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Elliptically contoured measures on infinite-dimensional Banach spaces

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Abstract. In the setting of an infinite-dimensional Banach space we derive sufficient conditions for the existence of a Gaussian measure with a given covariance function. Elliptically contoured measures are characterized as averages of Gaussian measures. This characterization is then used to prove existence and weak covergence results for elliptically contoured measures.

1. In this paper $\mathscr{X}$ will denote a real separable infinite-dimensional Banach space with norm $\|\cdot\|$. $\mathscr{X}^*$ will denote the topological dual space of $\mathscr{X}$.

A mean 0 measure μ satisfying

$$\int_{\mathscr{X}} \|x\|^2 \, d\mu(x) < \infty \tag{1.1}$$

induces a continuous, bilinear functional S on $\mathscr{X}^* \times \mathscr{X}^*$ given by

$$S(x^*, y^*) = \int_{\mathscr{X}} x^*(x) y^*(x) \, d\mu(x) \qquad (x^*, y^* \in \mathscr{X}^*) \tag{1.2}$$

called the *covariance function of* μ.

A measure μ is said to be *mean* 0 *Gaussian* if every continuous linear function x^* on $\mathscr{X}$ has a mean 0 Gaussian distribution with variance parameter $\int_{\mathscr{X}} [x^*(x)]^2 d\mu(x)$ on the real line. It is known that a mean 0 Gaussian measure is uniquely determined by its covariance function.

Let μ be a mean 0 measure satisfying (1.1) with covariance S. In Section 2 of this paper we will obtain sufficient conditions under which there is a mean 0 Gaussian measure with covariance function S. When such a Gaussian measure exists, μ and S are called *pre-Gaussian*.

Not all covariance operators arising from measures satisfying (1.1) are pre-Gaussian as can be seen from the following example.

EXAMPLE 1.1. Let $\mathscr{X} = l^1$. Let $e_k = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 appears as the kth element. Let $\{Z_k(\omega): k \geqslant 1\}$ be independent real-valued random variables with distribution

$$Z_k = \begin{cases} 1 & \text{with prob. } p_k, \\ -1 & \text{with prob. } p_k, \\ 0 & \text{with prob. } 1-2p_k, \end{cases}$$

where $\sum_k p_k < \infty$. Let $X = \sum_{k=1}^{\infty} Z_k(\omega) e_k$ and $\mu = \mathscr{L}(X)$. Then

$$\|X\|_1 = \sum_{k=1}^{\infty} |Z_k(\omega)| < \infty \quad \text{a.s.},$$

by the Borel–Cantelli lemma. μ satisfies (1.1) because

$$\mathrm{E}\|X\|_1^2 = \sum_{k,j} \mathrm{E}|Z_k(\omega)|\,|Z_j(\omega)| = \sum_{k\neq j} \mathrm{E}|Z_k(\omega)|\mathrm{E}|Z_j(\omega)| + \sum_k \mathrm{E}|Z_k(\omega)|^2$$
$$= \sum_{k\neq j} 2p_k 2p_j + 2p_k \leqslant 4\Big(\sum p_k\Big)^2 + 2\sum p_k < \infty.$$

Let $f_j \in l^\infty$ be defined by $f_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in jth position. Then

$$S(f_j, f_k) = \mathrm{E} f_j(x) f_k(x) = \begin{cases} 0 & j \neq k, \\ 2p_k & j = k. \end{cases}$$

Vakhania [9] has shown that the covariance of μ is pre-Gaussian iff

$$\sum S(f_k, f_k)^{1/2} = 2^{1/2} \sum p_k^{1/2} < \infty.$$

The technique of proof used in Section 2 has been previously employed by Wichura [10]. For completeness his result on the tightness of Gaussian measures will be restated and proven. Also we give two simple corollaries.

Wichura has also observed that his tightness result about Gaussian measures can be generalized to the case where the measures are "elliptically contoured". His technique is basically the same as in Section 2. We will prove this Section 4 using a new technique.

More important, however, are the results in Section 3 which characterize the class of all "elliptically contoured" measures on $\mathscr{X}$. It is surprising that this class of measures is so meager, and that all such measures are averages of Gaussian measures. To be more precise, we now give some definitions. A measure μ_n on $\boldsymbol{R}^n$ is *elliptically contoured with parameters* f *and* Σ ($\mu_n = \mathrm{EC}(f, \Sigma, n)$) if the density of μ_n is given by $|\Sigma|^{-1/2} f(\vec{x}\Sigma^{-1}\vec{x}^t)$, where $f\colon [0, \infty) \to [0, \infty)$ satisfies

$$\int_0^\infty r^{n-1} f(r^2)\,dr < \infty \tag{1.3}$$

and Σ is an $n\times n$ symmetric, positive definite matrix.

A measure μ is *elliptically contoured on the infinite-dimensional, separable Banach space* $\mathscr{X}$, if for every linearly independent $\{x_1^*, \ldots, x_n^*\} \subseteq \mathscr{X}^*$, the projection $\pi_n\colon x \to (x_1^*(x), \ldots, x_n^*(x))$ induces an elliptically contoured measure on $\boldsymbol{R}^n$. That is, if μ_n is this measure, then $\mu_n = \mathrm{EC}(f, \Sigma, n)$

for some f and Σ, as above. In particular, we see that the smallest closed subspace supporting such a measure must be $\mathscr{X}$.

Any mean 0 measure satisfying (1.1) has a Hilbert space H_μ contained in $\mathscr{X}$ constructed in the following way. Let $A\colon \mathscr{X}^* \to \mathscr{X}$ be defined in the Bochner sense by $Ax^* = \int_{\mathscr{X}} x\cdot x^*(x)\,d\mu(x)$. Clearly, then $\langle Ax^*, y^*\rangle = S(x^*, y^*)$. The range of A can then be completed under the norm induced by the inner product

$$(Ax^*, Ay^*)_\mu = \int_{\mathscr{X}} x^*(x) y^*(x)\,d\mu(x) = S(x^*, y^*). \tag{1.4}$$

Kuelbs [6] has shown that this completion H_μ can be realized as a subset of $\mathscr{X}$ and that the identity $i\colon H_\mu \to \mathscr{X}$ is continuous. Also he points out that, if H_μ^* is identified with H_μ in the usual way, then $\mathscr{X}^* \subseteq H_\mu \subseteq \mathscr{X}$. If the original measure was Gaussian, then H_μ is precisely the generating Hilbert space for μ in the sense that H_μ is the unique Hilbert space in $\mathscr{X}$ so that when we extend the Canonical Normal Cylinder Measure (CNCM) with variance parameter one on H_μ to $\mathscr{X}$ we get the Gaussian measure μ.

In Section 3 of this paper we will establish that, if μ is elliptically contoured on $\mathscr{X}$, then μ is an average of Gaussian measures in the sense that

$$\mu(A) = \int_0^\infty \mu_t(A)\,da(t),$$

where μ_t is the extension of the CNCM with variance parameter t and a is a probability measure on $(0, \infty)$ with first moment equal to 1.

2. In this section we state and prove some results about the existence and weak convergence of Gaussian measures. First we will need to establish three lemmas.

LEMMA 2.1. *Let* $x_1^*, \ldots, x_n^* \in \mathscr{X}^*$ *and* $\mathscr{A}_n$ *be a cylinder set determined by* $(x_1^*, \ldots, x_n^*)$. *Let* K *be a compact, convex balanced set such that* $K \subseteq \mathscr{A}_n$. *Then there is a convex, balanced cylinder set* $\mathscr{B}_n$ *determined by* $(x_1^*, \ldots, x_n^*)$ *such that* $K \subseteq \mathscr{B}_n \subseteq \mathscr{A}_n$.

Proof. Let $\pi_n(x) = (x_1^*(x), \ldots, x_n^*(x))$. Then $K \subseteq \mathscr{A}_n$ implies that $\pi_n(K) \subseteq \pi_n(\mathscr{A}_n)$. Hence $\pi_n^{-1}(\pi_n(K)) \subseteq \pi_n^{-1}(\pi_n(\mathscr{A}_n)) = \mathscr{A}_n$. Let $\mathscr{B}_n = \pi_n^{-1}(\pi_n(K))$; then $\mathscr{B}_n$ is clearly a cylinder set because $\pi_n(K)$ is compact. It is also convex and balanced because K is.

LEMMA 2.2. *Let* $\mathscr{A}_n$ *be a convex, balanced cylinder set determined by* $(x_1^*, \ldots, x_n^*)$, *where* $x_1^*, \ldots, x_n^* \in \mathscr{X}^*$. *Let* μ, ν *be two Gaussian cylinder measures with covariance operators* S *and* T, *respectively, such that*

$$T(x^*, x^*) \leqslant S(x^*, x^*) \quad (x^* \in \mathscr{X}^*). \tag{2.1}$$

Then

$$\nu(\mathscr{A}_n) \geqslant \mu(\mathscr{A}_n). \tag{2.2}$$

Proof. Let $\pi_n(\mathscr{A}_n) = A_n$ which is Borel measurable in R^n. Let $\mu^{(n)}$ and $\nu^{(n)}$ be the measures on $\boldsymbol{R}^n$ induced by the projection π_n. If we let S^n and T^n be the covariance matrices for $\mu^{(n)}$ and $\nu^{(n)}$, respectively, then it is easy to see that

$$(2.3)\qquad \begin{aligned}(S^n)_{ij} &= S(x_i^*, x_j^*),\\ (T^n)_{ij} &= T(x_i^*, x_j^*),\end{aligned} \qquad (i, j = 1, \ldots, n).$$

From (2.1) and (2.3) it follows that $S^n - T^n$ is positive semi-definite. According to Anderson ([1], p. 173), $\nu^{(n)}(A_n) \geqslant \mu^{(n)}(A_n)$ since A_n is convex and balanced. Thus we have (2.2).

LEMMA 2.3. *Let K be a compact set in $\mathscr{X}$. Then there is a $\{\mathscr{A}_k\colon k \geqslant 1\} \subseteq \mathscr{C}(\mathscr{X}, \mathscr{X}^*)$, the $\mathscr{X}^*$ induced cylinder sets in $\mathscr{X}$, such that*

$$(2.4)\qquad \mathscr{A}_{k+1} \subseteq \mathscr{A}_k \qquad (k \geqslant 1)$$

and

$$(2.5)\qquad \bigcap_{k=1}^{\infty} \mathscr{A}_k = K.$$

Proof. Let $\{x_k^*\colon k \geqslant 1\}$ be a weak-star dense subset of the unit ball of $\mathscr{X}^*$ and let $\pi_k(x) = \big(x_1^*(x), \ldots, x_k(x)\big)$ as before. Define $\mathscr{A}_k = \pi_k^{-1}\big(\pi_k(K)\big)$. Then $\mathscr{A}_k = K + \mathrm{Null}(\pi_k)$. Clearly, we will have (2.4). The $\mathscr{A}_k$ will decrease to K. To see this, let $\{x_k\}$ be a sequence such that each $x_k \in \mathscr{A}_k$ and $\lim_k x_k = x$. Then $x_k = y_k + z_k$, where $y_k \in K$ and $z_k \in \mathrm{Null}(\pi_k)$. Then, since K is compact, there is a subsequence $\{y_{k'}\}$ such that $\lim_{k'} y_{k'} = y$. Let $z = x - y$. Then $x_n^*(z) = 0$ for all x_n^* in our weak-star dense set. If $z \neq 0$, then by the Hahn–Banach theorem there is an $x^* \in \mathscr{X}^*$ so that $\|x^*\| = 1$ and $x^*(z) \neq 0$. For some subsequence, $\lim_{n'} x_{n'}^* = x^*$ in the weak-star topology. Thus,

$$(2.6)\qquad x^*(z) = \lim_n x_n^*(z) = 0.$$

Since the weak-star topology separates points, $z = 0$.

We are now ready to prove the main result of this section.

THEOREM 2.4. *Let μ be a mean 0 Gaussian measure on $\mathscr{X}$ with covariance S. Let $T\colon \mathscr{X}^* \times \mathscr{X}^* \to R$ be a positive definite bilinear functional satisfying*

$$(2.7)\qquad T(x^*, x^*) \leqslant S(x^*, x^*) \qquad (x^* \in \mathscr{X}^*).$$

Then there is a mean 0 Gaussian measure on $\mathscr{X}$ with covariance T.

Proof. The operator T is clearly continuous in the weak-star sense along the diagonal at the origin and so by the infinite-dimensional version of Bochner's theorem, see Prohorov [8], there is a mean 0 Gaussian cylinder measure ν defined on $\mathscr{C}(\mathscr{X}, \mathscr{X}^*)$ with covariance T.

We wish to show that ν extends to a countably additive measure on the Borel sets of $\mathscr{X}$. One way to do this is from Prohorov [8]. Define ν^* on the Borel sets by

$$(2.8)\qquad \nu^*(E) = \inf_{\substack{\mathscr{A} \supseteq E \\ \mathscr{A} \in \mathscr{C}(\mathscr{X}, \mathscr{X}^*)}} \nu(\mathscr{A}).$$

Then ν will extend to a countably additive measure, if for every $\varepsilon > 0$, there is a compact set K such that

$$(2.9)\qquad \nu^*(K) \geqslant 1 - \varepsilon.$$

Since μ is a measure on $\mathscr{X}$, there is a compact set K such that

$$(2.10)\qquad \mu(K) \geqslant 1 - \varepsilon.$$

We lose no generality by assuming that K is convex and balanced because the convex, balanced closure of a compact set is compact in a Banach space. Thus we have from (2.8), (2.10) and Lemmas 2.1 and 2.2 that

$$\nu^*(K) \geqslant \inf_{\substack{\mathscr{A} \supseteq K \\ \mathscr{A} \in \mathscr{C}(\mathscr{X}, \mathscr{X}^*)}} \nu(\mathscr{A}) = \inf_{\substack{\mathscr{B} \supseteq K \\ \mathscr{B} \in \mathscr{C}(\mathscr{X}, \mathscr{X}^*) \\ \mathscr{B}\ \text{convex, balanced}}} \nu(\mathscr{B}) \geqslant \inf_{\substack{\mathscr{B} \supseteq K \\ \mathscr{B} \in \mathscr{C}(\mathscr{X}, \mathscr{X}^*) \\ \mathscr{B}\ \text{convex, balanced}}} \mu(\mathscr{B})$$
$$\geqslant \mu(K) \geqslant 1 - \varepsilon.$$

Thus we have (2.9) and the theorem is proved.

THEOREM 2.5. *Let $\{\nu_n\colon n \geqslant 1\}$, μ be a collection of Gaussian measures with covariance operators $\{T_n\colon n \geqslant 1\}$ and S, respectively. Suppose there is a function T on $\mathscr{X}^* \times \mathscr{X}^*$ such that*

$$(2.11)\qquad \lim_n T_n(x^*, y^*) = T(x^*, y^*) \qquad (x^*, y^* \in \mathscr{X}^*),$$

$$(2.12)\qquad T_n(x^*, x^*) \leqslant S(x^*, x^*) \qquad (x^* \in \mathscr{X}^*, n \geqslant 1).$$

Then (a) *T is the covariance operator of a mean 0 Gaussian measure ν, and* (b)

$$(2.13)\qquad \nu_n \Rightarrow \nu.$$

Proof. (a) Condition (2.11) tells us that T is positive definite and conditions (2.11) and (2.12) together tell us that the hypotheses of Theorem 2.4 are satisfied and hence there is a mean 0 Gaussian measure ν with covariance T.

(b) The Levy continuity theorem easily yields that the finite-dimensional distributions of the ν_n converge to those of ν. Thus for weak convergence we need only verify that $\{\nu_n\colon n \geqslant 1\}$ are tight. That is for every $\varepsilon > 0$ there is a compact set K such that

$$(2.14)\qquad \sup_n \nu_n(K) \geqslant 1 - \varepsilon.$$

So fix $\varepsilon > 0$. As before, since μ is a measure, we can find a compact, convex and balanced set K such that (2.10) is satisfied. It is this K that we will use in showing (2.14). Let $\{\mathscr{A}_k: k \geqslant 1\}$ be as in Lemma 2.3. By Lemma 2.1, we can assume that each is convex and balanced. Hence, by Lemma (2.2), we have

$$\nu_n(K) = \lim_k \nu_n(\mathscr{A}_k) \geqslant \lim_k \mu(\mathscr{A}_k) = \mu(K) \geqslant 1-\varepsilon.$$

This completes the proof of Theorem 2.5.

Next we have some corollaries of the previous theorems.

Corollary 2.6. *Let X be a mean 0 Gaussian $\mathscr{X}$-valued random variable. Define*

$$Y = \begin{cases} X & \text{if } \|X\| \leqslant c, \\ 0 & \text{elsewhere.} \end{cases}$$

Let S and T be the covariance operators of the measures induced by X and Y. Let μ be the measure on $\mathscr{X}$ induced by X. Then

(a) $\mathrm{E}Y = 0$;

(b) *There is a mean 0 Gaussian measure ν with covariance T.*

Proof. (a) To establish this part, we show that Y is symmetric

$$\begin{aligned} P(Y \epsilon B) &= P(Y \epsilon B \cap \{x: \|x\| \leqslant c\}) \\ &= P(X \epsilon B \cap \{x: \|x\| \leqslant c\}) + \begin{cases} 0 & \text{if } 0 \epsilon B \\ P(\|X\| > c) & \text{if } 0 \notin B \end{cases} \\ &= P(-X \epsilon B \cap \{x: \|x\| \leqslant c\}) + \begin{cases} 0 & \text{if } 0 \epsilon B \\ P(\|-X\| > c) & \text{if } 0 \notin B \end{cases} \\ &= P(X \epsilon -B \cap \{x: \|x\| \leqslant c\} + \begin{cases} 0 & \text{if } 0 \epsilon B \\ P(\|X\| > c) & \text{if } 0 \notin B \end{cases} \\ &= P(Y \epsilon -B). \end{aligned}$$

(b) This part is a trivial consequence of Theorem 2.2 because $T(x^*, x^*) \leqslant S(x^*, x^*)$.

Corollary 2.7. *Let X be a mean 0 Gaussian random variable taking values in $\mathscr{X}$. Let $\{c_n: n \geqslant 1\}$ be any sequence increasing to $+\infty$. Define Y_n by*

$$Y_n = \begin{cases} X & \text{if } \|X\| \leqslant c_n, \\ 0 & \text{elsewhere.} \end{cases}$$

Let $\{T_n\}$ and S be the covariance operators of the measures induced by $\{Y_n\}$ and X, respectively. Let μ = the measure induced by X and for each n let ν_n be the mean 0 Gaussian measure with covariance T_n. Then

$$\nu_n \Rightarrow \mu.$$

Proof. This is obvious from Theorem 2.5 since $T_n(x^*, x^*) \leqslant S(x^*, x^*)$ and

$$\lim_n |S(x^*, y^*) - T_n(x^*, y^*)| = \lim_n \Big| \int_{\|x\|>c_n} x^*(x) y^*(x)\, d\mu(x) \Big|$$
$$\leqslant \lim_n \|x^*\| \, \|y^*\| \int_{\|x\|>c_n} \|x\|^2 d\mu(x) = 0.$$

3. In this section we will characterize elliptically contoured measures on $\mathscr{X}$. We first show that if μ_n is elliptically contoured on $\boldsymbol{R}^n$ then f and Σ can be chosen so that Σ is its covariance matrix.

Lemma 3.1. *Let $\mu_n = \mathrm{EC}(f, \Sigma, n)$ on $\boldsymbol{R}^n$ have covariance matrix T_{μ_n}. Then*

(a)

$$T_{\mu_n} = \sigma^2 \Sigma, \tag{3.1}$$

where

$$\sigma^2 = \int_{\boldsymbol{R}^n} y_1^2 f(y_1^2 + \ldots + y_n^2)\, dy_1 \ldots dy_n; \tag{3.2}$$

(b)

$$\mu_n = \mathrm{EC}(\tilde{f}, \tilde{\Sigma}, n), \tag{3.3}$$

where

$$1 = \int_{\boldsymbol{R}^n} y_1^2 \tilde{f}(y_1^2 + \ldots + y_n^2)\, dy_1 \ldots dy_n \tag{3.4}$$

and

$$\tilde{\Sigma} = T_{\mu_n}. \tag{3.5}$$

Proof. (a) Define the radial measure ν_n by

$$\nu_n(E) = \int_E f(\vec{y}\vec{y}^t)\, d\vec{y} = \int_{EA} |\Sigma|^{-1/2} f(\vec{x}\Sigma^{-1}\vec{x}^t)\, d\vec{x} \tag{3.6}$$
$$= \mu_n(EA), \quad \text{where} \quad EA = \{\vec{x}A: \vec{x} \epsilon E\},$$

where $\Sigma = A^t A$. Since ν_n is radial, its covariance matrix T_{ν_n} is also radial in the sense that

$$T_{\nu_n}(\vec{x}^*, \vec{x}^*) = T_{\nu_n}(\vec{y}^*, \vec{y}^*) \quad \text{if} \quad |\vec{x}^*| = |\vec{y}^*|.$$

Hence,

$$|\vec{x}^*|^2 \int_{\boldsymbol{R}^n} y_1^2 f(\vec{y}\vec{y}^t) d\vec{y} = T_{\nu_n}(|\vec{x}^*|\vec{e}_1, |\vec{x}^*|\vec{e}_1) = T_{\nu_n}(\vec{x}^*, \vec{x}^*)$$
$$= \int_{\boldsymbol{R}^n} (\vec{x}^*, \vec{y})^2 f(\vec{y}\vec{y}^t)\, d\vec{y}$$

$$= \int_{R^n} (\vec{x}^*, \vec{x}A^{-1})^2 |\Sigma|^{-1/2} f(\vec{x}\Sigma^{-1}\vec{x}^t)\, d\vec{x}$$

$$= \int_{R^n} \big(\vec{x}^*(A^{-1})^t, \vec{x}\big)^2 |\Sigma|^{-1/2} f(\vec{x}\Sigma^{-1}\vec{x}^t)\, d\vec{x}$$

$$= T_{\mu_n}\big(\vec{x}^*(A^{-1})^t, \vec{x}^*(A^{-1})^t\big).$$

Thus we have

$$T_{\mu_n}(\vec{x}^*, \vec{x}^*) = T_{\mu_n}\big(\vec{x}^*A^t(A^t)^{-1}, \vec{x}^*A^t(A^t)^{-1}\big) = T_{\nu_n}(\vec{x}^*A^t, \vec{x}^*A^t)$$

$$= |\vec{x}^*A^t|^2 \int_{R^n} y_1^2 f(\vec{y}\vec{y}^t)\, d\vec{y} = \vec{x}^*A^tA\vec{x}^{*t}\cdot \int_{R^n} y_1^2 f(\vec{y}\vec{y}^t)\, d\vec{y}$$

$$= \vec{x}^*\Sigma\vec{x}^{*t}\cdot \int_{R^n} y_1^2 f(\vec{y}\vec{y}^t)\, d\vec{y} = \Sigma(\vec{x}^*, \vec{x}^*)\cdot \int_{R^n} y_1^2 f(\vec{y}\vec{y}^t)\, d\vec{y}.$$

(b) Suppose that σ^2 in (3.2) is not 1. Let $\tilde{f}(r^2) = \sigma f(\sigma^2 r^2)$ and $\tilde{\Sigma} = \sigma^2\Sigma$. Then we will have (3.4) because

(3.7) $$\int_{R^n} y_1^2 \tilde{f}(\vec{y}\vec{y}^t)\, d\vec{y} = \int_{R^n} y_1^2 \sigma f(\sigma^2\vec{y}\vec{y}^t)\, d\vec{y} = \frac{1}{\sigma^2}\int_{R^n} x_1^2 f(\vec{x}\vec{x}^t)\, d\vec{x} = 1.$$

Also $\mu_n = \mathrm{EC}(\tilde{f}, \tilde{\Sigma}, n)$ because

$$|\tilde{\Sigma}|^{-1/2}\tilde{f}(\vec{y}\tilde{\Sigma}^{-1}\vec{y}^t) = \frac{1}{\sigma}\, |\Sigma|^{-1/2}\sigma f\left(\frac{\sigma^2\vec{y}\Sigma^{-1}\vec{y}^t}{\sigma^2}\right) = |\Sigma|^{-1/2} f(\vec{y}\Sigma^{-1}\vec{y}^t).$$

Thus we have (3.3). Part (a) together with (3.7) yields (3.5).

Henceforward we will assume that any given elliptically contoured measure will have its parameters adjusted so that the matrix parameter is in fact the covariance matrix for the measure. Under this assumption it is clear that finite-dimensional elliptically contoured measures are uniquely determined by their parameters.

LEMMA 3.2. *Let $\mu_n = \mathrm{EC}(f, \Sigma, n)$ with $\Sigma = A^tA$. Let B be an $n\times n$ non-singular matrix. Then*

(a) *if $\lambda_n(E) \equiv \mu_n(EB^{-1}A)$, then $\lambda_n = \mathrm{EC}(f, B^tB, n)$, and*

(b) *if $\lambda_n(E) = \mu_n(EB^{-1})$, then $\lambda_n = \mathrm{EC}(f, B^t\Sigma B, n)$.*

Proof. (a) Let ν_n be as in (3.6), then

$$\lambda_n(E) = \mu_n(EB^{-1}A) = \nu_n(EB^{-1}) = \int_{EB^{-1}} f(\vec{y}\vec{y}^t)\, d\vec{y}$$

$$= \int_{E} |B|^{-1} f\big(\vec{x}B^{-1}(B^{-1})^t\vec{x}^t\big)\, d\vec{x}.$$

(b) Apply (a), and we have $\lambda_n(E) = \mu_n(EB^{-1}A^{-1}A)$. Hence,

$$\lambda_n = \mathrm{EC}\big(f, (AB)^tAB, n\big) = \mathrm{EC}(f, B^tA^tAB, n) = \mathrm{EC}(f, B^t\Sigma B, n).$$

LEMMA 3.3. *Let μ be elliptically contoured on $\mathscr{X}$ as defined in Section 1 with covariance S. Then for each n there is a function f_n: $[0, \infty)\to[0, \infty)$ such that, if $x_1^*, \ldots, x_n^*$ are linearly independent in $\mathscr{X}^*$ and μ_n is the measure induced by $(x_1^*, \ldots, x_n^*)$, then $\mu_n = \mathrm{EC}(f_n, \Sigma, n)$ with $(\Sigma)_{ij} = S(x_i^*, x_j^*)$.*

Proof. By definition, there is some f_n (possibly depending on $(x_1^*, \ldots, x_n^*)$) so that $\mu = \mathrm{EC}(f_n, \Sigma, n)$. By Lemma 3.1, $(\Sigma)_{ij} = S(x_i^*, x_j^*)$. Thus we will be done if we can show that f_n does not depend on $(x_1^*, \ldots, x_n^*)$ but only on n. So let $y_1^*, \ldots, y_n^*$ be some other linearly independent set in $\mathscr{X}^*$ which induces a measure $\tilde{\mu}_n$ which is $\mathrm{EC}(\tilde{f}_n, \tilde{\Sigma}, n)$.

Case 1. Assume that both $x_1^*, \ldots, x_n^*$ and $y_1^*, \ldots, y_n^*$ are orthonormal in the H_μ inner product. Take $z_1^*, \ldots, z_n^*$ orthonormal and orthogonal to both $x_1^*, \ldots, x_n^*$ and $y_1^*, \ldots, y_n^*$ and suppose it induces a measure $\tilde{\tilde{\mu}}_n$ which is $\mathrm{EC}(\tilde{\tilde{f}}_n, \tilde{\tilde{\Sigma}}, n)$. Clearly, $\Sigma = \tilde{\Sigma} = \tilde{\tilde{\Sigma}} =$ Identity. We will show that $f_n = \tilde{\tilde{f}}_n$ and hence, by symmetry, $\tilde{f}_n = \tilde{\tilde{f}}_n$ and thus $f_n = \tilde{f}_n$. Let μ_{2n} be the measure induced by $(x_1^*, \ldots, x_n^*, z_1^*, \ldots, z_n^*)$ which is $\mathrm{EC}(f_{2n}, \text{Identity}, 2n)$. Let $F_n, \tilde{\tilde{F}}_n, F_{2n}$ be the distribution functions for $\mu_n, \tilde{\tilde{\mu}}_n$ and μ_{2n}, respectively. Then we have, using the consistency of F_{2n}, with both F_n and $\tilde{\tilde{F}}_n$, that

$$F_n(x_1, \ldots, x_n)$$

$$= F_{2n}(x_1, \ldots, x_n, \infty, \ldots, \infty)$$

$$= \int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_{2n}(y_1^2+\ldots+y_{2n}^2)\, dy_{2n}\ldots dy_1$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\Big[\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n} f_{2n}(y_1^2+\ldots+y_{2n}^2)\, dy_n\ldots dy_1\Big] dy_{2n}\ldots dy_{n+1}$$

$$= F_{2n}(\infty, \ldots, \infty, x_1, \ldots, x_n) = \tilde{\tilde{F}}_n(x_1, \ldots, x_n).$$

Hence $f = \tilde{\tilde{f}}$.

Case 2. Let $\{x_1^*, \ldots, x_n^*\}$, $\{y_1^*, \ldots, y_n^*\}$ be arbitrary linearly independent sets in $\mathscr{X}^*$. Let $\Sigma = A^tA$ and $\nu_n(E) = \mu_n(EA)$. Then $\nu_n = \mathrm{EC}(f_n, \text{Identity}, n)$ = measure induced on R^n by $(x_1^*, \ldots, x_n^*)A$. Similarly, let $\tilde{\Sigma} = \tilde{A}^t\tilde{A}$ and $\tilde{\nu}_n(E) = \tilde{\mu}_n(E\tilde{A})$ then $\tilde{\nu}_n = \mathrm{EC}(\tilde{f}_n, \text{Identity}, n)$ = measure on R^n induced by $(y_1^*, \ldots, y_n^*)\tilde{A}$. Then Case 1 applies to $(x_1^*, \ldots, x_n^*)A$ and $(y_1^*, \ldots, y_n^*)\tilde{A}$ and thus $f_n = \tilde{f}_n$. This completes the proof of Lemma 3.3.

Let $\{a_k^*: k\geqslant 1\}$ be a complete orthonormal basis of H_μ obtained by applying the Gram–Schmidt orthonormalization process to $\mathscr{X}^*$. Thus all the a_k^* are also in $\mathscr{X}^*$. Let V = linear span of $\{a_k^*\}$ which is an infinite-dimensional subspace of $\mathscr{X}^*$. Let $\mathscr{C}(\mathscr{X}, V)$ be the cylinder sets of $\mathscr{X}$ determined by V. We will use the notation $\hat{\mu}(x^*)$ to mean $\int \exp\{ix^*(x)\}\, d\mu(x)$.

LEMMA 3.4. *Let* $\varphi\colon V\to R$ *be given by*

$$\varphi(c_1a_1^*+\ldots+c_na_n^*)=\hat{\mu}(c_1a_1^*+\ldots+c_na_n^*).$$

Then

$$\varphi(c_1a_1^*+\ldots+c_na_n^*)=\psi(c_1^2+\ldots+c_n^2), \tag{3.8}$$

where

$$\psi(t)=\int_R e^{itz}f_1(z^2)\,dz.$$

Proof. Assume for now that all the $c_j\neq 0$, then

$$\varphi(c_1a_1^*+\ldots+c_na_n^*)=\int_{R^n} e^{i\Sigma c_jy_j}f_n(\vec{y}\vec{y}^t)\,d\vec{y}.$$

Then change variables by letting $z_1=\frac{\sum c_jy_j}{\sum c_j^2}$ and $z_2,\ldots,z_n$ be an orthonormal completion of R^n. Then $d\vec{z}=d\vec{y}$ and thus

$$\varphi(c_1a_1^*+\ldots+c_na_n^*)=\int_{R^n} e^{i\sum_{j=1}^n c_j^2z_1}f_n(\vec{z}\vec{z}^t)\,d\vec{z}$$

$$=\int_R e^{i\sum_{j=1}^n c_j^2z_1}f_1(z_1^2)\,dz_1=\psi\Big(\sum_{j=1}^n c_j^2\Big).$$

The case when some of the c_j are 0 follows similarly since f_n does not depend on the choice of the $a_1^*,\ldots,a_n^*$.

We are now ready to state the major theorem.

THEOREM 3.5. (a) *If a measure* μ *with covariance* S *satisfying* (1.1) *is elliptically contoured on* $\mathscr{X}$, *then the* CNCM *on* H_μ *extends to a Gaussian measure* μ_1 *on* $\mathscr{X}$ *and there is a probability measure* α *on* $(0,\infty)$ *such that*

$$\int_0^\infty t\,d\alpha(t)=1 \tag{3.9}$$

and

$$\mu(E)=\int_0^\infty \mu_t(E)\,d\alpha(t)\quad (E\ \textit{Borel set in}\ \mathscr{X}), \tag{3.10}$$

where $\mu_t(E)=\mu_1\left(\frac{E}{\sqrt{t}}\right)$.

(b) *Conversely, if* μ_1 *is a mean* 0 *Gaussian measure supported by* $\mathscr{X}$ *with covariance* S *and* α *is a probability measure on* $(0,\infty)$ *so that* (3.9) *holds, then* (3.10) *defines an elliptically contoured measure on* $\mathscr{X}$ *with covariance* S. *Also* (1.1) *is satisfied.*

Proof. (a) Let $\{a_n^*\colon n\geq 1\}$, V, φ and ψ be as in Lemma 3.4 Then from Lemma 3.4 we have

$$\hat{\mu}(x^*)=\varphi(x^*)=\psi\big(S(x^*,x^*)\big)\quad (x^*\in V).$$

Then, according to Kuelbs ([7], p. 415), we have for $x^*\in V$

$$\hat{\mu}(x^*)=\int_0^\infty \exp\left\{-\frac{S(x^*,x^*)t}{2}\right\}d\alpha(t), \tag{3.11}$$

where α is a finite non-negative measure on $[0,\infty)$. The function $\varphi_1(\lambda)=\hat{\mu}(\lambda a_1^*)$ is the characteristic function of a probability measure on R with a density so by the Rieman–Lebesgue lemma $\varphi_1(\lambda)\to 0$ as $|\lambda|\to\infty$. Hence $\alpha(0)=0$ and thus α is a measure on $(0,\infty)$. It is also evident that α is a probability measure since both sides of (3.12) are continuous at $\lambda=0$,

$$\varphi_1(\lambda)=\hat{\mu}(\lambda a_1^*)=\int_0^\infty \exp\left\{-\frac{\lambda^2t}{2}\right\}d\alpha(t). \tag{3.12}$$

Now define a cylinder measure ν on $\mathscr{C}(\mathscr{X},V)$ by

$$\nu(E)=\int_0^\infty \mu_t(E)\,d\alpha(t), \tag{3.13}$$

where μ_t is the CNCM with variance parameter t defined on H_μ. Then we have

$$\hat{\nu}(x^*)=\int_{\mathscr{X}} e^{ix^*(x)}\,d\nu(x) \tag{3.14}$$

$$=\int_0^\infty\int_R e^{iy}\big(2\pi tS(x^*,x^*)\big)^{-1/2}\exp\left\{-\frac{y^2}{2tS(x^*,x^*)}\right\}dy\,d\alpha(t)$$

$$=\int_0^\infty \exp\left\{-\frac{tS(x^*,x^*)}{2}\right\}d\alpha(t).$$

So if we pick $x^*\neq 0$ in V, then the measures induced on R by x^* by ν and μ are identical. Thus

$$S(x^*,x^*)=\int_{\mathscr{X}}[x^*(x)]^2\,d\nu(x) \tag{3.15}$$

$$=\int_R y^2\int_0^\infty \big(2\pi tS(x^*,x^*)\big)^{-1/2}\exp\left\{-\frac{y^2}{tS(x^*,x^*)}\right\}d\alpha(t)\,dy$$

$$= \int_0^\infty \int_R y^2 \big(2\pi t S(x^*, x^*)\big)^{-1/2} \exp\left\{-\frac{y^2}{2tS(x^*, x^*)}\right\} dy\, da(t)$$

$$= S(x^*, x^*) \int_0^\infty t\, da(t).$$

Thus we have (3.9).

It is also clear from (3.14) that for $E \in \mathscr{C}(\mathscr{X}, V)$ we have

$$\mu(E) = \nu(E) = \int_0^\infty \mu_t(E)\, da(t) \qquad (E \in \mathscr{C}(\mathscr{X}, V)). \tag{3.16}$$

In order to show that (3.16) holds for all Borel sets of $\mathscr{X}$, we first show that ν extends to a countably additive measure on the Borel sets. In order to do this, it suffices to show μ_t extends for all t which will be so iff μ_1 extends. Assume μ_1 does not extend. Then there is an $\varepsilon > 0$ such that for every compact set K

$$\mu_1^*(K) \leqslant 1 - \varepsilon.$$

Thus for every $t > 0$ and compact set K

$$\mu_t^*(K) = 1 - \varepsilon,$$

where

$$\mu_1^*(K) = \inf_{\substack{E \in \mathscr{C}(\mathscr{X}, V)\\ E \supseteq K}} \mu_t(E).$$

The set $\{\pi_n^{-1}(\pi_n(K)) \colon n \geqslant 1\}$, where $\pi_n(x) = (a_1^*(x), \ldots, a_n^*(x))$ forms a fundamental system for K. That is, for any $E \in \mathscr{C}(\mathscr{X}, V)$ such that $K \subseteq E$ we know there is an n so that $K \subseteq \pi_n^{-1}(\pi_n(K)) \subseteq E$. Now take a K_1 compact so that

$$\mu(K_1) \geqslant 1 - \frac{\varepsilon}{2}. \tag{3.17}$$

Then, combining (3.16) and (3.17),

$$1 - \frac{\varepsilon}{2} \leqslant \lim_n \mu\big(\pi_n^{-1}(\pi_n(K_1))\big) = \lim_n \int_0^\infty \mu_t\big(\pi_n^{-1}(\pi_n(K_1))\big)\, da(t)$$
$$= \int_0^\infty \lim_n \mu_t\big(\pi_n^{-1}(\pi_n(K_1))\big)\, da(t) = \int_0^\infty \mu_t^*(K_1)\, da(t)$$
$$\leqslant \int_0^\infty (1 - \varepsilon)\, da(t) = 1 - \varepsilon.$$

Thus we have a contradiction and so μ_1 and hence ν can be extended to the smallest σ-algebra containing $\mathscr{C}(\mathscr{X}, V)$ which is the Borel set. Since $\mu = \nu$ on $\mathscr{C}(\mathscr{X}, V)$, $\mu = \nu$ on all Borel sets and (3.13) holds on the Borel sets.

(b) It is clear that if (3.9) is valid (3.10) defines a countably additive measure on $\mathscr{X}$. Let H_{μ_1} be the generating space for μ_1 and $\{a_k^* \colon k \geqslant 1\}$ be a CON sequence in H_{μ_1}. Let ν_n be the radial measure induced by $\pi_n = (a_1^*, \ldots, a_n^*)$

$$\nu_n(E) = \mu\big(x \in \mathscr{X} \colon \pi_n(x) \in E\big) = \int_0^\infty \mu_t\big(\pi_n^{-1}(E)\big)\, da(t) \tag{3.18}$$
$$= \int_0^\infty \left(\int_E (2\pi t)^{-n/2} \exp\left\{-\frac{u_1^2 + \ldots + u_n^2}{2}\right\} d\mu_1 \ldots d\mu_n\right) da(t)$$
$$= \int_E \left(\int_0^\infty (2\pi t)^{-n/2} \exp\left\{-\frac{u_1^2 + \ldots + u_n^2}{2}\right\} d\mu_1 \ldots d\mu_n\right) da(t).$$

We see then that $\nu_n = \mathrm{EC}(f_n, \mathrm{Identity}, n)$, where

$$f(r^2) = \int_0^\infty (2\pi t)^{-n/2} \exp\left\{-\frac{r^2}{2}\right\} da(t).$$

It is clear that (3.18) is independent of the orthonormalization used so $(x_1^*, \ldots, x_n^*)$ will induce an elliptically contoured measure on $\boldsymbol{R}^n$ if the $\{x_1^*, \ldots, x_n^*\}$ are linearly independent. Thus μ is elliptically contoured on $\mathscr{X}$. A computation similar to (3.5) shows that the covariance of S is μ.

We will now show that (1.1) is satisfied. From Kuelbs [6] we know there is a sequence $\{x_n^* \colon n \geqslant 1\}$ in the unit ball of $\mathscr{X}^*$ so that for $x \in \mathscr{X}$

$$\|x\| = \sup_n |x_n^*(x)| \qquad (x \in \mathscr{X}). \tag{3.19}$$

Thus, if $(\Sigma)_{ij} = S(x_i^*, x_j^*)$, then, using the Monotone and Dominated Convergence Theorems, we have

$$\int_{\mathscr{X}} \|x\|^2 d\mu(x) = \int_{\mathscr{X}} \sup_n |x_n^*(x)|^2 d\mu(x)$$
$$= \lim_n \int_{\boldsymbol{R}^n} \int_0^\infty \sup_{1 \leqslant j \leqslant n} |y_j|^2 (2\pi t)^{-n/2} |\Sigma|^{-1/2} \exp\left\{-\frac{\vec{y}\Sigma\vec{y}^t}{2t}\right\} da(t)\, d\vec{y}$$
$$= \lim_n \int_0^\infty t \int_{\boldsymbol{R}^n} \sup_{1 \leqslant j \leqslant n} |y_j|^2 (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp\left\{-\frac{\vec{y}\Sigma^{-1}\vec{y}^t}{2}\right\} d\vec{y}\, da(t)$$
$$= \int_0^\infty t \lim_n \int_{\boldsymbol{R}^n} \sup_{1 \leqslant j \leqslant n} |y_j|^2 (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp\left\{-\frac{\vec{y}\Sigma^{-1}\vec{y}^t}{2}\right\} d\vec{y}\, da(t)$$

$$= \int_0^\infty t \lim_n \int_{\mathcal{X}} \sup_{1\leqslant j\leqslant n} |x_j^*(x)|^2 \, d\mu_1(x)\, da(t)$$

$$= \int_0^\infty t \int_{\mathcal{X}} \|x\|^2 d\mu_1(x)\, da(t) = \int_{\mathcal{X}} \|x\|^2 d\mu_1(t) < \infty.$$

It is interesting to note that Theorem 3.5 can be extended to the case where μ is elliptically contoured on some infinite-dimensional closed subspace of $\mathcal{X}$.

THEOREM 3.6. (a) *Let the smallest closed subspace containing the support of μ be M. Suppose μ satisfies* (1.1) *and is elliptically contoured on M. Then there is a probability measure on* $(0, \infty)$ *so that* (3.9) *and* (3.10) *hold.*

(b) *Conversely, let μ_1 be an infinite-dimensional Gaussian measure on $\mathcal{X}$ and a a probability measure on* $(0, \infty)$ *so that* (3.9) *holds. Then* (3.10) *defines an elliptically contoured measure on M = support of μ_1.*

Proof. (a) This can be proved easily by applying Theorem 3.5 to the Banach space M and noting that $M^* \subseteq H_\mu \subseteq M$. (3.10) will then be valid for Borel sets of M. We extend (3.10) to $\mathcal{X}$ by letting $\mu_1(M^c) = \mu(M^c) = 0$.

(b) is similar.

Another point worth mentioning is that the most common examples of elliptically contoured measures on $\mathcal{X}$ are the Gaussian measures on $\mathcal{X}$. In this case the measure induced by $(x_1^*, \ldots, x_n^*)$ is $\mathrm{EC}(f_n, \Sigma, n)$, where $f_n(r^2) = (2\pi)^{-n/2}\exp\{-r^2/2\}$. One might ask are there any other elliptically contoured measures where $f_n = c_n f$, where c_n is come constant depending only on n? The answer is that this property characterizes Gaussian measures.

THEOREM 3.7. *Let μ be elliptically contoured on some infinite-dimensional subspace M in $\mathcal{X}$. Suppose the elliptically contoured measure induced by any linearly independent set $(x_1^*, \ldots, x_n^*)$ in M^* is denoted by $\mathrm{EC}(f_n, \Sigma, n)$. (As before $(\Sigma)_{ij} = S(x_i^*, x_j^*)$). Then $f_n = c_n f$ iff μ is Gaussian.*

Proof. As before take $\{a_k^*\colon k \geqslant 1\}$ CON in H_μ. The the measure μ_n induced by $(a_1^*, \ldots, a_n^*)$ will be $\mathrm{EC}(f_n, \mathrm{Identity}, n)$. Because the area of S^{n-1} is $2\pi^{n/2}/\Gamma(n/2)$ and $\mu_n(\boldsymbol{R}^n) = 1$ it is easy to show

$$c_n = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}\int_0^\infty r^{n-1}f(r^2)\,dr}. \tag{3.20}$$

Next we verify that all moments of f are determined. The set of radial measures $\{\mu_n\colon n \geqslant 1\}$ must all be consistent, that is,

$$\mu_n(A\times \boldsymbol{R}) = \mu_{n-1}(A) \quad (A \in B^{n-1}),$$

$$\int_A \!\!\ldots\! \int \int_{\boldsymbol{R}} c_n f(x_1^2+\ldots+x_n^2)\,dx_1\ldots dx_n = \int_A\!\!\ldots\!\int c_{n-1} f(x_1^2+\ldots+x_{n-1}^2)\,dx_1\ldots dx_{n-1}. \tag{3.21}$$

Thus for each n,

$$\int_{\boldsymbol{R}} c_n f(x_1^2+\ldots+x_n^2)\,dx_n = c_{n-1} f(x_1^2+\ldots+x_{n-1}^2) \quad \text{a.s. } [dx_1\ldots dx_{n-1}].$$

By letting $x_1^2+\ldots+x_{n-1}^2 = y$, $x_n = x$, we have

$$\int_{\boldsymbol{R}} c_n f(y^2+x^2)\,dx = c_{n-1} f(y^2) \quad \text{a.s. } [dy].$$

Therefore there is a common set $B \subseteq \boldsymbol{R}$ such that $m(B^c) = 0$ and

$$\int_{\boldsymbol{R}} f(y^2+x^2)\,dx_n = \frac{c_{n-1}}{c_n} f(y^2) \quad (y \in B,\ n \geqslant 2).$$

In particular, we have c_{n-1}/c_n is independent of n

$$\frac{c_{n-1}}{c_n} = K = \text{constant}. \tag{3.22}$$

Thus, using (3.20) and (3.22),

$$\int_0^\infty r^{n-1}f(r^2)\,dr = \frac{K\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\pi^{1/2}}\int_0^\infty r^{n-2}f(r^2)\,dr \tag{3.23}$$

and, by induction, we will know all moments of f when we compute K because $\int_0^\infty f(r^2)\,dr = \frac{1}{2}\int_{\boldsymbol{R}} f(r^2)\,dr = \frac{1}{2}$. But, by Lemma 3.1, $\int_0^\infty r^2 f(r^2)\,dr = \frac{1}{2}\int_{\boldsymbol{R}} r^2 f(r^2)\,dr = \frac{1}{2}$. Therefore, by (3.23) applied twice,

$$\frac{1}{2} = \frac{K\Gamma(\frac{3}{2})}{\Gamma(1)\pi^{1/2}}\int_0^\infty r^1 f(r^2)\,dr = \frac{K^2\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})\pi}\cdot\frac{1}{2}.$$

Therefore,

$$K = \left(\frac{\pi\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}\right)^{1/2} = (2\pi)^{1/2}. \tag{3.24}$$

Thus formula (3.24) dictates all moments of f and because the moments satisfy

$$\overline{\lim_n}\frac{\left(\int r^k f(r^2)\,dr\right)^{1/k}}{k} < \infty$$

we know from Brieman ([2], p. 182) that f is determined by its moments. Since $f(r^2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{r^2}{2}\right)$ generates a consistent family of radial measures, it is the only possible function. Hence our original measure μ was Gaussian.

4. In this section we establish some consequences of the results contained in Section 3. First, we are justified in using the notation $\mu = \mathrm{EC}(\alpha, S)$ for a measure which is elliptically contoured on some infinite-dimensional closed subspace M contained in $\mathscr{X}$, where α is probability measure on $(0, \infty)$ with first moment equal to 1 and S is a pre-Gaussian covariance operator so that $S(x^*, x^*) \neq 0$ if $x^* \in M^*$. If μ_1 is the Gaussian measure with covariance S, then (3.9) translates into

$$\mu(A) = \int_0^\infty \mu_1\left(\frac{A}{\sqrt{t}}\right) d\alpha(t). \tag{4.1}$$

Wichura has indicated that his weak covergence result given by Theorem 3.5 can be generalized for appropriately parametered elliptically contoured measures. His method of proof is exactly as in Theorem 3.5, except in proving the crucial Lemma 3.2 when we employ Anderson's inequality he substitutes a result of Das Gupta, et al. [3] that given two elliptically contoured measures on $\boldsymbol{R}^n$, $\mu_1 = \mathrm{EC}(f, \Sigma_1, n)$ and $\mu_2 = \mathrm{EC}(f, \Sigma_2, n)$, and $\Sigma_2 - \Sigma_1$ is positive semi-definite, then

$$\mu_1(A) \geqslant \mu_2(A), \tag{4.2}$$

where A is a convex, balanced set in $\boldsymbol{R}^n$.

We shall establish his theorem as well as the analogous existence-type result of Section 2 by combining results of Sections 2 and 3.

THEOREM 4.1. *Suppose* $\mu = \mathrm{EC}(\alpha, S)$ *on* $M \subseteq \mathscr{X}$ *and suppose* $T: \mathscr{X}^* \times \mathscr{X}^* \to R$ *is a bilinear operator such that*

$$T \text{ is positive semi-definite}, \tag{4.3}$$

and

$$T(x^*, x^*) \leqslant S(x^*, x^*) \quad (x^* \in \mathscr{X}^*). \tag{4.4}$$

Then there is an elliptically contoured measure $\mu = \mathrm{EC}(\alpha, T)$ *with support equal to* $T(x^*, x^*)^\perp$.

Proof. Because of (4.3) and (4.4) and Theorem 2.4, it is clear that there is a Gaussian measure with covariance T. By Theorem 3.6, μ exists.

THEOREM 4.2. *Let* $\{\nu_n: n \geqslant 1\}$ *be a sequence of elliptically contoured measures so that* $\nu_n = \mathrm{EC}(\alpha, T_n)$. *Suppose* S *is a pre-Gaussian covariance operator such that*

$$T_n(x^*, x^*) \leqslant S(x^*, x^*) \quad (x^* \in \mathscr{X}^*). \tag{4.3}$$

Also let

$$\lim_n T_n(x^*, x^*) = T(x^*, x^*) \quad (x^* \in \mathscr{X}^*). \tag{4.4}$$

Then

$$\nu_n \Rightarrow \nu = \mathrm{EC}(\alpha, T). \tag{4.5}$$

Proof. From Theorem 4.1 we know that ν exists. We know the finite-dimensional distributions of ν_n converge to those ν by looking at the characteristic functions,

$$\begin{aligned}\lim_n \hat{\nu}_n(x^*) &= \lim_n \int_{\mathscr{X}} e^{ix^*(x)} d\nu_n(x)\\ &= \lim_n \int_0^\infty \int_R \left(2\pi t T_n(x^*, x^*)\right)^{-1/2} \exp\left\{-\frac{y^2}{2tT_n(x^*, x^*)}\right\} dy\, d\alpha(t)\\ &= \lim_n \int_0^\infty \exp\left\{-\frac{tT_n(x^*, x^*)}{2}\right\} d\alpha(t)\\ &= \int_0^\infty \exp\left\{-\frac{tT(x^*, x^*)}{2}\right\} d\alpha(t) = \hat{\nu}(x^*).\end{aligned}$$

Next we show that the $\{\nu_n: n \geqslant 1\}$ are tight. Let $\mu = \mathrm{EC}(\alpha, S)$. Fix $\varepsilon > 0$. Let K be a compact, convex symmetric set so that

$$\mu(K) \geqslant 1 - \varepsilon.$$

As in Lemma 2.1 and Lemma 2.3 we take $\mathscr{A}_m \in \mathscr{C}(\mathscr{X}, \mathscr{X}^*)$ so that $\mathscr{A}_m$ are convex and symmetric and decrease to K. Then apply Lemma 2.2 and we have

$$\begin{aligned}\nu_n(K) &= \lim_m \nu_n(\mathscr{A}_m) = \lim_m \int_0^\infty \nu_{n,t}(\mathscr{A}_m)\, d\alpha(t) \geqslant \lim_m \int_0^\infty \mu_t(\mathscr{A}_m)\, d\alpha(t)\\ &= \lim_m \mu(\mathscr{A}_m) = \mu(K) \geqslant 1 - \varepsilon,\end{aligned}$$

where $\nu_{n,t}$ and μ_t are the Gaussian measures with covariances tT_n an tS, respectively.

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Interpolation of Orlicz spaces

by

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Abstract. Let φ, φ_0 and φ_1 be positive increasing functions on $[0, \infty)$ connected by the formula $\varphi^{-1} = \varphi_0^{-1}\varrho(\varphi_1^{-1}/\varphi_0^{-1})$ with a suitable ϱ. Consider the corresponding Orlicz spaces L^φ, L^{φ_0} and L^{φ_1}. It is shown that L^φ is an interpolation space with respect to L^{φ_0} and L^{φ_1} provided ϱ is "a little more than concave".

0. Introduction. In this paper we give a contribution to the following problem: *Given three Orlicz spaces L^φ, L^{φ_0} and L^{φ_1} on some measure space M, under what conditions is it true that L^φ is an interpolation space with respect to L^{φ_0} and L^{φ_1}* ? Roughly speaking, we show that, assuming that φ is expressed in terms of φ_0 and φ_1 in the form

$$\varphi^{-1} = \varphi_0^{-1}\varrho(\varphi_1^{-1}/\varphi_0^{-1}) \tag{0.1}$$

(where φ^{-1} is the inverse of φ, etc.), it is sufficient to assume that ϱ is "a little more than concave". In particular, our result applies when

$$\varrho(x) = x^\theta \quad (0 < \theta < 1), \tag{0.2}$$

in which case (0.1) specializes to

$$\varphi^{-1} = (\varphi_0^{-1})^{1-\theta}(\varphi_1^{-1})^\theta, \tag{0.1'}$$

covering thus the case treated by Rao [17] (cf. Kraynek [10]). As an example of a function ϱ, more general than the one in (0.2), to which our theory applies, we mention

$$\varrho(x) = x^\theta\big(\log(e+x)\big)^\alpha\big(\log(e+1/x)\big)^\beta \quad (0 < \theta < 1,\ \alpha,\ \beta \text{ arbitrary real}). \tag{0.2'}$$

Whereas that author uses Thorin's proof conveniently adapted, we shall instead rely on an idea of Gagliardo [5], in the special case of L^p (cf. Peetre [13]). More precisely, given any quasi-Banach couple $\vec{A} = \{A_0, A_1\}$ we define interpolation spaces $\langle\vec{A}, \varrho\rangle = \langle A_0, A_1, \varrho\rangle$. In the special case when A_0 and A_1 are both rearrangement invariant spaces of measurable