Since $\varphi\left(\frac{1}{r_k}\zeta^{k,\nu}\right) = f(\zeta^{k,\nu}) = 0$ for $\nu \geqslant 1$, we have $\varphi = 0$, and consequently $f = 0$.

Remark 4. Our Theorem remains true for real analytic functions. The above proof may be repeated without any change.

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A counterexample to several questions about Banach spaces

by

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Abstract. There exists a separable Banach space X with nonseparable dual such that l^1 does not imbed in X and such that every normalized weakly null sequence in X has a subsequence equivalent to the usual basis of c_0. Weak sequential convergence and norm convergence in X^* coincide. Other properties of X and X^* are investigated.

1. Introduction. In this paper we construct a Banach space X which provides answers to many open questions about the isomorphic structure of Banach spaces. Our main result is

THEOREM 1. *There exists a separable Banach space X satisfying the following:*

(a) *Every sequence in X which converges weakly but not in norm to zero has a c_0 subsequence.*

(b) *There exists a separable subspace F of X^* such that X^*/F is isometrically isomorphic to $c_0(\Gamma)$, where Γ has cardinality* $\mathfrak{c}$.

(c) *X is hereditarily c_0.*

(d) *There exists a subspace Y of X with Y^* separable such that Y does not imbed in c_0.*

(e) *X^* has the Schur property; i.e., weak sequential convergence and norm convergence in X^* coincide. In particular, X^* is hereditarily l^1.*

(f) *There exists a bounded set Γ in X^* of cardinality $\mathfrak{c}$ such that no sequence in Γ is a weak Cauchy sequence. Yet, no subspace of the closed linear span of Γ is isomorphic to $l^1(\Lambda)$ for any uncountable set Λ.*

Of course, (b) and (c) of Theorem 1 show that X is another example of a separable space with X^* nonseparable such that l^1 does not imbed in X. The first of these examples was given by James in [7]. Later, Lindenstrauss and Stegall [11] gave a second example of such a space (which they called JF) and studied the duality properties of the above mentioned example of James (which they called JT). The space JT is hereditarily l^2, while the space JF has many subspaces isomorphic to l^2 and many isomorphic to c_0. Both the spaces JT and JF are closely related to the non-reflexive space J isometric to J^{**} introduced by James in [6]. In the space X of Theorem 1, c_0 plays the role that J does in JT and JF. (The influence of the papers [7] and [11] on this paper is considerable.

The definition of the space X is a variation of the definition of the space JT. The study of the duality properties of X is inspired by a similar study in [11].)

Part (c) of Theorem 1 shows that if l^1 does not imbed in the separable space Z and Z^* is nonseparable, then Z need not contain a reflexive subspace. Parts (a) and (d) answer this question: If Z is a Banach space (even with Z^* separable) such that every sequence going weakly but not in norm to zero has a c_0 subsequence, must Z imbed into c_0? (The author was asked this question by W. Johnson.) To prove that there are subspaces of X with separable dual which do not imbed into c_0, we use the Szlenk index [22]. (Possible use of the Szlenk index in this setting was suggested by P. Wojtaszczyk.) As we indicate following the proof of (d), the methods we use actually show the following: For any countable ordinal α, there exists a subspace Y of X with Y^* separable such that Y does not imbed in $C(\alpha)$.

An easy application of part (e) shows that a Banach space with the Schur property need not have the Radon–Nikodym property. (H. Rosenthal pointed out that the proof that X^* is hereditarily l^1 actually shows that X^* has the Schur property.) In fact, the proof of (e) (together with a trivial modification in the nonseparable case) actually shows that *any* Banach space satisfying (a) also satisfies (e).

Part (f) shows that an uncountable version of this theorem of H. Rosenthal [18] fails: If a bounded sequence $\{x(n): n = 1, 2, \ldots\}$ in a Banach space Z has no weak Cauchy subsequence, then a subsequence of $\{x(n): n = 1, 2, \ldots\}$ is equivalent to the usual basis of l^1. Rosenthal deduces this from a combinatorial result (Theorem 2 of [18]) concerning the behavior of a sequence of pairs of sets $\{(A_n, B_n): n = 1, 2, \ldots\}$. The uncountable version of this result fails also, as we show following the proof of (f) in § 3.

One other interest in (f) is that it shows the difficulty of finding general conditions on a Banach space Z (or its duals) which imply that $l^1(\Lambda)$ imbeds in Z for some uncountable set Λ. For example, if $L^1\{0,1\}^{\mathfrak{m}}$ imbeds in Z^* and $\mathfrak{m} \geqslant \mathfrak{c}$, then it is easy to show (cf. [5]) that Z contains a bounded set of cardinality $\mathfrak{c}$ out of which no sequence is weak Cauchy. It is not yet known if $l^1(\Lambda)$ imbeds in Z for some uncountable Λ whenever $L^1\{0,1\}^{\mathfrak{m}}$ imbeds in Z^*. (There are some positive results in this direction. For example, in [4] we show that every nonseparable subspace of $C\{0,1\}^{\mathfrak{m}}$ contains an isomorph of $l^1(\Lambda)$ for some uncountable set Λ.)

An interesting use of (b) and (e) is that they illustrate the sharpness of this result of Rosenthal [17]: If $c_0(\Lambda)$ imbeds in Z^* for some set Λ, then $l^\infty(\Lambda)$ imbeds in Z^*. If X is the space of Theorem 1, X^* contains no isomorph of c_0. But the quotient of X^* by a separable space is isometric to $c_0(\Lambda)$, where Λ is uncountable.

Let us briefly discuss the ideas involved in the definition and study of the space X. First, as in the space JT, X is a space of functions defined on a dyadic tree T. However, we define the norm in X in terms of "admissible" segments (see § 2). Next, at the beginning of § 3, we define the notion of a strongly incomparable sequence of nodes of the tree T. Such a sequence has a stronger property than that its terms are pairwise incomparable (with respect to the partial order on T). Lemma 2 shows the existence of many such sequences of nodes of T, and may be of independent interest. The crucial component in the proof of Theorem 1 is contained in Proposition 5, which gives a decomposition of an element $x \in X$ and is used to analyze the most general sequence in X which is equivalent to the usual basis of c_0. This "diagonal decomposition" which is used in Proposition 5 may be useful in other connections.

Let us indicate the organization of the remainder of this paper. § 2 contains basic Banach space definitions and notation, the definition of the tree and related notions, and the definition of the space X which satisfies Theorem 1. The key results here are Lemma 2 and Proposition 5. The proof of (c) of Theorem 1 follows closely the analysis of the duality properties of the space JT given in [11], so we do not present the proof in great detail. Following the proof of (f) of Theorem 1, we show how to obtain an uncountable counterexample to the "independence" lemma of Rosenthal mentioned above. § 4 contains some remarks and open problems.

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2. Preliminaries. For the most part our notation and terminology are standard, or can be found in references [1], [2], [3], or [4].

All Banach spaces will be real Banach spaces. If X is a Banach space, X^* denotes its dual space. A subspace of X is a closed linear submanifold of X. An operator $T: X \to Y$ is a bounded linear operator. The Banach spaces X and Y are *isomorphic* if they are linearly homeomorphic, i.e., if there is an operator $T: X \to Y$ which is one-to-one and onto. For a set D in X, $[D]$ is the *closed linear span* of the set D, i.e., the smallest subspace of X containing D. A sequence $x(1), x(2), \ldots$ in X is *weakly null* if $x(n) \to 0$ weakly in X, i.e., if $\lim_n x^*(x(n)) = 0$ for all $x^* \in X^*$. Let Y be a Banach space. Then we say that Y *imbeds in* X if Y is isomorphic to a subspace of X. Also, we say that a *Banach space X is hereditarily Y* if Y imbeds in every infinite-dimensional subspace of X.

The Banach space c_0 is the space of all sequences $t_1, t_2, \ldots$ of real numbers with $\lim_n t_n = 0$ and $\|(t_n)\| = \max_n |t_n|$. A sequence $x(1), x(2), \ldots$ in X is called a c_0 *sequence* if it is equivalent to the usual basis of c_0, i.e., if there are constants $\delta, K > 0$ such that $\delta \max_j |t_j| \leqslant \|\sum_{j=1}^{s} t_j x(j)\| \leqslant K \max_j |t_j|$ for all s, and scalars $t_1, \ldots, t_s$.

The example we shall study is a space of functions on a dyadic tree, which we now define. The dyadic tree T is the set $T = \bigcup_{n=0}^{\infty} \{0,1\}^n$ together with the partial order defined below. Elements $\varphi \in T$ will be called *nodes*. If φ is a node and $\varphi \in \{0,1\}^n$, we write $|\varphi| = n$ and $\varphi = (\varepsilon_1, \ldots, \varepsilon_n)$. Let $m \geqslant n$ and let $\psi = (\delta_1, \ldots, \delta_m)$ be a node. Then $\psi \geqslant \varphi$ if $\delta_i = \varepsilon_i$ for $i = 1, \ldots, n$. If $\psi \geqslant \varphi$ and $|\psi| > |\varphi|$, then we write $\psi > \varphi$. If φ and ψ are nodes such that neither $\varphi \geqslant \psi$ nor $\psi \geqslant \varphi$, then φ and ψ are *incomparable*.

Let integers n and m be given with $n \leqslant m$. Then we say that a subset S of T is an *n-m segment* if

(1) for every k with $n \leqslant k \leqslant m$ there exists a unique $\varphi \in S$ with $|\varphi| = k$;

(2) if $\varphi, \psi \in S$ and $|\varphi| > |\psi|$, then $\varphi > \psi$. A subset S of T is a *segment* if it is an *n-m* segment for some n and m. We say that S *passes through a node* φ if $\varphi \in S$. It is clear from these definitions that there is a segment S passing through φ and ψ if and only if $\varphi \geqslant \psi$ or $\psi \geqslant \varphi$.

A *branch* B of T is a sequence $\varphi(0), \varphi(1), \ldots$ of nodes such that

(1) $|\varphi(n)| = n$ for each n;

(2) if $m > n$, $\varphi(m) > \varphi(n)$. It is clear that each branch of T can be identified uniquely with an infinite sequence of 0's and 1's.

Now let $x\colon T \to \mathbf{R}$ be a function. We will denote $x = \{t_\varphi\colon \varphi \in T\}$, where $t_\varphi = x(\varphi)$ for all $\varphi \in T$. Let us define some (algebraic) linear functionals and projections on the vector space of finitely nonzero functions on T. Fix such an $x = \{t_\varphi\colon \varphi \in T\}$. If S is a segment, then we define $S^*(x) = \sum_{\varphi \in S} t_\varphi$. (If $S = \{\varphi\}$, then we write φ^* instead of $\{\varphi\}^*$.) If B is a branch, then we define $B^*(x) = \sum_{\varphi \in B} t_\varphi$. We now define the *projections*. For $\psi \in T$, define P_ψ by

$$P_\psi x(\varphi) = \begin{cases} t_\varphi & \text{if } \varphi \geqslant \psi, \\ 0 & \text{otherwise.} \end{cases}$$

For an integer n, define P_n by

$$P_n x(\varphi) = \begin{cases} t_\varphi & \text{if } |\varphi| \geqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, for a branch B of T, define P_B by

$$P_B x(\varphi) = \begin{cases} t_\varphi & \text{if } \varphi \in B, \\ 0 & \text{otherwise.} \end{cases}$$

(Once the Banach space X is defined, it will be clear that $\|S^*\| = \|B^*\| = \|P_\psi\| = \|P_n\| = \|P_B\| = 1$.)

To define the Banach space X, we need the notion of an admissible family of segments. Let $S_1, \ldots, S_r$ be segments in T. These segments are admissible provided

(1) there exist integers $m \leqslant n$ such that each S_i is an *m-n* segment;

(2) $S_i \cap S_j = \emptyset$ if $i \neq j$. For a finitely nonzero function $x\colon T \to \mathbf{R}$, define

$$\|x\| = \max \sum_{i=1}^{r} |S_i^*(x)|,$$

where the max is taken over all families $S_1, \ldots, S_r$ of admissible segments. The Banach space X is the completion of the space of finitely nonzero functions on T in the above norm.

For $\varphi \in T$, let $e_\varphi \in X$ be defined by

$$e_\varphi(\psi) = \begin{cases} 1 & \text{if } \varphi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

If we list the elements in T as $\varphi(0,1); \varphi(1,1), \varphi(1,2); \ldots$, where for each n, $\varphi(n,1), \ldots, \varphi(n, 2^n)$ is an enumeration of $\{0,1\}^n$, then it is easily checked that the sequence of unit vectors $e_{\varphi(0,1)}; e_{\varphi(1,1)}, e_{\varphi(1,2)}; \ldots$ is a Schauder basis for X.

Let S be a p-q segment and $x \in X$ such that $x = (P_k - P_j)x$. We say that S *passes through the support of* x if $p \leqslant k < j \leqslant q$. We say that S *begins in* (*below*) *the support of* x if $k \leqslant p < j$ ($j \leqslant p$) *and ends in* (*above*) *the support of* x if $k \leqslant q < j$ ($q > k$).

Let $\Delta = \{0,1\}^{\aleph_0}$ denote the Cantor set, $C(\Delta)$ the Banach space of continuous real valued functions on Δ. Then the "natural" base of closed open sets of Δ can be denoted by $U_\varphi = \{(\varepsilon_1, \varepsilon_2, \ldots)\colon (\varepsilon_1, \ldots, \varepsilon_n) = \varphi\}$ as φ runs through T. The we identify Δ with the set Γ of branches of T and the sets U_φ with the nodes φ. To make this a little more precise, we observe that Γ is weak* homeomorphic to the Cantor set. Thus, if $B(n)^* \to B^*$ weak* and $\varphi \in T$, then there exists an N such that, for $n \geqslant N$, either

(i) if $\varphi \in B$, then $B(n)$ passes through φ; or

(ii) if $\varphi \notin B$, $B(n)$ does not pass through φ.

By a slight abuse of language, we will let Γ denote both the set of branches B of T and the associated set of functionals B^*. No confusion will arise from this practice.

Many of the proofs involve the selection of subsequences of given sequences. As the constructions can involve several parameters at one time, we adopt the following conventions: Sequences of vectors will be denoted by $x(1), x(2), \ldots$ and sequences of nodes by $\varphi(1), \varphi(2), \ldots$ (Sometimes, these sequences may be indexed as functions of two or more variables.) There will be no confusion between $x(n)$, the nth term of a sequence, and $x(\varphi)$, the value of the function x at the node φ.

Whenever possible, we index sequences by infinite subsets of the integers N. If M is an infinite subset of N, we will consider M as a subsequence of N.

On the other hand, we will index sequences of scalars by the traditional subscripts, e.g., a sequence of scalars $t_1, t_2, \ldots$ or $\{t_j: j \in N\}$.

3. Proofs of the main results. This section contains the analysis of the space X. Before proving any of the assertions of Theorem 1, we need some preliminary lemmas. The first of these concerns the behavior of a sequence of nodes of T. Let us say that a sequence of nodes $\{\varphi(n): n \in N\}$ is a *strongly incomparable sequence* if

(i) $\varphi(n)$ and $\varphi(m)$ are incomparable if $n \neq m$; and

(ii) no family of admissible segments passes through more than two of the $\varphi(n)$'s, $n \in N$.

LEMMA 2. *Let $\{\varphi(n): n \in N\}$ be a sequence of nodes of T such that $|\varphi(n)| > |\varphi(m)|$ if $n > m$. Then there exists a subsequence N' of N such that either*

(i) *$\{\varphi(n): n \in N'\}$ determines a unique branch of T; or*

(ii) *$\{\varphi(n): n \in N'\}$ is a strongly incomparable sequence.*

Proof. If $n > m$, then either $\varphi(n) > \varphi(m)$ or $\varphi(n)$ and $\varphi(m)$ are incomparable. So by a direct use of Ramsey's theorem (cf. [12], for example) or by translating the proof of Lemma 4 of [13] into the appropriate terminology, we can pick a subsequence M of N such that either

(i) $\{\varphi(n): n \in M\}$ determines a unique branch of T; or,

(ii)′ if $m, n \in M$ and $m \neq n$, then $\varphi(m)$ and $\varphi(n)$ are incomparable.

What remains to be shown is that if (ii)′ holds, then a subsequence N' of M satisfies (ii).

First, inductively pick a sequence $\psi(0), \psi(1), \ldots$ in T satisfying, for each i,

(a) $|\psi(i)| = i$;

(b) $\psi(i+1) > \psi(i)$;

(c) $\{n \in M: \varphi(n) \geqslant \psi(i)\}$ is infinite.

Now inductively pick sequences $n_1, n_2, \ldots$ in M and $k_1, k_2, \ldots$ such that $k_i = |\varphi(n_{i-1})|$ and $\varphi(n_i) \geqslant \psi(k_i)$ (put $\varphi(n_1) = \varphi(1)$). We show that $N' = \{n_1, n_2, \ldots\}$ satisfies (ii).

Let $S_1, S_2, \ldots, S_r$ be an admissible family of p-q segments. Let k_j be the smallest integer for which $k_j \geqslant p$. Then if $i > j$, $\varphi(n_i) \geqslant \psi(k_j)$. Since at most one of the segments can pass through $\psi(k_j)$, since every p-q segment which passes through $\varphi(n_i)(i \geqslant j)$ must pass through $\psi(k_j)$, and since no segment can pass through two $\varphi(n_i)$'s, we have that $S_1, \ldots, S_r$ can pass through at most one $\varphi(n_i)$, $i \geqslant j$. Also, one of the segments $S_1, \ldots, S_r$ can pass through $\varphi(n_{j-1})$. Finally, for $i \leqslant j-2$, $m_{i+1} < p$ so the segments $S_1, \ldots, S_r$ cannot pass through $\varphi(n_i)$. ■

Remark. The conclusion of Lemma 2 is still valid if we assume only that the segments $S_1, \ldots, S_r$ are pairwise disjoint and begin at the same level (i.e., if there is a p such that each S_i is a p-q_i segment for $i = 1, \ldots, r$).

The proof of part (a) of Theorem 1, which will be our first concern, is accomplished by a decomposition of an element of X into what are roughly its "isomorphic" and "non-isomorphic" components. We turn our attention first to what will turn out to be the non-isomorphic part of the decomposition.

Recall the identification between Γ and the Cantor set Δ (see § 2). Define an operator $R: X \to C(\Delta)$ by $Rx(B) = B^*(x)$. It is easy to check that R is well defined and $\|R\| = 1$.

LEMMA 3. *Let $w(1), w(2), \ldots$ be a bounded sequence in X and $n_1 < n_2 < \ldots$ a sequence of integers such that $w(k) = (P_{n_k} - P_{n_{k+1}})w(k)$. Assume also that $\|Rw(k)\| < 2^{-n_k-1}$.*

Let $S_1, \ldots, S_r$ be an admissible family of segments passing through the supports of $w(c), \ldots, w(d)$. Then, given scalars $t_c, \ldots, t_d$, we have

$$\sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{d} t_j w(j) \Big) \Big| \leqslant \max_j |t_j|.$$

Proof. First, observe that $r \leqslant 2^{n_c}$ and that if $j \geqslant c$, we have $r 2^{-(n_j+1)} \leqslant 2^{n_c-(n_j+1)} \leqslant 2^{-(j-c-1)}$. Also, observe that since $w(j) = (P_{n_j} - P_{n_{j+1}})w(j)$, then

$$2^{-(n_j+1)} \geqslant \|Rw(j)\| = \sup \big|S^*\big(w(j)\big)\big|$$

where the sup is taken over all segments passing through the support of $w(j)$.

Using these observations, we have that

$$\sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{d} t_j w(j) \Big) \Big| \leqslant \sum_{j=c}^{d} |t_j| \sum_{i=1}^{r} \big| S_i^* \big(w(j) \big) \big| \leqslant \sum_{j=c}^{d} |t_j| r \|Rw(j)\|$$
$$\leqslant \sum_{j=c}^{d} |t_j| 2^{-(j-c-1)} \leqslant \max_j |t_j|. \ ■$$

Remark. If we assume only that $\|Rw(j)\| \to 0$ in Lemma 3, then a subsequence of $\{w(j): j \in N\}$ (and a *different* sequence of n_j's) will satisfy the assumptions of Lemma 3. It is also immediate that a sequence satisfying Lemma 3 is equivalent to the usual basis of c_0, but we will not prove this here.

The final preliminary lemma concerns the "isomorphic" part of the decomposition of an element of X.

LEMMA 4. *Let $x(1), x(2), \ldots$ be a bounded sequence in X, $n_1 < n_2 < \ldots$ a sequence of integers and $\delta > 0$ satisfy the following:*

(1) *For each k, $x(k) = (P_{n_k} - P_{n_{k+1}})x(k)$;*

(2) *For each k, there exists $\varphi(k) \in T$ with $|\varphi(k)| = n_k$ such that $x(k) = P_{\varphi(k)}x(k)$;*

(3) *For every branch B, $|B^*(x(k))| \leqslant \delta$.*

Let $S_1, \ldots S_r$ be an admissible family of segments passing through the supports of $x(c), \ldots, x(d)$, and let scalars $t_c, \ldots, t_d$ be given.

(a) *If $\{\varphi(k): k \in N\}$ determines a branch B_0 of T and $|B_0^*(x(k))| \leqslant 2^{-k}$ for each k, then*

$$\sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{d} t_j x(j) \Big) \Big| \leqslant (\delta + 2^{-c+1}) \max_j |t_j|;$$

(Note: $\{\varphi(k): k \in N\}$ determines a branch B_0 if there is a unique branch B_0 of T with $\varphi(k) \in B_0$ for each $k \in N$.)

(b) *If $\{\varphi(k): k \in N\}$ is a strongly incomparable sequence, then*

$$\sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{d} t_j x(j) \Big) \Big| \leqslant 2\delta \max_j |t_j|.$$

Proof. To prove case (a), observe that at most one of the S_i's can have the property that

$$S_i^* \Big(\sum_{j=c}^{d} t_j x(j) \Big) \neq 0.$$

If S denotes this segment, and if S passes through $\varphi(c), \ldots, \varphi(k)$, but not $\varphi(k+1)$ (hence, not through $\varphi(j)$ for $j \geqslant k+1$), we have

$$\sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{d} t_j x(j) \Big) \Big| \leqslant \sum_{j=c}^{k-1} |t_j B_0^*(x(j))| + |t_k|\,|S^*(x(k))| \leqslant \sum_{j=c}^{k-1} |t_j| 2^{-j} + |t_k| \delta$$
$$\leqslant (\delta + 2^{-c+1}) \max |t_j|.$$

To prove (b), observe that the segments $S_1, \ldots, S_r$ can pass through at most two of the $\varphi(k)$'s, and each segment can pass through at most one $\varphi(k)$. So, if S_α and S_β pass through $\varphi(k)$ and $\varphi(m)$, then

$$\sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{d} t_j x(j) \Big) \Big| = |S_\alpha^*(t_k x(k))| + |S_\beta^*(t_m x(m))| \leqslant 2\delta \max_j |t_j|.$$

The cases where $S_1, \ldots, S_r$ pass through zero or one node are handled similarly. ■

The key to the proof of (a) of Theorem 1 is Proposition 5 which follows. To obtain (a) of Theorem 1, we must only apply a standard perturbation argument, which we omit.

PROPOSITION 5. *Let $x(1), x(2), \ldots$ be a sequence of norm one elements in X and $n_1 < n_2 < \ldots$ a sequence of integers such that $x(k) = (P_{n_k} - P_{n_{k+1}})x(k)$ for each k. Assume also that $x(k) \to 0$ weakly as $k \to \infty$. Then there exist integers $k_1 < k_2 < \ldots$ such that*

$$\max_j |t_j| \leqslant \Big\| \sum_{j=1}^{s} t_j x(k_j) \Big\| \leqslant 7 \max_j |t_j|$$

for all s, and scalars $t_1, \ldots, t_s$.

Proof. For each $m = 1, 2, \ldots$ and fixed k, let $F(k, m)$ be the set of those $\varphi \in \{0, 1\}^{n_k}$ such that there exists at least one branch B passing through φ with $|B^*(x(k))| > 2^{-m}$ and, for all branches B passing through φ, $|B^*(x(k))| \leqslant 2^{-m+1}$. Define

$$x(k, m) = \sum_{\varphi \in F(k,m)} P_\varphi x(k).$$

Then for any m, we have a unique decomposition

$$x(k) = x(k, 1) + \ldots + x(k, m) + w(k, m),$$

where $w(k, m)$ has the property that for any branch B, $|B^*(w(k, m))| \leqslant 2^{-m}$. Observe that $x(k, j)$ is well defined independent of m, and that $F(k, m)$ has fewer than 2^m elements independent of k.

For $m = 1$, consider the sequences $\{x(k, 1): k \in N\}$ and $\{F(k, 1): k \in N\}$. Pick a subsequence N_1 of N such that the cardinality of the set $F(k, 1)$ is an integer b_1 independent of $k \in N_1$. If $b_1 = 0$, then put $N_1'' = N_1' = N_1$. If $b_1 > 0$, then $b_1 = 1$, and we write $F(k, 1) = \{\varphi(k, 1; 1)\}$ for $k \in N_1$. Pick a subsequence N_1' of N_1 and a subset $I_1 \subset \{1\}$ (so $I_1 = \{1\}$ or $\emptyset$) such that $\{\varphi(k, 1; 1): k \in N_1'\}$ either determines a branch $B(1,1)$ if $1 \in I_1$ or is strongly incomparable if $1 \notin I_1$. Pick $k_1 \in N_1'$ such that, if $1 \in I_1$, $|B(1, 1)^*(x(k_1, 1))| < 2^{-1}$ (recall that the sequence $x(k) \to 0$ weakly; if $I_1 = \emptyset$, any $k_1 \in N_1'$ will work). Let $N_1'' = \{k \in N_1': k > k_1\}$.

Assume now that we have selected for $m = 1, \ldots, j$ infinite subsets N_m, N_m', and N_m'' of N, integers b_m, and, for those m with $b_m > 0$, an integer $k_m \in N_m'$ and a subset $I_m \subset \{1, \ldots, b_m\}$ satisfying the following properties:

(i) $N_1 \supset N_1' \supset N_1'' \supset \ldots \supset N_j \supset N_j' \supset N_j''$.

(ii) The set $F(k, m)$ has cardinality b_m for all $k \in N_m$.

(iii) If $b_m = 0$, then $N_m' = N_m'' = N_m$.

(iv) If $b_m > 0$, then

(a) if $k \in N'_m$, then $x(k, m) = \sum_{i=1}^{b_m} P_{\varphi(k,m;i)} x(k)$ (where

$$F(k, m) = \{\varphi(k, m; i): i = 1, \ldots, b_m\});$$

(b) for fixed $i \in \{1, \ldots, b_m\}$, if $i \in I_m$, then $\{\varphi(k, m; i): k \in N'_m\}$ determines a branch $B(m, i)$ of T, and if $i \notin I_m$, then $\{\varphi(k, m; i): k \in N'_m\}$ is a strongly incomparable sequence;

(c) for each $n = 1, \ldots, m$ and $i \in I_n$, $|B(n, i)^*(x(k_m, n))| < 2^{-m}$;

(d) $N''_m = \{k \in N'_m: k > k_m\}$.

We must pick $N_{j+1}, N'_{j+1}, N''_{j+1}, b_{j+1}$, and, if $b_{j+1} > 0$, an integer $k_{j+1} \in N'_{j+1}$ and a subset $I_{j+1} \subset \{1, \ldots, b_{j+1}\}$ all satisfying (i)–(iv) above.

Consider $\{x(k, j+1): k \in N''_j\}$ and $\{F(k, j+1): k \in N''_j\}$. Pick a subsequence N_{j+1} of N''_j and an integer b_{j+1} such that, for each $k \in N_{j+1}$, the cardinality of the set $F(k, j+1)$ is b_{j+1}. If $b_{j+1} = 0$, put $N''_{j+1} = N'_{j+1} = N_{j+1}$. If $b_{j+1} > 0$ then enumerate $F(k, j+1) = \{\varphi(k, j+1; i): i = 1, \ldots, b_{j+1}\}$. Then, by b_{j+1} repeated applications of Lemma 2, we can select a subsequence N'_{j+1} of N_{j+1} and a subset $I_{j+1} \subset \{1, \ldots, b_{j+1}\}$ such that if $i \in I_{j+1}$, then $\{\varphi(k, j+1; i): k \in N_{j+1}\}$ determines a branch $B(j+1, i)$ of T, and if $i \notin I_{j+1}$, then $\{\varphi(k, j+1; i): k \in N_{j+1}\}$ is a strongly incomparable sequence.

Now select $k_{j+1} \in N'_{j+1}$ such that, for $m = 1, \ldots, j+1$ and $i \in I_m$, $|B(m, i)^*(x(k_{j+1}, m))| < 2^{-j-1}$. (This is possible since $x(k) \to 0$ weakly and since $\bigcup_{m=1}^{j+1} I_m$ is finite.) Finally, let $N''_{j+1} = \{k \in N'_{j+1}: k > k_{j+1}\}$. This completes the induction process.

Let us write $x(k_j) = \sum_{m=1}^{j} x(k_j, m) + w(k_j, j)$. By construction, $\|Rw(k_j, j)\| \leqslant 2^{-j}$. So, passing to a subsequence M of those j with $b_j > 0$ so that $\{w(k_j, j): j \in M\}$ satisfies the assumptions of Lemma 3, we have the following:

For any admissible family of segments $S_1, \ldots, S_r$ passing through the supports of $x(c), \ldots, x(d)$ and any scalars $t_c, \ldots, t_d$,

$$\sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{d} t_j w(k_j, j) \Big) \Big| \leqslant \max_j |t_j|.$$

For the remainder of the proof, we adopt the following convention to avoid introducing additional notation. Any statement about j's refers only to those j's in M. For example, if we pick scalars $t_c, \ldots, t_d$, then we pick them only for those $j \in M$ between c and d. Similarly, $\sum_{j=c}^{d} t_j x(k_j)$ denotes a sum only for those $j \in M$ between c and d.

Next, we consider (for fixed m) the elements $x(k_j, m)$ for $j \geqslant m$. By definition of $x(\cdot, m)$ and by (iv) of the induction process, if we fix an

$i \in \{1, \ldots, b_m\}$, then the elements $\{P_{\varphi(k_j,m;i)} x(k_j): j \geqslant m\}$ satisfy the assumptions of either (a) or (b) of Lemma 4 with $\delta = 2^{-m+1}$. But now, since each $x(k_j, m)$ is a sum of b_m terms of the form $P_{\varphi(k_j,m;i)} x(k_j)$, a direct application of Lemma 4, the triangle inequality, and (when (a) of Lemma 4 applies), the fact that if $c \geqslant m$, $(\delta + 2^{-c+1}) = (2^{-m+1} + 2^{-c+1}) \leqslant 2^{-m+2}$ yield the following:

For any family of admissible segments $S_1, \ldots, S_r$ passing through the supports of $x(c), \ldots, x(d)$ and any scalars $t_c, \ldots, t_d$,

$$\sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{d} t_j x(k_j, m) \Big) \Big| \leqslant b_m 2^{-m+2} \max_j |t_j|.$$

At last, we are ready to estimate $\|\sum_{j=1}^{s} t_j x(k_j)\|$. Let s and scalars $t_1, \ldots, t_s$ be given. Let $S_1, \ldots, S_r$ be an admissible family of segments. To be precise, we should distinguish five separate (but obvious) cases. We consider only the most illustrative of these, which is the following: The segments begin in the support of $x(k_{c-1})$ and end in the support of $x(k_{d+1})$, where $1 \leqslant c-1 \leqslant d+1 \leqslant s$. Then,

$$\begin{aligned}
&\sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=1}^{s} t_j x(k_j) \Big) \Big| \\
&\leqslant |t_{c-1}| \sum_{i=1}^{r} \big| S_i^*(x(k_{c-1})) \big| + |t_{d+1}| \sum_{i=1}^{r} \big| S_i^*(x(k_{d+1})) \big| + \sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{d} t_j x(k_j) \Big) \Big| \\
&\leqslant |t_{c-1}| \, \|x(k_{c-1})\| + |t_{d+1}| \, \|x(k_{d+1})\| + \sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{d} t_j w(k_j, j) \Big) \Big| + \\
&\qquad + \sum_{i=1}^{r} \Big| S_i^* \Big(\sum_{j=c}^{r} t_j \Big(\sum_{m=1}^{j} x(k_j, m) \Big) \Big) \Big| \\
&\leqslant 3 \max_j |t_j| + \sum_{m=1}^{d} \Big| \sum_{i=1}^{r} S_i^* \Big(\sum_{j=\max(m,c)}^{d} t_j x(k_j, m) \Big) \Big| \\
&\leqslant \Big(3 + \sum_{m=1}^{d} 2^{-m+2} b_m \Big) \max_j |t_j|.
\end{aligned}$$

To complete this part of the inequality, we show that $\sum_{m=1}^{\infty} b_m 2^{-m} \leqslant 1$. For fixed j, consider the norm one element $x(k_j)$. Write $x(k_j) = x(k_j, 1) + \ldots + x(k_j, j) + w(k_j, j)$. Recall that $F(k_j, m) \cap F(k_j, n) = \emptyset$ if $m \neq n$. So, for each $m = 1, \ldots, j$, there exist b_m distinct n_{k_j}-$n_{k_{j+1}}$ segments $S_{m,1}, \ldots, S_{m,b_m}$ such that $2^{-m} < |S_{m,i}^*(x(k_j))| \leqslant 2^{-m+1}$ for each i. But then, $S_{1,1}, \ldots, S_{1,b_1}, S_{2,1}, \ldots, S_{j,1}, \ldots, S_{j,b_j}$ forms an admissible family of segments. Thus,

$$1 \geqslant \|x(k_j)\| \geqslant \sum_{m=1}^{j} \sum_{i=1}^{b_m} |S_{m,i}^*(x(k_j))| \geqslant \sum_{m=1}^{j} b_m 2^{-m},$$

as desired.

It remains only to show that $\|\sum_{j=1}^{s} t_j x(k_j)\| \geqslant \max_j |t_j|$. For fixed j, pick admissible p-q segments $S_1, \ldots, S_r$ such that $1 = \|x(k_j)\| = \sum_{i=1}^{r} |S_i^*(x(k_j))|$ and $n_{k_j} \leqslant p \leqslant q < n_{k_{j+1}}$. Then, $\sum_{i=1}^{r} |S_i^*(\sum_{j=1}^{s} t_j x(k_j))| = |t_j|$. Thus, $\|\sum_{j=1}^{s} t_j x(k_j)\| \geqslant \max_j |t_j|$, which completes the proof of Proposition 5. ■

Remark. If in Proposition 5 we do not assume that $\{x(k)\colon k \in N\}$ is weakly null, we can prove directly that there are subsequences (k_j) and (m_j) of N with $k_1 < m_1 < k_2 < m_2 < \ldots$ such that $\{x(k_j) - x(m_j)\colon j \in N\}$ is equivalent to the usual basis of c_0. The proof is similar to that of Proposition 5, but involves more notation. (In particular, (c) can be proved without the use of (b) and Rosenthal's characterization of Banach spaces containing l^1 [18].)

We begin now to assemble the components which will be used in the proof of (b). Let $F = [\{\varphi^*\colon \varphi \in T\}]$. We will show that X^*/F is isometrically isomorphic to $c_0(\Gamma)$.

Let E be the completion of the normed space of all finitely non-zero sequences $(t_0, t_1, \ldots, t_s, 0, 0, \ldots)$ with

$$\|(t_n)\| = \max_{k \leqslant m} \Big|\sum_{i=k}^{m} t_i\Big|.$$

Letting $\{e(n)\colon n \geqslant 0\}$ be the unit vector basis for E, $\{f(n)\colon n \geqslant 0\}$ the biorthogonal functionals to the $e(n)$'s, we have the following easy result, whose proof we omit.

LEMMA 6. (a) *The space E is isomorphic to c_0.* (b) *If $f \in E^*$, then $\lim_n f(e(n))$ exists. If $\lim_n f(e(n)) = 0$, then f is in the closed linear span of the set $\{f(n)\colon n \in N\}$.*

Since for every branch B, $P_B(X)$ is isometrically isomorphic to E, this lemma implies that if $x^* \in X^*$, then $\lim_n x^*(e_{\varphi(n)}) = \lim_n P_B^* x^*(e_{\varphi(n)})$ exists and is zero if and only if $P_B^* x^*$ is in the closed span of $\{\varphi(n)^*\colon \varphi(n) \in B\}$.

Let us define an operator $Q\colon X^* \to l^\infty(\Gamma)$ by $Qx^*(B) = \lim_n x^*(e_{\varphi(n)})$, where $B = \{\varphi(0), \varphi(1), \ldots\}$. This next lemma shows that $Q(X^*) \subset c_0(\Gamma)$.

LEMMA 7. *Let $x^* \in X^*$ and $\varepsilon > 0$ be given. Then, $\{B \in \Gamma\colon |Qx^*(B)| > \varepsilon\}$ is finite.*

Proof. Assume that $\{B(n)\colon n \in N\}$ is a sequence of distinct branches of T with $|Qx^*(B(n))| > \varepsilon$ for each n. By passing to a subsequence and reindexing, we may assume that there is a $B \in \Gamma$ such that $B(n)^* \to B^*$ weak* and that $Qx^*(B(n)) > \varepsilon$ for all n.

Let $n_1 = 1$. Pick $\varphi(1) \in B_1 \setminus B$ with $x^*(e_{\varphi(1)}) > \varepsilon$. Pick $\psi(1) \in B$, $|\psi(1)| \geqslant |\varphi(1)|$. Then since $B(n)^* \to B^*$ weak*, there exists an infinite subset N_1 of N such that $B(n)$ passes through $\psi(1)$ for all $n \in N_1$. Now pick $n_2 \in N_1$ and $\varphi(2) \in B(n_2) \setminus B$ such that $x^*(e_{\varphi(2)}) > \varepsilon$. Continuing this process inductively in the obvious fashion, we obtain a subsequence $N' = \{n_1, n_2, \ldots\}$ of N and a sequence $\{\varphi(j)\colon j \in N\}$ of nodes such that $\varphi(j) \in B(n_j)$ and $x^*(e_{\varphi(j)}) > \varepsilon$ for each j, and such that $\{\varphi(j)\colon j \in N\}$ is a strongly incomparable sequence. (The proof that this sequence is strongly incomparable is the same as that given in Lemma 2.) But then, $\{e_{\varphi(j)}\colon j \in N\}$ is a c_0 sequence, hence is weakly null. Thus, $x^*(e_{\varphi(j)}) \to 0$ as $j \to \infty$, which contradicts $x^*(e_{\varphi(j)}) > \varepsilon$ for all j. ■

Thus, we may regard the operator $Q\colon X^* \to c_0(\Gamma)$. For a fixed branch B, $|Qx^*(B)| = |\lim_{\varphi \in B} x^*(e_\varphi)| \leqslant \|x^*\|$, so $\|Q\| \leqslant 1$. Since $|QB^*(B)| = 1$, we conclude that $\|Q\| = 1$.

To see that Q is a quotient map, pick distinct branches $B_1, \ldots, B_r$ in Γ and scalars $t_1, \ldots, t_r$ with $\max_j |t_j| = 1$. Pick m so that, if $\varphi(i) \in B_i$, $|\varphi(i)| = m$, then $\varphi(i)$ and $\varphi(j)$ are distinct if $i \neq j$. Define $x^* = \sum_{j=1}^{r} t_j P_m^*(B_j^*)$. Then $\|x^*\| = 1$ and

$$Qx^*(B) = \begin{cases} t_j & \text{if } B = B_j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, Q is a quotient map.

Let G be the kernel of Q. It is clear that $F \subset G$. To complete the proof we must show that $F = G$. The next result is proved in the same manner as Lemma 1 of [11] combined with the idea of Lemma 7 above, so we omit the proof.

LEMMA 8. *Let $x^* \in G$. Then $\lim_n \max_{|\varphi|=n} \|P_\varphi^* x^*\| = 0$.*

Proof of (b) of Theorem 1. Pick $0 < \delta < 1/8$ (so that $2 + 4\delta < 3 - 4\delta$). Assume that $x^* \in G$, $\|x^*\| = 1$, and $\inf\{\|x^* - f\|\colon f \in F\} > 1 - \delta$. Pick $x \in X$, $\|x\| = 1$, such that $P_m(x) = 0$ (for some m) and $x^*(x) > 1 - \delta$.

Let $\varepsilon > 0$ be such that $2^m \varepsilon < \delta$. By Lemma 8, there exists an $n > m$ such that $\|P_\varphi^* x^*\| < \varepsilon$ for all φ, $|\varphi| = n$, and such that $n \geqslant 2^{m+1}$. Pick $y \in X$, $\|y\| = 1$, such that $P_n y = y$, $x^*(y) > 1 - \delta$ and $P_k y = 0$ for some $k > n$. Finally, pick $z \in X$, $\|z\| = 1$, such that $P_k(z) = z$ and $x^*(z) > 1 - \delta$.

Then $\|x+y+z\| \geqslant x^*(x+y+z) > 3(1-\delta)$. To reach a contradiction, we consider two cases.

Case 1. Assume that for any admissible segments $S_1, \ldots, S_{2m}$ passing through the support of y,

$$\sum_{j=1}^{2^m} |S_j^*(y)| \leqslant 1-4\delta.$$

Then it follows easily from the definition of the norm that $\|x+y+z\| \leqslant 3-4\delta$, and since $3-4\delta < 3-3\delta$, we have a contradiction.

Case 2. For some admissible segments $S_1, \ldots, S_{2m}$ passing through the support of y,

$$\sum_{j=1}^{2^m} |S_j^*(y)| > 1-4\delta.$$

Let $\varphi(j) = S_j \cap \{0,1\}^n$ for $j = 1, \ldots, 2^m$. Put $y(1) = \sum_{j=1}^{2^m} P_{\varphi(j)} y$ and $y(2) = y - y(1)$.

Then for any family $R_1, \ldots, R_{2m}$ of admissible segments passing through the support of y but disjoint from the segments $S_1, \ldots, S_{2m}$, we have the following:

$$\sum_{j=1}^{2^m} |R_j^*(y)| = \sum_{j=1}^{2^m} |R_j^*(y(2))| < 4\delta.$$

This is true, since by suitably truncating above and below the support of y, we can make the segments $S_1, \ldots, S_{2m}, R_1, \ldots, R_{2m}$ into an admissible family (recall that $2^{m+1} \leqslant n$). If the inequality above were false, we would then have $\|y\| > (1-4\delta)+4\delta = 1$, which is impossible.

Thus, for *any* admissible family of segments $R_1, \ldots, R_{2m}$ passing through the support of $y(2)$, we have

$$\sum_{j=1}^{2^m} \left|R_j^*\big(y(2)\big)\right| < 4\delta.$$

On the other hand, $\left|x^*\big(y(1)\big)\right| = \left|x^*\Big(\sum_{j=1}^{2^m} P_{\varphi(j)} y\Big)\right| \leqslant \sum_{j=1}^{2^m} \|P^*_{\varphi(j)} x^*\| \, \|y\| \leqslant 2^m \varepsilon < \delta$. Thus, $x^*\big(y(2)\big) = x^*(y) - x^*\big(y(1)\big) \geqslant (1-\delta)-\delta = 1-2\delta$. Therefore, $x^*\big(x+y(2)+z\big) \geqslant 3-4\delta$. But if $R_1, \ldots, R_s$ (where $s \leqslant 2^m$) is any admissible family of segments passing through the support of $y(2)$, we have that

$$\sum_{j=1}^{s} \left|R_j^*\big(x+y(2)+z\big)\right| \leqslant \|x\| + \sum_{j=1}^{s} \left|R_j^*\big(y(2)\big)\right| + \|z\| < 2+4\delta.$$

This implies that $\|x+y(2)+z\| < 2+4\delta$, which contradicts $x^*\big(x+y(2)+z\big) \geqslant 3-4\delta$. Therefore, $x^* \in F$, and the proof of (b) of Theorem 1 is complete.

Proof of (c) of Theorem 1. It clearly suffices to show (by virtue of (a)) that every bounded sequence in X has a weak Cauchy subsequence. By the Main Theorem of [18], we need only show that X has no subspace isomorphic to l^1, and to do this, it is clearly enough to prove that the cardinality of X^{**} is $\mathfrak{c}$. This is immediate from

LEMMA 9. *X^{**} is isomorphic to $F^* \oplus l^1(\Gamma)$.*

Proof. Let $Q\colon X^* \to c_0(\Gamma)$ be the quotient map defined in the proof of (b) above. We are done once we show that $Q^*\big(l^1(\Gamma)\big)$ is complemented in X^{**}.

Let $B_1, \ldots, B_n$ be distinct branches of T. Pick m so that if $i \neq j$ and $\varphi(i) \in B_i, \varphi(j) \in B_j$, and $|\varphi(i)| = m$, $|\varphi(j)| = m$, then $\varphi(i) \neq \varphi(j)$. Define $G = [\{P_m^*(B_i^*)\colon i = 1, \ldots, n\}]$. It is clear that G is isometrically isomorphic to l_n^∞ and $Q^*|_G$ is an isometry onto $[\{Q^*(B_i^*)\colon i = 1, \ldots, n\}]$. Thus, by Lemma 1 of [20], $Q^*\big(l^1(\Gamma)\big)$ is complemented in X^{**}. ∎

To finish the proof of (c), we observe that since F is separable, F^* has cardinality $\mathfrak{c}$, as does $l^1(\Gamma)$, since Γ has cardinality $\mathfrak{c}$. Thus, X^{**} has cardinality $\mathfrak{c}$.

Proof of (d) of Theorem 1. Recall that the set Γ is weak* homeomorphic to the Cantor set. So, by Theorem 2, p. 285 of [9], we can pick a subset Ω of Γ with Ω weak* homeomorphic to ω^ω. Let $Y = \{x \in X\colon x(\varphi) = 0 \text{ if } \varphi \notin \bigcup \{B\colon B \in \Omega\}\}$. Since Ω is weak* closed and countable, if follows from part (b) of Theorem 1 that Y^* is separable.

To prove that Y does not imbed in c_0, we use properties of the sets used in the definition of the Szlenk index (cf. [22]). We define these sets as follows: Let E be a Banach space, $\varepsilon > 0$. Define $A_0(\varepsilon, E) = S_{E^*} = \{f \in E^*\colon \|f\| \leqslant 1\}$. If $A_\beta(\varepsilon, E)$ has been defined for all ordinal numbers $\beta < \alpha$, we define $A_\alpha(\varepsilon, E)$ by

(i) if $\alpha = \beta+1$ for some β, then $A_\alpha(\varepsilon, E) = \{f \in E^*\colon$ there is a sequence $f(n) \in A_\beta(\varepsilon, E)$, $f(n) \to f$ weak*, and a weakly null sequence $x(n) \in E$, $\|x(n)\| \leqslant 1$, such that $\limsup_n \left|f(n)\big(x(n)\big)\right| \geqslant \varepsilon\}$;

(ii) if α is a limit ordinal, then $A_\alpha(\varepsilon, E) = \bigcap_{\beta<\alpha} A_\beta(\varepsilon, E)$.

In [22], Szlenk shows that if E^* is separable and $\alpha > \beta$, then $A_\alpha(\varepsilon, E)$ is a closed nowhere dense subset of $A_\beta(\varepsilon, E)$.

We need only the following restatement of Proposition 2.3 of [22].

LEMMA 10. *Let E and G be separable Banach spaces with E isomorphic to a subspace of G. If $A_\alpha(1, E) \neq \emptyset$ for some ordinal number α, then $A_\alpha(\varepsilon, G) \neq \emptyset$ for some $\varepsilon > 0$.*

To prove that Y does not imbed in c_0, we use the following obvious lemma, which we state without proof.

LEMMA 11. *Let $f(n), f \in l^1 = c_0^*$. Assume that $\lambda > \varepsilon > 0$, that $\|f(n)+f\| \leqslant \lambda$ for all n, that $f(n) \to 0$ weak*, and that there exists a weakly null sequence $\{x(n): n \in N\}$ in c_0, $\|x(n)\| \leqslant 1$ for all n, such that $\limsup_n |f(n)(x(n))| \geqslant \varepsilon$. Then, $\|f\| \leqslant \lambda - \varepsilon$.*

An easy application of Lemma 11 yields that for every $\varepsilon > 0$, there exists an n such that $A_n(\varepsilon, c_0) = \emptyset$. Therefore, $A_\omega(\varepsilon, c_0) = \emptyset$ for all $\varepsilon > 0$. To complete the proof of (d), we will show that $A_\omega(1, Y) \neq \emptyset$ and then apply Lemma 10.

To prove that $A_\omega(1, Y) \neq \emptyset$, we show that $\Omega^{(\alpha)} \subset A_\alpha(1, Y)$ for every ordinal α (where $\Omega^{(\alpha)}$ is the αth derived set of Ω). Since the ωth derived set of ω^ω is non-empty, this shows that $A_\omega(1, Y) \neq \emptyset$. The fact that $\Omega^{(\alpha)} \subset A_\alpha(1, Y)$ for all α follows easily from

LEMMA 12. *Let $B(n)^*, B^* \in \Omega$, $B(n)^* \to B^*$ weak*. Then there exists a c_0 sequence $x(j) \in Y$, $\|x(j)\| = 1$ for all j, and a subsequence $\{B(n_j)^*: j \in N\}$ of $\{B(n)^*: n \in N\}$ such that $B(n_j)^*(x(j)) = 1$ for all $j \in N$.*

Proof. Let $n_1 = 1$. Pick $\varphi(1) \in B_1 \setminus B$ and $\psi(1) \in B$, $|\psi(1)| \geqslant |\varphi(1)|$. Then, since $B(n)^* \to B^*$ weak*, there exists an infinite subset N_1 of N such that $B(n)$ passes through $\psi(1)$ for all $n \in N_1$. Now pick $n_2 \in N_1$ and $\varphi(2) \in B(n_2) \setminus B$. Continuing this process inductively in the obvious fashion, we obtain a subsequence $N' = \{n_1, n_2, \ldots\}$ of N and a sequence of nodes $\{\varphi(j): j \in N\}$ with $\varphi(j) \in B(n_j)$ for all j such that $\{\varphi(j): j \in N\}$ is a strongly incomparable sequence. Then, $\{e_{\varphi(j)}: j \in N\}$ is a c_0 sequence, hence converges weakly to zero. Finally, $e_{\varphi(j)} \in Y$ and $B(n_j)^*(e_{\varphi(j)}) = 1$ for all $j \in N$. ■

Remark. Using the same techniques for higher countable ordinals, we can show the following:

PROPOSITION 13. *For every countable ordinal α, there exists a subspace Y of X with Y^* separable such that Y does not imbed in $C(\alpha)$.*

Proof of (e) of Theorem 1. It suffices to show that if $f(n) \in X^*$, $f(n) \to 0$ weak* and not in norm, then a subsequence of $\{f(n): n \in N\}$ is equivalent to the usual basis of l^1. For then, if $\{g(n): n \in N\}$ converges weakly to $g \in X^*$ but not in norm, then there exist a pair of subsequences $\{m_i: i \in N\}$ and $\{n_i: i \in N\}$ with $m_1 < n_1 < m_2 < \ldots$ such that $g(n_i) - g(m_i) \to 0$ weakly and not in norm. In particular, if $f(i) = g(n_i) - g(m_i)$, then $f(i) \to 0$ weak* and not in norm. Since $\{f(i): i \in N\}$ has a subsequence equivalent to the usual basis of l^1, $\{f(i): i \in N\}$ cannot converge weakly to zero. This contradiction will complete the proof. So we prove

LEMMA 14. *Let $\{f(n): n \in N\}$ be a sequence in X^*, $\|f(n)\| = 1$ for all n. If $f(n) \to 0$ weak*, then $\{f(n): n \in N\}$ has a subsequence equivalent to the usual basis of l^1.*

Proof. By Theorem III. 1 of [8], $\{f(n): n \in N\}$ has a subsequence (which we do not reindex) which is a weak* basic sequence. In particular, it follows from the proof of this result that there exists a sequence $\{x(n): n \in N\}$ in X, $\|x(n)\| \leqslant 2$ for all n, such that $f(n)(x(m)) = \delta_{nm}$.

Since l^1 does not imbed in X, $\{x(n): n \in N\}$ has a subsequence (which we again do not reindex) which is weak Cauchy. It follows that $z(n) = x(2n) - x(2n+1) \to 0$ weakly and $\|z(n)\| \geqslant f(2n)(z(n)) = 1$. Thus, by part (a) of Theorem 1, $\{z(n): n \in N\}$ has a c_0 sequence (which we continue to call $\{z(n): n \in N\}$). Let K be a constant such that $\left\|\sum_{i=1}^{s} \delta_i z(i)\right\| \leqslant K$ whenever $|\delta_i| \leqslant 1$ for all s, $i = 1, \ldots, s$. The following easy computation shows that $\{f(2n): n \in N\}$ is equivalent to the usual basis of l^1. Let s and scalars $t_1, \ldots, t_s$ be given. Let $\delta_i = 1$ if $t_i \geqslant 0$ and -1 if $t_i < 0$. Then, $K^{-1}\left\|\sum_{i=1}^{s} \delta_i z(i)\right\| \leqslant 1$, and so

$$\left\|\sum_{j=1}^{s} t_j f(2j)\right\| \geqslant K^{-1} \sum_{j=1}^{s} t_j f(2j)\Big(\sum_{i=1}^{s} \delta_i z(i)\Big) \geqslant K^{-1} \sum_{j=1}^{s} |t_j|,$$

which completes the proof of (e).

Proof of (f) of Theorem 1. We first show that if $\{B(n)^*: n \in N\}$ is a sequence in Γ, then there is a subsequence N' of N such that $\{B(n)^*: n \in N'\}$ is equivalent to the usual basis of l^1. First, pick a subsequence N_1 of N and a $B^* \in \Gamma$ such that $\{B(n)^*: n \in N_1\}$ converges weak* to B^*. Then, as in the proof of Lemma 12, there exists a subsequence N' of N_1 and a sequence of nodes $\{\varphi(n): n \in N'\}$ such that $\varphi(n) \in B(n) \setminus \bigcup \{B(m): m \in N, m \neq n\}$ and such that $\{\varphi(n): n \in N'\}$ is a strongly incomparable sequence. But then, $\{e_{\varphi(n)}: n \in N'\}$ is equivalent to the usual basis of c_0 and $B(n)^*(e_{\varphi(n)}) = \delta_{nm}$, so the same computation as in the proof of Lemma 14 shows that $\{B(n)^*: n \in N'\}$ is equivalent to the usual basis of l^1.

Thus, no sequence chosen out of the set Γ is a weak Cauchy sequence. On the other hand, it follows from Theorem 1 of [3] (cf. also [13]) that $l^1(\Lambda)$ does not imbed in X^* if Λ is uncountable. In particular, for no uncountable subset Λ of Γ is Λ equivalent to the usual basis of $l^1(\Lambda)$. ■

At this point, let us indicate how to translate (f) of Theorem 1 into a statement concerning families of pairs of sets. For a branch B of T, let

$$C_B = \{x^{**} \in X^{**}: \|x^{**}\| \leqslant 1 \text{ and } x^{**}(B^*) \geqslant 1/2\};$$

and,

$$D_B = \{x^{**} \in X^{**}: \|x^{**}\| \leqslant 1 \text{ and } x^{**}(B^*) \leqslant -1/2\}.$$

Since no subspace of X^* is isomorphic to $l^1(\Lambda)$ for any uncountable set Λ, and since X^* is isometric to a subspace of $C(S_{X^{**}})$, for no uncountable $\Lambda \subset \Gamma$ is $\{(C_B, D_B): B \in \Lambda\}$ an independent family of pairs of sets. (Of course, $S_{X^{**}} = \{x^{**}: \|x^{**}\| \leqslant 1\}$. A family of pairs of sets $\{(U_\alpha, V_\alpha): \alpha \in \Lambda\}$ is independent if, given distinct $\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_m \in \Lambda$, then

$\left(\bigcap_{i=1}^{n} U_{a_i}\right)\cap\left(\bigcap_{i=n+1}^{m} V_{a_i}\right) \neq \emptyset$.) So to show that the uncountable version of Theorem 3 of [18] is false, we need only show that if $\{B(n)\colon n\in N\}$ is a sequence of distinct branches, then there exists an x^{**} such that $x^{**} \in C_{B(n)}$ for infinitely many $n \in N$ and $x^{**} \in D_{B(n)}$ for infinitely many $n \in N$. (Thus, in the language of [18], no sequence chosen from the set $\{(C_B, D_B)\colon B \in \Gamma\}$ is convergent.)

By passing to a first subsequence, we may assume that $B(n)^* \to B^*$ weak* for some $B^* \in \Gamma$. As in the proof of Lemma 12, we can select a strongly incomparable sequence $\{\varphi(j)\colon j \in N\}$ and a subsequence $n_1 < n_2 < \ldots$ of N such that $B(n_j)^*(e_{\varphi(k)}) = \delta_{kj}$. For $m = 1, 2, \ldots$ define $x(m) = 2^{-1}\sum_{i=1}^{m}(-1)^i e_{\varphi(i)}$. Then $\|x(m)\| \leqslant 1$ and if $m \geqslant j$, we have that $B(n_j)^*\big(x(m)\big) = (-1)^j/2$.

Let x^{**} be a weak* accumulation point in X^{**} of the sequence $\{x(m)\colon m \in N\}$. Then $\|x^{**}\| \leqslant 1$ and $x^{**}\big(B(n_j)^*\big) = (-1)^j/2$, so $x^{**} \in C_{B(n_j)}$ if j is even and $x^{**} \in D_{B(n_j)}$ if j is odd.

4. Remarks and open problems. We begin by mentioning other possible norms related to the norm on the space X. For a finitely non-zero function x on the tree T, we define

$$\|x\|_2 = \max\Big(\sum_{i=1}^{n}\big(S_i^*(x)\big)^2\Big)^{1/2},$$

where the max is taken over all families of admissible segments $S_1, \ldots, S_n$; and

$$\|x\|_3 = \max\sum_{i=1}^{n}|S_i^*(x)|,$$

where the max is taken over all families of segments which are pairwise disjoint and *begin* at the same level. Let X_i be the completion of the finitely non-zero functions in $\|\cdot\|_i$ for $i = 2, 3$. H. Rosenthal has shown (using a similar but somewhat more sophisticated analysis than that in Proposition 5) that both spaces X_2 and X_3 are hereditarily c_0.

We conclude by stating some open problems related to or suggested by our study.

PROBLEM 1. *Does X have property* u?

PROBLEM 2. *Let Z be a separable Banach space having property* u *such that l^1 does not imbed in Z. Is Z^* separable?*

It is clear that an affirmative answer to the first problem gives a negative answer to the second.

Now, it follows from results in [19] that the unit ball of X is weak* sequentially dense in the unit ball of X^{**}. So, according to the definition of property u given in [14], an affirmative answer to Problem 1 is equivalent to one for this problem:

If $x^{**} \in X^{**}$, is there a weakly unconditionally converging (w.u.c.) *series $\sum_{n=1}^{\infty} x(n)$ in X such that $\big\{\sum_{n=1}^{m} x(n)\colon m \in N\big\}$ converges weak* to x^{**}?*

(A series $\sum_{n=1}^{\infty} x(n)$ is a w.u.c. if for every $x^* \in X^*$, $\sum_{n=1}^{\infty}\big|x^*\big(x(n)\big)\big| < \infty$.)

Recall the decomposition (in Lemma 9) of X^{**} as $X^{**} = Z \oplus Q^*\big(l^1(\Gamma)\big)$, where Z is isomorphic to F^*. It is not difficult to show that every $x^* \in Q^*\big(l^1(\Gamma)\big)$ is the limit of a w.u.c. So the functionals which remain to be checked are those $x^* \in Z$.

PROBLEM 3. *Let Z be a separable Banach space such that l^1 does not imbed in Z. Does there exist a separable subspace F of Z^* such that Z^*/F is weakly compactly generated?*

If Z^* is separable, the answer is trivially yes. Also, the answer to this problem is yes for the spaces JT, X, and X_2. We do not know the answer for the space JF.

Added in proof: In the article *On Banach spaces which contain $l^1(\tau)$ and types of measures on compact spaces* (to appear), R. Haydon has shown the following: Let $\mathfrak{m}$ be a cardinal number such that $\mathfrak{n} < \mathfrak{m}$ implies $\mathfrak{n}^{\aleph_0} < \mathfrak{m}$. Then if $L^1\{0,1\}^{\mathfrak{m}}$ imbeds in X^*, $l^1_{\mathfrak{m}}$ imbeds in X.

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