[3] Ч. Фояш, *О некоторых полугруппах сжатий, связанных с представлением конволюционных алгебр, I*, Rev. Math. Pures et Appl. 7 (1962), pp. 319–325.
[4] B. Sz-Nagy, C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, Amsterdam–Budapest 1970.
[5] —, — *Modèle de Jordan pour une classe d'opérateurs de l'espace de Hilbert*, Acta Sci. Math. 31 (1970), pp. 91–115.
[6] W. Sierpiński, *Cardinal and Ordinal Numbers*, Warszawa 1958.

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A property of determining sets for analytic functions

by

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Abstract. Any locally determining set at $0 \in \boldsymbol{K}^n$ ($\boldsymbol{K} = \boldsymbol{R}$ or $\boldsymbol{K} = \boldsymbol{C}$) for analytic functions (1) contains a sequence convergent to $0 \in \boldsymbol{K}^n$, which is itself locally determining at $0 \in \boldsymbol{K}^n$.

1. In this note we prove the following theorem.

THEOREM. *Let $E \subset \boldsymbol{C}^n$ be a locally determining set at $0 \in \boldsymbol{C}^n$ for holomorphic functions. Then there is a sequence $\{a_\nu\} \subset E$ convergent to $0 \in \boldsymbol{C}^n$ which is a locally determining set at $0 \in \boldsymbol{C}^n$ for holomorphic functions.*

This theorem is an answer to Question 2 posed in [2]. Its proof is based on a lemma concerning locally complete sets in separable Hilbert spaces. The lemma seems to be interesting by itself.

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2. Let H be a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. A subset $A \subset H$ is called *complete* if and only if the equations $\langle x, a\rangle = 0$ for each $a \in A$ imply $x = 0$.

LEMMA. *Assume that A is a subset of a separable Hilbert space such that for every constant $0 < r < 1$ the set $A_r := \{x \in A: \|x\| < r\}$ is complete. Then there is a sequence $\{a_\nu\} \subset A$ convergent to $0 \in H$ which is complete.*

Proof. Let $\{e_i\}$ be an orthonormal base in H and let r_1 be a positive number. Then the set A_{r_1} is complete. Hence the closure of the linear subspace spanned by A_{r_1} is equal to H. Thus there exist scalars $\beta_1^{(1)}, \ldots, \beta_{s_1}^{(1)}$ and vectors $a_1^{(1)}, \ldots, a_{s_1}^{(1)} \in A_{r_1} - \{0\}$ such that

$$\|\beta_1^{(1)} a_1^{(1)} + \ldots + \beta_{s_1}^{(1)} a_{s_1}^{(1)} - e_1\| < 2^{-1}.$$

Put $r_2 = \min\{\|a_1^{(1)}\|, \ldots, \|a_{s_1}^{(1)}\|, 2^{-2}\}$. Then $a_k^{(1)} \notin A_{r_2}$ for $k = 1, \ldots, s_1$. As before, we can choose scalars $\beta_1^{(2)}, \ldots, \beta_{s_2}^{(2)}$ and vectors $a_1^{(2)}, \ldots, a_{s_2}^{(2)}$

(1) A subset E of $\boldsymbol{K}^n$ ($n \geqslant 1$) is called a *locally determining set at* $0 \in \boldsymbol{K}^n$ for analytic functions, if for each connected neighbourhood U of $0 \in \boldsymbol{K}^n$ the subset $E \cap U$ is determining for the analytic functions in U, i.e. if a function f is analytic in U and vanishing on $E \cap U$, the function f is identically zero ([1]).

$\in A_{r_2} - \{0\}$ such that

$$\|\beta_1^{(2)} a_1^{(2)} + \ldots + \beta_{s_2}^{(2)} a_{s_2}^{(2)} - e_2\| < 2^{-2}.$$

By recurrence we choose a sequence of positive numbers r_k such that

(1) $$r_k \leqslant 2^{-k} \quad \text{for} \quad k = 1, 2, \ldots,$$

and scalars $\beta_1^{(k)}, \ldots, \beta_{s_k}^{(k)}$ and vectors $a_1^{(k)}, \ldots, a_{s_k}^{(k)} \in A_{r_k} - \{0\}$ with the property

(2) $$\|\beta_1^{(k)} a_1^{(k)} + \ldots + \beta_{s_k}^{(k)} a_{s_k}^{(k)} - e_k\| < 2^{-k} \quad \text{for} \quad k \geqslant 1.$$

Consider the sequence $a_1^{(1)}, \ldots, a_{s_1}^{(1)}, a_1^{(2)}, \ldots, a_{s_2}^{(2)}, a_1^{(3)}, \ldots$, which we shortly denote by $a_1, a_2, \ldots$ Since $a_j^{(k)} \in A_{r_k}$ for $j = 1, \ldots, s_k$ and $r_k \to 0$, we have $\lim_{j\to\infty} a_j = 0$. We claim that the sequence $\{a_j\}$ is complete. Indeed, put $u_k = \sum_{j=1}^{s_k} \beta_j^{(k)} a_j^{(k)}$ and let $b \in H$, $\|b\| = 1$, be a vector such that $\langle b, u_k\rangle = 0$ for $k \geqslant 1$. Then $b = \sum_{k=1}^{\infty} \alpha_k e_k$, where $\alpha_k = \langle b, e_k - u_k\rangle$. From the Parseval equation, the Cauchy inequality and from (2) we obtain

$$1 = \|b\|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2 \leqslant \sum_{k=1}^{\infty} \|b\|^2 \|e_k - u_k\|^2 < \sum_{k=1}^{\infty} 4^{-k} = 3^{-1},$$

a contradiction which completes the proof.

3. Put $I := \{\alpha \in \mathbf{Z}_+^n : \alpha_1 + \ldots + \alpha_n > 0\}$, $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, $\zeta = (\zeta_1, \ldots, \zeta_n)$ for $z, \zeta \in \mathbf{C}^n$.

The vector space of complex multisequences

$$\mathscr{H}_2 := \Big\{(c_\alpha)_{\alpha \in I} : \sum_{\alpha \in I} |c_\alpha|^2 < \infty\Big\}$$

is a separable Hilbert space with the scalar product $\langle f, g\rangle := \sum_{\alpha \in I} f_\alpha \bar{g}_\alpha$, where $f = (f_\alpha)$, $g = (g_\alpha) \in \mathscr{H}_2$. If $f = (f_\alpha) \in \mathscr{H}_2$, then the function $f(z) := \sum_{\alpha \in I} f_\alpha z^\alpha$ is holomorphic in the open polydisc $P_1 = \{z \in \mathbf{C}^n : |z| < 1\}$ ($|z| := \max_{1\leqslant j\leqslant n} |z_j|$). If $f \in \mathscr{H}_2$ and $g_\zeta := (\bar{\zeta}^\alpha)_{\alpha \in I}$, where $\zeta \in P_1$, then $\langle f, g_\zeta\rangle = f(\zeta)$. If a function $f(z) := \sum_{\alpha \in I} f_\alpha z^\alpha$ is holomorphic in a neighbourhood of the closed polydisc $\bar{P}_1$, then $f = (f_\alpha) \in \mathscr{H}_2$.

4. Remark 1. If $E \subset \mathbf{C}^n$ is a locally determining set at $0 \in \mathbf{C}^n$ for holomorphic functions, then the set

$$\frac{1}{r} E_\varrho := \Big\{\frac{1}{r} z : z \in E, |z| < \varrho\Big\},$$

where $0 < r \leqslant \varrho < 1$, is determining for functions holomorphic in P_1.

Indeed, if $f \in O(P_1)$ and $f(z) = 0$ for $z \in \frac{1}{r} E_\varrho$, then $f\left(\frac{1}{r}\zeta\right) = 0$ for $\zeta \in E_\varrho$. Since the function $\zeta \to f\left(\frac{1}{r}\zeta\right)$ is holomorphic in a neighbourhood of 0, we have $f\left(\frac{1}{r} z\right) = 0$. Thus $f = 0$.

Remark 2. If $E \subset \mathbf{C}^n$ is a locally determining set at $0 \in \mathbf{C}^n$ for holomorphic functions, then the set

$$\mathscr{E}_{r\varrho} := \Big\{g_\zeta : \zeta \in \frac{1}{r} E_r, \|g_\zeta\| < \varrho\Big\}$$

where $0 < r < 1$, $0 < \varrho < 1$ and $g_\zeta := (\bar{\zeta}^\alpha)_{\alpha \in I}$, is complete in $\mathscr{H}_2$.

Indeed, the equations

$$\|g_\zeta\|^2 = \sum_{\alpha \in I} \bar{\zeta}^\alpha \zeta^\alpha = \frac{1}{1-|\zeta_1|^2} \ldots \frac{1}{1-|\zeta_n|^2} - 1$$

imply that $g_\zeta \to 0$ in $\mathscr{H}_2$ if and only if $\zeta \to 0 \in \mathbf{C}^n$. So, if $f \in \mathscr{H}_2$ is such that $f(\zeta) = \langle f, g_\zeta\rangle = 0$ for every $g_\zeta \in \mathscr{E}_{r,\varrho}$, then there is a constant $\varrho' \in (0, r]$ such that $f(\zeta) = 0$ for every $\zeta \in \frac{1}{r} E_{\varrho'}$. Thus, according to Remark 1, we have $f = 0$ (because $f(z) = \sum_{\alpha \in I} c_\alpha z^\alpha$ is holomorphic in P_1).

Lemma and Remark 2 yield immediately the following

Remark 3. If E is a locally determining set at $0 \in \mathbf{C}^n$ for holomorphic functions, then for every constant $0 < r < 1$ there is a sequence $\{\zeta^\nu\} \subset \frac{1}{r} E_r$ convergent to $0 \in \mathbf{C}^n$ with the property

$$f \in \mathscr{H}_2 \quad \text{and} \quad f(\zeta^\nu) = \langle f, g_{\zeta^\nu}\rangle = 0 \quad \text{for } \nu \geqslant 1 \Rightarrow f = 0.$$

5. Proof of Theorem. Let $r_k = \frac{1}{k}$ and let $\{\zeta^{k,\nu}\}_{\nu\geqslant 1} \subset E_{r_k}$ be a sequence such that, if $f \in \mathscr{H}_2$ and $f\left(\frac{1}{r_k}\zeta^{k,\nu}\right) = 0$ for $\nu \geqslant 1$, then $f = 0$. Obviously, the origin is the only accumulation point of the countable set

$$D := \{\zeta^{k,\nu} : k, \nu \geqslant 1\} \subset E.$$

We claim that D is a locally determining set at $0 \in \mathbf{C}^n$ for holomorphic functions. Indeed, if a function f is holomorphic in a connected neighbourhood U of $0 \in \mathbf{C}^n$ such that $f|_{D\cap U} = 0$, then there is a positive integer k such that the function $\varphi(z) := f(r_k z)$ is holomorphic in a neighbourhood of $\bar{P}_1$, so $\varphi \in \mathscr{H}_2$.

Since $\varphi\left(\frac{1}{r_k}\zeta^{k,\nu}\right) = f(\zeta^{k,\nu}) = 0$ for $\nu \geqslant 1$, we have $\varphi = 0$, and consequently $f = 0$.

Remark 4. Our Theorem remains true for real analytic functions. The above proof may be repeated without any change.

References

[1] J. Chmielowski, *Ensembles déterminants pour les fonctions analytiques*, C. R. Acad. Sci. Paris, Sér. A 279 (1974), pp. 639–641.

[2] — *Constructions des ensembles déterminants pour les fonctions analytiques*, Studia Math. 57(1976), pp. 141–146.

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A counterexample to several questions about Banach spaces

by

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Abstract. There exists a separable Banach space X with nonseparable dual such that l^1 does not imbed in X and such that every normalized weakly null sequence in X has a subsequence equivalent to the usual basis of c_0. Weak sequential convergence and norm convergence in X^* coincide. Other properties of X and X^* are investigated.

1. Introduction. In this paper we construct a Banach space X which provides answers to many open questions about the isomorphic structure of Banach spaces. Our main result is

THEOREM 1. *There exists a separable Banach space X satisfying the following:*

(a) *Every sequence in X which converges weakly but not in norm to zero has a c_0 subsequence.*

(b) *There exists a separable subspace F of X^* such that X^*/F is isometrically isomorphic to $c_0(\Gamma)$, where Γ has cardinality* $\mathfrak{c}$.

(c) *X is hereditarily c_0.*

(d) *There exists a subspace Y of X with Y^* separable such that Y does not imbed in c_0.*

(e) *X^* has the Schur property; i.e., weak sequential convergence and norm convergence in X^* coincide. In particular, X^* is hereditarily l^1.*

(f) *There exists a bounded set Γ in X^* of cardinality $\mathfrak{c}$ such that no sequence in Γ is a weak Cauchy sequence. Yet, no subspace of the closed linear span of Γ is isomorphic to $l^1(\Lambda)$ for any uncountable set Λ.*

Of course, (b) and (c) of Theorem 1 show that X is another example of a separable space with X^* nonseparable such that l^1 does not imbed in X. The first of these examples was given by James in [7]. Later, Lindenstrauss and Stegall [11] gave a second example of such a space (which they called JF) and studied the duality properties of the above mentioned example of James (which they called JT). The space JT is hereditarily l^2, while the space JF has many subspaces isomorphic to l^2 and many isomorphic to c_0. Both the spaces JT and JF are closely related to the non-reflexive space J isometric to J^{**} introduced by James in [6]. In the space X of Theorem 1, c_0 plays the role that J does in JT and JF. (The influence of the papers [7] and [11] on this paper is considerable.