u)}_{f,x_0,\mathrm{loc}}$) a *local stable* (*unstable*) *manifold* contained in a ball $B(x, a)$ (or contained in some small ball with a centre in x_0). If f is fixed, we denote $W^{s(u)}_{f,x_0,a}$ by $W^{s(u)}_{x_0,a}$.

Define an equivalence relation in $\mathrm{Per}(f)$ as follows:

$x \sim y$ if for some points $a \in W^u_{x,\mathrm{loc}}$, $b \in W^u_{y,\mathrm{loc}}$ and for some positive integers m, n the following conditions are satisfied:

$$f^m(a) \in W^s_{y,\mathrm{loc}} \qquad f^n(b) \in W^s_{x,\mathrm{loc}},$$

$f^m|_{W^u_{x,\mathrm{loc}}}$ is transverse to the $W^s_{y,\mathrm{loc}}$ in the point a,

$f^n|_{W^u_{y,\mathrm{loc}}}$ is transverse to the $W^s_{x,\mathrm{loc}}$ in the point b.

Let sets $\Omega_j(f)$ be defined as closures of equivalence classes of the relation $\sim$. The sets $\Omega_j(f)$ are invariant. This decomposition of Ω into a sum $\bigcup_{j=1}^{I} \Omega_j(f)$ is usually called the *Spectral Decomposition*.

Denote

$$W^u(\Omega_j) = \{y \in M\colon \text{there exist } (y_n) \in \tilde{M}(f) \text{ such that } y = y_0 \text{ and } \operatorname{dist}(y_n, \Omega_j) \xrightarrow[n\to-\infty]{} 0\},$$

$$W^s(\Omega_j) = \{y \in M\colon \operatorname{dist}(f^n(y), \Omega_j) \xrightarrow[n\to+\infty]{} 0\}.$$

We have

$$W^{u(s)}(\Omega_j) = \bigcup_{(x_n) \in \tilde{\Omega}_j(f)} W^{u(s)}_{x_0}$$

(see [6]).

We say that f satisfies the no-cycle condition iff there exists no sequence of numbers $j_1, \ldots, j_k$ $(k \geqslant 1)$ such that

$$\big(W^s(\Omega_{j_r}) - \Omega_{j_r}\big) \cap \big(W^u(\Omega_{j_{r+1}}) - \Omega_{j_{r+1}}\big) \neq \emptyset$$

for $1 \leqslant r \leqslant k$ and $j_1 = j_k$. (For the assumption that $k \geqslant 1$ see the example in Remark 1.6.)

Under the no-cycle condition one can choose a simple ordering $<$ on the Ω_j, using indices such that $\Omega_1 < \Omega_2 < \ldots < \Omega_I$ and $i < j$ implies that $W^s(\Omega_j) \cap W^u(\Omega_i) = \emptyset$.

Call $f|_{\Omega_j}$ a *quasi-expanding map* iff $\dim E^u_{x_0} = \dim M$ for an f-trajectory $(x_n) \in \tilde{\Omega}_j(f)$ (it is independent of the choice of (x_n)).

In Sections 3 and 4 we prove the following Theorems B and C:

Theorem B. *If condition 2° in Theorem A does not hold, then in any C^r-neighbourhood of f there exists an infinite collection of pairwise non-Ω-conjugate endomorphisms.*

It seems interesting to describe more precisely topological types of endomorphisms in a small neighbourhood of an Axiom A endomorphism which do not satisfy condition 2° (see Theorem 4.11 [6]).

Theorem C. *If a map f satisfying Axiom A is structurally stable, then condition 2° from Theorem A holds together with the following:*

if for some $i_1 \neq i_2$

$$W^u(\Omega_{i_1}) \cap \Omega_{i_2} \neq \emptyset,$$

then $f|_{\Omega_{i_1}}$ is a quasi-expanding map.

In Section 5 we give some examples of sets of endomorphisms; they are open in C^1-topology and consist of endomorphisms satisfying Axiom A and the no-cycle condition which are not C^r Ω-stable. We also give a non-trivial example of a non-Ω-stable endomorphism which can be perturbed to an Ω-stable one.

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1. No-cycle condition, filtration. Let $f\colon M\to M$ be a C^r-endomorphism satisfying Axiom A.

1.1. PROPOSITION. *If f satisfies the no-cycle condition, then for any family of compact neighbourhoods U_i of Ω_i there exists an adapted filtration, i.e. there exist a finite sequence of compact sets $(M_0, \ldots, M_I)$ and a sequence of positive integers $(m_1, \ldots, m_I)$ such that*

(1) $\emptyset = M_0 \subset \ldots \subset M_I = M$;

(2) $f(M_i) \subset \operatorname{int}(M_i)$ *for every* i;

(3) $\Omega_i \subset M_i - f^{-m_i}(M_{i-1}) \subset \operatorname{int}(U_i)$.

Remark. Because of singularities, f need not be an open map. This is the reason why we introduce a filtration adapted to U_i instead of a fine filtration.

Proof of Proposition 1.1. One can proceed as in [8] but with some modifications:

1° We use the following topological lemma:

Let $f\colon M\to M$ be a continuous map. If $Q \subset M$ is a compact neighbourhood of a compact set P and, for every $N \geqslant 0$,

$$\bigcap_{n\geqslant 1} \underbrace{f\big(Q\cap f\big(Q\cap \ldots \cap f(Q\cap f(Q))\ldots\big)\big)}_{f\ n\text{-times}} = P,$$

then there exists a compact set V such that

$$P \subset \operatorname{int}(V) \subset V \subset Q \quad \text{and} \quad f(V) \subset \operatorname{int}(V).$$

Our proof differs from the proof in [8] at the beginning. Observe that there exists an n such that $\underbrace{f\big(Q\cap \ldots \cap f(Q)\ldots\big)}_{f\ n\text{-times}} \subset Q$. This implies

$$f\big(f^{-(n-1)}(Q)\cap f^{-(n-2)}(Q)\cap \ldots \cap Q\big) \subset f^{-(n-1)}(Q);$$

hence $f(W)\subset W$, where $W = f^{-(n-1)}(Q)\cap \ldots \cap Q$. Moreover, W is a compact neighbourhood of P. Hence $f^m(W) \subset \operatorname{int}(W)$ for some positive integer m. Now one can proceed in almost the same way as in [8].

2° We define a fundamental domain of f on $W^u(\Omega_i)$ in the following way. For $(x_n)\in\tilde{\Omega}_i(f)$ denote

$$\tilde{W}^u_{(x_n)} = \{(y_n)\in \tilde{M}(f)\colon\ \varrho(y_n, x_n) \xrightarrow[n\to-\infty]{} 0\}.$$

For sufficiently small $\varepsilon > 0$ $W^u_{x_i,\varepsilon}$ is an embedded disc (see [6]). Hence one can consider the metric ϱ^u in $W^u_{x_i,\varepsilon}$ induced by the Riemannian metric on M restricted to $W^u_{x_i,\varepsilon}$. Denote

$$\tilde{W}^u_{(x_n),\varepsilon,\varrho^u} = \{(y_n)\in \tilde{W}^u_{(x_n)}\colon\ \varrho^u(y_i, x_i) \leqslant \varepsilon \text{ for } i\leqslant 0\},$$

$$\tilde{W}^u_{(x_n),\varepsilon,\varrho^u,\mathrm{open}} = \{(y_n)\in \tilde{W}^u_{(x_n)}\colon\ \varrho^u(y_i, x_i) < \varepsilon \text{ for } i\leqslant 0\}.$$

We claim that the set

$$\tilde{F}^u_i(\varepsilon,\delta) = \bigcup_{(x_n)\in\tilde{\Omega}_i} \tilde{W}^u_{(x_n),\varepsilon,\varrho^u} - \bigcup_{(x_n)\in\tilde{\Omega}_i} \tilde{W}^u_{(x_n),\delta,\varrho^u,\mathrm{open}} \quad \text{for} \quad \varepsilon > \delta$$

is compact.

Indeed, for $k = 1, 2, \ldots$ let

(1) $$(y_n^k)\in \tilde{F}^u_i(\varepsilon,\delta) \quad \text{and} \quad (y_n^k)\to(y_n^0).$$

Let

$$(y_n^k)\in \tilde{W}^u_{(z_n^k),\varepsilon,\varrho^u},$$

where $(z_n^k)\in\tilde{\Omega}_i$. There exists a subsequence $(z_n^{p_k})$ of (z_n^k) which converges to a $(z_n^0)\in\tilde{\Omega}_i$. For simplicity we shall denote $(z_n^{p_k})$ by (z_n^k). Hence, by the continuity of the following function L_j,

(2) $$\tilde{\Omega}_i(f)\ni(z_n) \overset{L_j}{\mapsto} W^u_{z_j,\varepsilon}\in\{C^1\text{-embeddings of the disc with } C^1\text{-topology}\}$$

(for details see [6], Theorem 2.5), we obtain

(3) $$(y_n^0)\in \tilde{W}^u_{(z_n^0),\varepsilon,\varrho^u}.$$

Suppose now that there exists an $(x_n)\in\tilde{\Omega}_i(f)$ such that

(4) $$(y_n^0)\in \tilde{W}^u_{(x_n),\delta,\varrho^u,\mathrm{open}}.$$

It follows from (3), (4) and [6], Theorem 2.1 (e) that if ε is sufficiently small, then

$$W^u_{z_j^0,4\varepsilon,\varrho^u} \supset W^u_{x_j,\varepsilon,\varrho^u} \quad \text{for all } j\leqslant 0.$$

Hence, if ε is sufficiently small, and k is large enough, one can define the f-trajectories (v_n^k) for large k's by the conditions

$$v_0^k\in W^s_{x_0,\mathrm{loc}}\cap W^u_{z_0^k,4\varepsilon}, \quad v_n^k\in W^u_{z_n^k,4\varepsilon,\varrho^u} \quad \text{for } n<0,$$

$$v_n^k = f^n(v_0^k) \quad \text{for } n\geqslant 0.$$

By a local product structure of $\tilde{\Omega}_i(f)$ (see [6], Proposition 3.7), we conclude that $(v_n^k)\in\tilde{\Omega}(f_i)$ and $(v_n^k)\xrightarrow[k\to+\infty]{}(x_n)$. Therefore, in view of (4) and by the continuity of the functions L_j, we get $(y_n^k)\in \tilde{W}^u_{(v_n^k),\delta,\varrho^u,\mathrm{open}}$ for sufficiently large k, which contradicts (1). Therefore $\tilde{F}^u_i(\varepsilon,\delta)$ is compact. This finishes the proof.

Define the fundamental domain of f on $W^u(\Omega_i)$ by $F^u_i(\varepsilon,\delta) = \pi_0\tilde{F}^u_i(\varepsilon,\delta)$ for δ/ε sufficiently small ($\pi_0\colon \tilde{M}(f)\to M$ $\pi_0\big((x_n)\big) = x_0$). The fact that $F^u_i(\varepsilon,\delta)\cap\Omega_i = \emptyset$ easily follows from the local maximality of $\Omega_i(f)$.

1.2. NOTATION. Let g be C^1-*near* f. There is a unique conjugacy

$$h_{gf}\colon\ (\tilde{\Omega}(f),\tilde{f}) \to (h_{gf}\tilde{\Omega}(f),\tilde{g}) \hookrightarrow (\tilde{\Omega}(g),\tilde{g})$$

near the inclusion $\tilde{\Omega}(f) \hookrightarrow \mathop{\mathsf{P}}\limits_{-\infty}^{+\infty} M_n,\ M_n = M.$

θ_{gf} is an induced conjugacy $\theta_{gf}\colon \operatorname{Per}(f) \to \theta_{gf}\operatorname{Per}(f) \hookrightarrow \operatorname{Per}(g)$. The existence and properties of h_{gf} and θ_{gf} are described in [6], Theorem 1.20 and [3].

The following Lemma is a simple conclusion of the local maximality of $\Omega(f)$ and of the theorem on ε-trajectories (see [2], [6]):

1.3. LEMMA. *There exist a neighbourhood U of $\Omega(f)$ and a neighbourhood $N \subset C^1(M, M)$ of f such that $g \in N$ implies that if (x_n) is a g-trajectory in U then $(x_n) \in h_{gf}\tilde{\Omega}(f)$.*

1.4. PROPOSITION. *Let $(M_0, \ldots, M_I)$ be a filtration for f adapted to sufficiently small sets U_i. Then there is a neighbourhood N_1 of f in $C^1(M, M)$ such that if $g \in N_1$ then h_{gf} maps $\tilde{\Omega}(f)$ onto $\tilde{\Omega}(g)$ and $\Omega(g)$ is g-invariant, i.e. $g(\Omega(g)) = \Omega(g)$.*

Proof. Let U and N be such as in Lemma 1.3. One can assume that U_i's are pairwise disjoint and $\bigcup_{i=1}^{I} U_i \subset U$. Let $N_1 \subset N$ be a neighbourhood of f such that if $g \in N_1$ then

(1) $$g^{m_i+1}\big(f^{-m_i}(M_i)\big) \subset \operatorname{int}(M_i),$$

(2) $$g(M_i) \subset \operatorname{int}(M_i),$$

(3) $$g\big(f^{-m_i}(M_i)\big) \subset f^{-m_i}(M_i).$$

For any $g \in N_1$, (1) and (2) imply $\Omega(g) \subset \bigcup_i \big((M_i) - f^{-m_i}(M_{i-1})\big) \subset \bigcup_i U_i$. So $\tilde{\Omega}(g) = h_{gf}\tilde{\Omega}(f)$.

Now we claim that $g(\Omega(g)) = \Omega(g)$. Suppose that $x \in \Omega(g)$ and $g^{-1}(x) \cap \Omega(g) = \emptyset$. For some i, $x \in \operatorname{int}(M_i) - f^{-m_i}(M_{i-1})$. Since $g^n(x) \in \Omega(g)$ for $n \geqslant 0$, we have $g^n(x) \in U$. Since $x \in \Omega(g)$, there exists a sequence of g-trajectories (z_n^k) such that $z_0^k \xrightarrow[k\to\infty]{} x$ and $z_{s(k)}^k \to x$ for some sequence of negative integers $s(k)$. Let $(z_n^{p_k})$ be a subsequence of (z_n^k) such that $(z_n^{p_k})$ converges to some g-trajectory (x_n). Of course $x_n = g^n(x)$ for $n \geqslant 0$. Lemma 1.3 yields the existence of a negative integer q such that $x_q \notin U_i$. This and (3) imply $x_q \notin M_i$. Hence there exists a p_k such that $z_q^{p_k} \notin M_i$ but $z_{s(p_k)}^{p_k} \in M_i$ $(s(p_k) < q)$. This contradicts (2).

In the similar way to that followed in [4] one can check the following:

1.5. PROPOSITION. *If, for all g C^r-near f, h_{gf} maps $\tilde{\Omega}(f)$ onto $\tilde{\Omega}(g)$, then f satisfies the no-cycle condition.*

1.6. COROLLARY. *If there is a cycle for f, then f is not C^r Ω-stable.*

Proof. Indeed for g C^r-near f let (x_n) be a g-trajectory such that $(x_n) \notin h_{gf}\tilde{\Omega}(f)$ and $(x_n) \in \tilde{\Omega}(g)$. By Lemma 1.3, there exists an integer N such that $x_N \notin \theta_{gf}\operatorname{Per}(f)$. For the existence of a conjugacy $\Omega(f)$ with $\Omega(g)$, x_N must be a limit of a sequence of g-periodic points; hence $\operatorname{Per}(g) \not\supseteq \theta_{gf}\operatorname{Per}(f)$. Now it is obvious that the conjugacy cannot exist because θ_{gf} is a one-one map and, for any positive integer k, f has only a finite number of points of period k.

1.7. Remark. In the no-cycle condition for endomorphisms satisfying Axiom A it is essential to consider the case of cycles of length one. It is well known that such cycles cannot exist for diffeomorphisms.

EXAMPLE. $f\colon S^1 \to S^1$, $f(z) = e^{i\cdot\varphi(-i\cdot\log(z))}$, where φ is as in Figure 1.

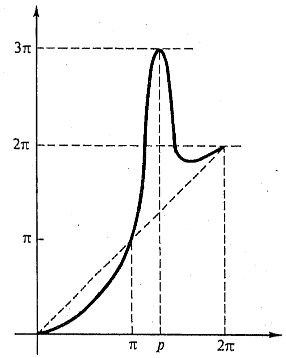

Fig. 1

Here $\Omega = \{-1, 1\}$, $\Omega_1 = \{1\}$, $\Omega_2 = \{-1\}$. A point $e^{i\cdot p} \in W^s(\Omega_2) \cap W^u(\Omega_2)$. However, we have

1.8. PROPOSITION. *Let $f\colon M \xrightarrow{\text{onto}} M$ satisfy Axiom A. Suppose that under Spectral Decomposition $\Omega(f)$ becomes a sum of only one set Ω_1. Then $\Omega(f) = M$.*

Proof. First we claim that

(1) $$M = \bigcup_{i\geqslant 0} f^{-i}(\Omega).$$

Let $x \in M$. If U is any neighbourhood of Ω, then there exists an $N > 0$ such that if $n \geqslant N$ then $f^n(x) \in U$. If U is sufficiently small, then $f^N(x) \in W^s_{y,\varepsilon}$ for a $y \in \Omega(f)$ and for an ε such that $f^i(W^s_{y,\varepsilon}) \cap \operatorname{Sing}(f) = \emptyset$ for $i \geqslant 0$. There exists a point z arbitrarily close to $f^N(x)$ such that $z \in W^s_{v,\varepsilon}$ for a periodic v and $f^i(W^s_{v,\varepsilon}) \cap \operatorname{Sing}(f) = \emptyset$ for $i \geqslant 0$.

If V is any neighbourhood of z, then the λ-Lemma (see [5]), the fact that $f^p|_{W^u_{v,\mathrm{loc}}} \pitchfork W^s_{w,\mathrm{loc}}$ for any periodic w and suitable p, the continuity of L_j (see 1.1 (2)) and the density of $\mathrm{Per}(f)$ in $\Omega(f)$ imply $W^u(\Omega) \subset \overline{\bigcup_{i\geqslant 0} f^i(V)}$. $z \in W^u(\Omega)$ because f maps M onto M. Thus $z \in \Omega$. So $f^N(x) \in \Omega$, which proves (1).

Denote $f^{-1}(\Omega) - \Omega$ by A. Suppose $A \neq \emptyset$. (1) yields $M = \Omega \cup \bigcup_{i\geqslant 0} f^{-i}(A)$. Arguments which prove (1) give us $A \cap U = \emptyset$; hence A is a closed set.

Thus we could decompose M into a union of a countable family of closed sets from which two at least are nonempty, but this is impossible. (Indeed, if a connected, locally arcwise connected, complete metric space $M = \bigcup_{i=1}^{\infty} K_i$, $K_i \cap K_j = \emptyset$ for $i \neq j$, K_i are closed sets and K_1, $K_2 \neq \emptyset$, then $\mathrm{Fr} K_1 \neq \emptyset$ or $\mathrm{Fr} K_2 \neq \emptyset$; hence $N = M - \bigcup_{i=1}^{\infty} \mathrm{int} K_i \neq \emptyset$. $\mathrm{Fr} K_i$ are nowhere dense in N and N is the complete metric space. This situation contradicts the Baire theorem.)

1.9. Remark. It f is not "onto", then the above proposition is not true. Here is an example: $f\colon S^1 \to S^1$, $f(z) = e^{(i/2)\cdot \sin(i\cdot \log(z))}$.

2. Proof of Theorem A. Let $f \in C^r(M, M)$ be an endomorphism satisfying Axiom A.

2.1. Lemma. *There exist numbers $R > 0$, $a > 0$, $A > 0$ such that if $\varrho_{C^1}(f, g_1) < a$ and $\varrho_{C^1}(f, g_2) < a$ then for any $\eta > 0$ $\varrho_{C^0}(g_1, g_2) < \eta$ implies*:

(1) $$\varrho\big(x_j, \pi_j h_{g_1 g_2}((x_n))\big) < A\eta,$$

(2) $$\varrho_H(W^u_{g_1, \pi_j h_{g_1 g_2}((x_n)), R},\ W^u_{g_2, x_j, R}) < A\eta,$$

for any g_2-trajectory $(x_n) \in h_{g_2 f}(\tilde\Omega(f))$ (ϱ_H — a Hausdorff metric between sets).

The proof is standard and will be omitted.

It is easy to prove that we can choose an R such that if $\varrho_{C^1}(f, g) < a$ then $g|_{B(\pi_0 h_{gf}((x_n)), R)}$ is a diffeomorphism onto its image and R has the properties described in [6], Theorem 2.1.

From Lemma 2.1 one can obtain by a standard procedure:

2.2. Lemma. *There exists an $a > 0$ such that for any $\vartheta > 0$ there exists a positive integer $p(\vartheta)$ which has the following property.*

For any $\eta > 0$ and $g_1, g_2 \in C^1(M, M)$ such that $\varrho_{C^1}(f, g_1) < a$, $\varrho_{C^1}(f, g_2) < a$ if $\varrho_{C^0}(g_1, g_2) < \eta$ then for every $(x_n) \in h_{g_2 f}\tilde\Omega(f)$ the conditions

(1) $g_1|_{W_q} = g_2|_{W_q}$ *for every* $q\colon 0 < q < p(\vartheta)$

(we define $W_q = (g_2|_{B(x_{-q}, R)})^{-1} \circ \ldots \circ (g_2|_{B(x_{-1}, R)})^{-1}(W^u_{g_2, x_0, R})$*),*

(2) $g_2(x_q) = g_1(x_q)$ *for* $-p(\vartheta) < q < p(\vartheta)$

imply
$$\varrho_H(W^u_{g_2, x_0, R},\ W^u_{g_1, \pi_0 h_{g_1 g_2}((x_n)), R}) < \vartheta\cdot\eta.$$

Assuming only (2), *we get*
$$\varrho\big(\pi_0 h_{g_1 g_2}((x_n)),\ x_0\big) < \vartheta\cdot\eta.$$

2.3. Lemma. *Let $\tilde\Omega_i(f)$ be infinite. Then for any $(x_n) \in \Omega_i(f)$ and any $\delta > 0$ there exists a $y \in \big(W^u_{x_0, \delta} \cap \Omega(f)\big) - \{x_0\}$.*

Proof. Of course, $\dim W^u_{x_0} > 0$, and so the lemma easily follows from the local product structure of $\Omega(f)$ (see [6], Proposition 3.9).

Now we shall prove our key proposition:

2.4. Proposition. *If $f|_{\Omega_i}$ is not a one-one map, then there exist: a $g \in C^r(M, M)$ arbitrarily close to f in C^r-topology and two different g-periodic points x, $y \in \pi_0 h_{gf} \tilde\Omega_i$ (of periods arbitrarily large) such that $y \in W^u_{x,\mathrm{loc}}$.*

Proof. Let (x_n), $(y_n) \in \tilde\Omega_i(f)$, $x_0 = y_0$, $x_{-1} \neq y_{-1}$, and let y_{-1} be a non-periodic point. (If x_0 is periodic, we consider the periodic trajectory of x_0 instead of (x_n).)

Let

(1) $$L = \max\big(2\cdot \sup_{x\in M} \|Df_x\|,\ 2\cdot \sup_{x\in U} \|(Df_x)^{-1}\|,\ 1\big)$$
for a U — a neighbourhood of $\Omega_i(f)$.

Let $\vartheta = 1/(8\cdot L^2)$ and let a, R, $p = p(\vartheta)$ be as in Lemma 2.2. Let V_1 and V_2 be neighbourhoods of x_{-p} and y_{-p}, respectively, such that $f^q|_{V_j}$ are diffeomorphisms onto their images, for $q\colon 0 \leqslant q \leqslant 2p$, $j = 1, 2$. Let $d > 0$ be a number such that
$$B(y_{-1}, d) \subset f^{p-1}(V_2) \quad\text{and}\quad B(x_0, d) \subset f^p(V_1).$$

Denote
$$B_1^q = \begin{cases} (f|_{f^{p+q}(V_1)})^{-1} \circ \ldots \circ (f|_{f^{p-1}(V_1)})^{-1}(B(x_0, d)) & \text{for } q < 0,\\ f^q(B(x_0, d)) & \text{for } q \geqslant 0,\end{cases}$$
$$B_2^q = \begin{cases} (f|_{f^{p+q}(V_2)})^{-1} \circ \ldots \circ (f|_{f^{p-2}(V_2)})^{-1}(B(y_{-1}, d)) & \text{for } q < -1,\\ f^{q+1}(B(y_{-1}, d)) & \text{for } q \geqslant -1.\end{cases}$$

Assume for d also the following:
$$B_1^q \cap B(y_{-1}, d) = \emptyset \quad\text{for}\quad -p \leqslant q \leqslant p,$$
$$B_2^q \cap B(y_{-1}, d) = \emptyset \quad\text{for}\quad -p \leqslant q \leqslant p,\ q \neq -1.$$

By Lemma 2.3, there exists a point $w_{-1} \in \big(W^u_{y_{-1}, d/10} \cap \Omega(f)\big) - \{y_{-1}\}$; for $q\colon -p \leqslant q < 0$ we define w_q by the formulas: $w_q \in B_2^q$, $f^{-q-1}(w_q) = w_{-1}$.

Let $d_1 < \varrho(w_{-1}, y_{-1})/2$.

All endomorphisms appearing further in this proof will be some perturbations of f inside the closed ball $B(y_{-1}, d_1) = V$.

If d_1 is sufficiently small, then one can fix coordinates on neighbourhoods of V and $f(V)$ such that the metric induced by a Riemannian metric on M is close to the Euclidean metric defined by the coordinates. One can introduce a standard metric $\varrho_{C^r}(\cdot, \cdot)$ between perturbations of f.

Moreover, one can obtain $f(V) \subset U$ (for the definition of U see (1)).

One can easily check that

$$f(V) \supset B(y_0, d_1/L). \tag{2}$$

Fix any number $\beta > 0$ such that $\beta < \alpha$ (α is defined in Lemmas 2.1, 2.2) and if $\varrho_{C^1}(f, g) < \beta$ then $g|_V$ is a diffeomorphism onto $f(V)$ and L is a Lipschitz constant for $g|_V$ and $(g|_V)^{-1}$.

Let $\delta > 0$ ($\delta < d_1$) be such that the following condition holds:

(3) *if $x, y \in B(y_{-1}, \delta)$ and, for a positive integer K, $\varrho(x, y) < \delta/K$, then for any $g \in B_{C^r}(f, \beta)$ there exists a $\hat{g}$ such that $g(y) = \hat{g}(x)$, $\varrho_{C^r}(g, \hat{g}) < \beta/K$ and $\varrho_{C^0}(g, \hat{g}) < 2L\varrho(x, y)$.*

Moreover, assume that δ satisfies the following condition:

$$A \cdot 2L\delta + \delta < \min\Big(\inf_{-p \leqslant q \leqslant p} \big(\operatorname{dist}(x_q, M - B_1^q)\big), \inf_{-p \leqslant q \leqslant p} \big(\operatorname{dist}(w_q, M - B_2^q)\big), \operatorname{dist}(w_{-1}, V)\Big). \tag{4}$$

Take periodic trajectories u_n^0 and v_n^0 with periods arbitrarily large, such that

$$\varrho(v_q^0, w_q) < \delta/2, \tag{5}$$

$$\varrho(u_q^0, x_q) < \delta, \tag{6}$$

for $|q| \leqslant p$ and

$$\varrho(u_0^0, y_0) < \delta/4L, \tag{7}$$

$$\operatorname{dist}(W^{u_0}_{v^0_{-1}, d/2}, \hat{u}_{-1}^0) < \delta/2^4 \tag{8}$$

where $\hat{u}_{-1}^0$ is defined as a unique counterimage (it exists by (2) and (7)) of u_0^0 under f which lies in V.

Of course,

$$\varrho(\hat{u}_{-1}^0, y_{-1}) < \delta/4. \tag{9}$$

Observe that $W^{u_0}_{v^0_{-1}, d/2} \subset B_2^{-1}$.

Let $g \in C^1(M, M)$ and

$$\varrho_{C^0}(g, f) < 2L\delta \quad \text{and} \quad \varrho_{C^1}(g, f) < \beta. \tag{10}$$

Then from (4), (5), (6) and Lemma 2.1 it follows that

$$\theta_{gf}(v_q^0) \in B_2^q, \quad \theta_{gf}(v_{-1}^0) \notin B(y_{-1}, d_1) = V \quad \text{and} \quad \theta_{gf}(u_q^0) \in B_1^q \quad \text{for } |q| \leqslant p. \tag{11}$$

Then (by Lemma 2.2 and from the definitions of ϑ and p):

$$\varrho\big(\theta_{gf}(u_0^0), u_0^0\big) < \delta/4L; \tag{12}$$

hence by (2) and (7) there exists a $u_{-1}(g)$ — a unique counterimage of $\theta_{gf}(u_0^0)$ under g which lies in V.

We shall construct by induction a sequence of endomorphisms f_k satisfying the following conditions:

$$\varrho_{C^r}(f_k, f_{k-1}) < \beta/2^{k+3}, \tag{13_k}$$

$$\varrho_{C^0}(f_k, f_{k-1}) < 2L\delta/2^{k+3}, \tag{14_k}$$

$$\operatorname{dist}\big(W^{u_k}_{v^k_{-1}, d/2}, u_{-1}(f_k)\big) < \delta/2^{k+4} \tag{15_k}$$

(we denote $\theta_{f_j f}(u_n^0)$ by u_n^j and $\theta_{f_j f}(v_n^0)$ by v_n^j),

$$\varrho\big(u_{-1}(f_k), u_{-1}(f_{k-1})\big) < \delta/2^{k+2}. \tag{16_k}$$

Of course, (13_k), (14_k), (16_k) make sense for $k \geqslant 1$.

Define $f_0 = f$; then (15_k) holds by (8). Assume that f_j are constructed for $j \leqslant k$ for which (13_j), (14_j), (15_j), (16_j) hold. We shall construct f_{k+1}.

By (15_k) there exists a $u^{*k} \in W^{u_k}_{v^k_{-1}, d/2}$ such that

$$\varrho\big(u^{*k}, u_{-1}(f_k)\big) < \delta/2^{k+4}. \tag{17}$$

By (9), (16_j) ($j = 1, \ldots, k$) and (17)

$$\varrho\big(u_{-1}(f_k), y_{-1}\big) < \delta/2 < \delta \quad \text{and} \quad \varrho(u^{*k}, y_{-1}) < \delta.$$

Then one can use (3) (put $K = 2^{k+4}$) to construct f_{k+1} such that

$$\varrho_{C^r}(f_{k+1}, f_k) < \beta/2^{k+4},$$

$$\varrho_{C^0}(f_{k+1}, f_k) < 2L\delta/2^{k+4} \quad \text{and} \quad f_{k+1}(u^{*k}) = f_k\big(u_{-1}(f_k)\big) = u_0^k.$$

Since for $g = f_{k+1}$ (10) is satisfied, formulas (11) allow us to use Lemma 2.2. So, by definitions of ϑ and p, one can obtain

$$\varrho\big(u_{-1}(f_{k+1}), u^{*k}\big) < \delta/2^{k+6} \quad \text{and} \quad \varrho_H(W^u_{f_{k+1}, v^{k+1}_{-1}, d/2}, W^u_{f_k, v^k_{-1}, d/2}) < \delta/2^{k+6}, \tag{18}$$

which implies (15_{k+1}).

(17) and (18) imply $\varrho\big(u_{-1}(f_{k+1}), u_{-1}(f_k)\big) < \delta/2^{k+3}$, i.e. (16_{k+1}). This finishes our induction.

Define $g = \lim_{k \to +\infty} f_k$. (13_k), (14_k), (15_k) imply that g is a C^r-map satisfying the conditions of our proposition.

This proposition and Lemma 1.3 imply the following

2.5. COROLLARY. *If $f|_{\Omega_i}$ is not a one-one map, then there exist a $g \in C^r(M, M)$ arbitrarily close to f in C^r-topology, two different periodic g-trajectories $(x_n), (y_n) \in h_{gf}(\tilde{\Omega}_i)$ of periods arbitrarily large and a g-trajectory $(z_n) \in h_{gf}(\tilde{\Omega}_i)$ such that $z_0 = x_0$ and $\varrho(z_n, y_n) \xrightarrow[n\to-\infty]{} 0$.*

2.6. PROPOSITION. *In some neighbourhood N of f in C^1-topology the set of endomorphisms g satisfying the condition*

(*) *if $x, y \in \mathrm{Per}(f)$ $x \neq y$ and $\dim(E^u_y) < \dim(M)$, then $\theta_{gf}(x) \notin W^u_{g,\theta_{gf}(y)}$*

is a residual (dense, G_δ) subset of N in C^r-topology.

Proof. One can proceed as in [6], Theorem 4.3.

We shall prove Theorem A.

Assume 1° for f. Then Corollary 1.6 yields the no-cycle condition. Now, if there exists an i such that $f|_{\Omega_i}$ is neither a one-one map nor a quasi-expanding map, then one can perturb f to g_1 which has properties described for g in Corollary 2.5. On the other hand, one can perturb f to a g_2 which has property (*) (see Proposition 2.6).

Now suppose that there exists an Ω-conjugacy h. One can easily check that h must preserve the relation between periodic points which defines the equivalence classes Ω_j; hence h induces a permutation σ of Ω_j's.

If g is C^1-close to f, we say that $g|_{\Omega_{j(g)}}$ satisfies property P if $g|_{\Omega_{j(g)}}$ is not a one-one map and we say $g|_{\Omega_{j(g)}}$ satisfies property Q if $y \in W^u_{x,\mathrm{loc}}$ for some periodic points $x, y \in \Omega_j(g)$ (see Proposition 2.4 and Corollary 2.5).

Observe that σ preserves properties P and Q. Observe also that if $g|_{\Omega_{j(g)}}$ is a quasi-expanding map which satisfies property P, then $g|_{\Omega_{j(g)}}$ satisfies property Q. Thus if $g_2|_{\Omega_{j(g_2)}}$ satisfies P, then this satisfies Q if ond only if this is a quasi-expanding map. Because in the case of g_1 the number of j's such that $g_1|_{\Omega_{j(g_1)}}$ satisfies P and Q is greater than in the case af g_2, we obtain a contradiction.

Assume that 2° holds. Then, in view of Proposition 1.4, it suffices to show that if g is C^1-close to f, then h_{gf} is a lift of a homeomorphism.

First suppose that $f|_{\Omega_i}$ is a quasi-expanding map. Then there are: a neighbourhood $N \subset C^1(M, M)$ of f and numbers $\alpha > 0$ and $\lambda > 1$ such that $g \in N$ implies:

(1) $$\|Dg(v)\| > \lambda\|v\| \quad \text{for} \quad v \in T\Big(\bigcup_{x\in\Omega_i} B(x, 3\alpha)\Big),$$

(2) $$\varrho(\pi_0 h_{gf}, \pi_0) < \alpha$$

and

(3) for $x \in \bigcup_{y\in\Omega(f)} \big(B(y, \alpha)\big)$, $g|_{B(x,2\alpha)}$ is a diffeomorphism onto its image.

Suppose that for some $g \in N$ there exist $(x_n), (y_n) \in \tilde{\Omega}_i(f)$ such that $x_0 = y_0$ and $\pi_0 h_{gf}\big((x_n)\big) \neq \pi_0 h_{gf}\big((y_n)\big)$. Denote

$$\hat{x_n} = \pi_n h_{gf}\big((x_n)\big), \quad \hat{y_n} = \pi_n h_{gf}\big((y_n)\big), \quad \varrho(\hat{x_0}, \hat{y_0}) = \tau.$$

(2) implies $\varrho(\hat{x_n}, \hat{y_n}) < 2\alpha$ for $n \geqslant 0$. Let K be such that

(4) $$\lambda^K \cdot \tau > 2\alpha.$$

Using (1), (2), (3), one can easily construct a family of curves $L_n\colon \langle 0, 1\rangle \to M$ such that $g \circ L_n = L_{n+1}$, $L_n(0) = \hat{x_n}$ and $L_n(1) = \hat{y_n}$ for $n\colon 0 \leqslant n \leqslant N$, $\mathrm{length}(L_n) < 2\alpha$. Thus, by (4), $\mathrm{length}(L_0) < \tau$, which is a contradiction. Therefore, if $g \in N$ then h_{gf} is a lift of some continuous map h_1. Similarly, $h_{fg} = h_{gf}^{-1}$ is a lift of a h_2. So $h_1 h_2 = h_2 h_1 = \mathrm{id}$, and hence h_1 is a homeomorphism.

Now suppose that $f|_{\Omega_i}$ is a one-one map. Then there exists a neighbourhood U of Ω_i such that $f|_U$ is a diffeomorphism onto its image. So if $g \in C^1(M, M)$ is C^1-close to f, then there exists a unique homeomorphism h close to identity which conjugates $\Omega_i(f)$ with a g-invariant subset of U (this is a well-known fact). Thus, the uniqueness of h_{gf} implies that h_{gf} is a lift of h.

2.7. Remark. Using a similar idea to that presented in [3] or in the Introduction of [6] and something like Lemma 2, one can easily prove that condition 2° is necessary for the $\varepsilon - C^r$ Ω-stability of an Axiom A endomorphism. This is of course weaker than Theorem A.

Theorem A implies the following

2.8. PROPOSITION. *If f has no cycles and for some i $\Omega_i(f)$ is a repeller (i.e. there exists a compact neighbourhood U of $\Omega_i(f)$ such that $f^{-1}(U) \subset \mathrm{int}\, U$ and $\bigcap_{n\geqslant 0} f^{-n} U = \Omega_i$ or equivalently $W^s(\Omega_i) \subset \Omega_i$) and if $f|_{\Omega_i}$ is neither a quasi-expanding nor a one-one map, then there exists a neighbourhood in C^1-topology $N \ni f$ such that $g \in N$ implies that g is not C^r Ω-stable.*

Proof. The above-mentioned properties of f are preserved under C^1-perturbations. The main thing is to prove that the property "the map is not one-one" is preserved.

3. Proof of Theorem B. Let $C^r(M, M) \supset B(f, d_1) \supset \ldots \supset B(f, d_n) \supset \ldots$ be a sequence of balls with a centre f in C^r-topology with radii $d_n \to 0$. Let

$$a_n = \inf\{m\colon \text{there is a map } g \in B(f, d_n) \text{ and there is a point } x \in \mathrm{Per}(g) - \theta_{gf}\mathrm{Per}(f) \text{ such that } m \text{ is a period of } x\}$$

(we assume that $\inf \emptyset = +\infty$).

By Lemma 1.3, $a_n \to +\infty$.

(a) Suppose that, for every n, $a_n < +\infty$. Let a_{j_n} be a strictly increasing subsequence of a_n. It is obvious that the maps g_{j_n} which realize numbers a_{j_n} are pairwise non-conjugate.

(b) Suppose that $a_q = \infty$ for an integer q. By Proposition 2.4, one can construct (by induction) maps $\hat{g_k} \in B(f, d_q) = N$ such that

(1_k) *$y^k \in W^u_{x^k,\mathrm{loc}}$ for some x^k, $y^k \in \theta_{\hat{g_k} f}(\mathrm{Per} f)$ with minimal periods s^k, r^k, respectively, and such that*

$$\max(s^k, r^k) < \min(s^{k+1}, r^{k+1})$$

for $k = 1, 2, \ldots$

Using an idea from the proof of Theorem 4.8 of [6], it is easy to perturb $\hat{g_k}$ to $g_k \in N$ with the property (1_k), where we replace $\hat{g_k}$ by g_k, and moreover $x \notin W^u_{g_k, y}$ for any $x, y \in \mathrm{Per}(g)$ such that $\dim(E^u_y) < \dim(M)$ and the minimal periods of x and y are smaller than $\min(s^k, r^k)$.

Proceeding as in the proof of Theorem A, one can check that g_k are pairwise non-conjugate.

4. Proof of Theorem C.

4.1. LEMMA. *If $W^u(\Omega_j(f)) \cap \Omega_i(f) \neq \emptyset$ for $i \neq j$, then there exist a $g \in C^r(M, M)$ arbitrarily C^r-close to f such that $x \in W^u_{g,y}$ for some $x \in \theta_{gf}(\Omega_i(f) \cap \mathrm{Per}(f))$ and $y \in \theta_{gf}(\Omega_j(f) \cap \mathrm{Per}(f))$.*

Proof. A proof of this lemma can be based on the same ideas as the proofs of Proposition 2.4 and Lemma 4.6 of [6] but is easier.

Proof of Theorem C. Theorem C immediately follows from Lemma 4.1 and Proposition 2.6.

5. Examples. Introduce the following symbols:

z^n — the standard expanding map of S^1,

h — a diffeomorphism of S^1 with a sharp source (i.e. the expansion coefficient large enough) at $z = -1$, a sharp sink at $z = +1$ and no other fixed points,

U — any Anosov diffeomorphism $M \to M$,

H — the Shub endomorphism (see [7]) described as follows:

$$H: S^1 \to S^1, \quad H(z) = e^{i \cdot \varphi(-i \cdot \log(z))},$$

where φ is given by Figure 1 (p. 67).

The point 1 is a sink. Denote other two fixed points by a and b. Of course, $\Omega_1 = \{1\}$, $\Omega_2 = S^1 - \bigcup_{n \geq 0} H^{-n}(P)$, where P is as in Figure 2.

Further we shall denote Ω_2 by ω.

We shall consider some examples of endomorphisms:

1. $H \times U$: $S^1 \times M \to S^1 \times M$.

Here $\Omega_1(H \times U) = \{1\} \times M$, $\Omega_2(H \times U) = \omega \times M$. $H \times U$ is Ω_1-stable but is not Ω_2-stable persistently (in view of Proposition 2.8).

2. $H \times z^n$: $T^2 \to T^2$.

Here $\Omega_1 = \{1\} \times S^1$, $\Omega_2 = \omega \times S^1$. $H \times z^n$ is Ω_2-stable since it is quasi-expanding but $H \times z^n$ is not Ω_1-stable persistently because it is not a one-one map persistently. Indeed, $\tilde{\Omega}_1(H \times z^n)$ is a solenoid, and so

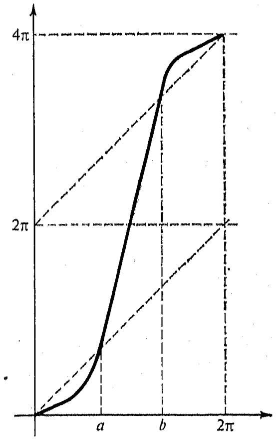

Fig. 2

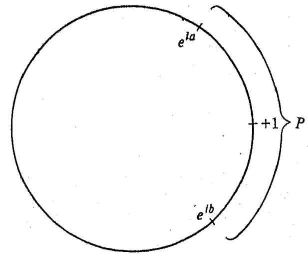

Fig. 3

it cannot be homeomorphically mapped into R^2 (no solenoid is a movable compact and only movable compacta can be homeomorphically mapped into R^2, see [1]).

The same arguments can be applied for a non-Ω-stability of the map $h \times z^n$: $T^2 \to T^2$.

3. $f = h \times h \times z^n$: $T^3 \to T^3$.

Here

$$\Omega_1 = \{1\} \times \{1\} \times S^1, \qquad \Omega_2 = \{1\} \times \{-1\} \times S^1,$$
$$\Omega_3 = \{-1\} \times \{1\} \times S^1, \qquad \Omega_4 = \{-1\} \times \{-1\} \times S^1.$$

f is Ω_4-stable as a quasi-expanding map, f is not Ω_1-stable but can be C^r-small perturbed to Ω_1-stable map by a standard construction of a solenoid inside a solid torus T^2. The map f is neither Ω_2-stable nor Ω_3-stable persistently. Indeed, let W denote the intersection of the set $S^1 \times \{-1\} \times S^1$ with some solid torus which is a neighbourhood of a circle $\{1\} \times \{-1\} \times S^1$. The point -1 is the sharp source for h, and so if g is a C^1-small perturbation of f, then there exists a 2-submanifold W' of T^3 C^1-close to W, invariant under g. W' contains $\Omega_2(g)$ — the image of the solenoid $\tilde{\Omega}_2(f)$.

Now, by the arguments applied in Example 2, $g|_{\Omega_2(g)}$ is not a one-one map.

4. $h\times H\colon T^2\to T^2$.

Here

$$\Omega_1 = \{1\}\times\{1\}, \qquad \Omega_2 = \{1\}\times\omega,$$
$$\Omega_3 = \{-1\}\times\{1\}, \qquad \Omega_4 = \{-1\}\times\omega.$$

$h\times H$ is not Ω-stable in view of the nature of Ω_2. However, it can be C^r-small perturbed to an Ω-stable map $H^\wedge$ which is defined as follows:

$$H^\wedge(z_1, z_2) = \left(h(z_1)\cdot e^{i\cdot\varepsilon\cdot\sin(i\cdot\log(z_2))\cdot\Phi(z_1)},\ H(z_2)\right),$$

where $\Phi\colon S^1\to\langle 0, 1\rangle$ is a smooth bump function such that Φ is equal to 0 in a neighbourhood of $z_1 = -1$ and is equal to 1 in a neighbourhood of $z_1 = 1$ and $\varepsilon > 0$ is an arbitrarily small number.

Since the point 1 is a *sharp* sink for h, there occurs a kind of Smale's "horseshoe" example near the circle $\{1\}\times S^1$:

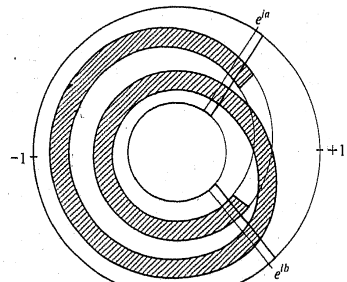

Fig. 4

So $H^\wedge|_{\Omega_2(H^\wedge)}$ is a one-one map; more exactly, it is a Bernoulli shift. Thus, owing to Theorem A, $H^\wedge$ is Ω-stable.

I do not know whether $H^\wedge$ is structurally stable or not.

Let $\Psi\colon R\to\langle -1, 1\rangle$ be a periodic, smooth bump function (with period equal to 2π) such that ψ is equal to 0 in the interval $\langle 0, \pi\rangle$ and is equal to the function sinus in the interval $(\pi+\alpha, 2\pi-\alpha)$ for a small number α.

If in the definition of $H^\wedge$ we put Ψ instead of sin, then $H^\wedge$ is not structurally stable by Theorem C, because $W^u\left(\Omega_2(H^\wedge)\right)\cap\Omega_1(H^\wedge) \neq \emptyset$.

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