A property of determining sets for analytic functions

by

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Abstract. Any locally determining set at $0 \in K^{n}(K = \mathbb{R} \text{ or } K = \mathbb{C})$ for analytic functions (1) contains a sequence convergent to $0 \in K^{n}$, which is itself locally determining at $0 \in K^{n}$.

1. In this note we prove the following theorem.

Theorem. Let $E \subset C^{n}$ be a locally determining set at $0 \in C^{n}$ for holomorphic functions. Then there is a sequence $(\{a_{k}\})_{k} \subset E$ convergent to $0 \in C^{n}$ which is a locally determining set at $0 \in C^{n}$ for holomorphic functions.

This theorem is an answer to Question 2 posed in [2]. Its proof is based on a lemma concerning locally complete sets in separable Hilbert spaces. The lemma seems to be interesting by itself.

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2. Let $H$ be a separable Hilbert space with scalar product $\langle \cdot , \cdot \rangle$ and the norm $\| \cdot \| = \langle \cdot , \cdot \rangle^{1/2}$. A subset $A \subset H$ is called complete if and only if the equations $(x, a) = 0$ for each $a \in A$ imply $x = 0$.

Lemma. Assume that $A$ is a subset of a separable Hilbert space such that for every constant $0 < r < 1$ the set $A_{r} := \{x \in A : \|x\| < r\}$ is complete. Then there is a sequence $(a_{k}) \subset A$ convergent to $0 \in H$ which is complete.

Proof. Let $(e_{1})$ be an orthonormal base in $H$ and let $r_{1}$ be a positive number. Then the set $A_{r_{1}}$ is complete. Hence the closure of the linear subspace spanned by $A_{r_{1}}$ is equal to $H$. Thus there exist scalars $\beta_{1}^{(1)}, \ldots, \beta_{m}^{(1)}$ and vectors $a_{1}^{(1)}, \ldots, a_{m}^{(1)} \in A_{r_{1}} - \{0\}$ such that

$$\|\beta_{1} a_{1} + \cdots + \beta_{m} a_{m} - e_{1}\| < 2^{-1}.$$ 

Put $r_{k} = \min \{\|a_{1}^{(k)}\|, \ldots, \|a_{m}^{(k)}\|, 2^{-k}\}$. Then $a_{1}^{(k)} \notin A_{r_{k}}$ for $k = 1, \ldots, s_{1}$. As before, we can choose scalars $\beta_{1}^{(k)}, \ldots, \beta_{m}^{(k)}$ and vectors $a_{1}^{(k)}, \ldots, a_{m}^{(k)}$.

(1) A subset $E$ of $K^{n}$ ($n \geq 1$) is called a locally determining set at $0 \in K^{n}$ for analytic functions, if for each connected neighbourhood $U$ of $0 \in K^{n}$ the subset $E \cap U$ is determining for the analytic functions in $U$, i.e. if a function $f$ is analytic in $U$ and vanishing on $E \cap U$, the function $f$ is identically zero ([1]).
Indeed, if \( f \in O(P_1) \) and \( f(z) = 0 \) for \( z \in \frac{1}{r} E_r \), then \( f \left( \frac{1}{r} z \right) = 0 \) for \( z \in E_r \). Since the function \( z \rightarrow f \left( \frac{1}{r} z \right) \) is holomorphic in a neighbourhood of \( 0 \), we have \( f \left( \frac{1}{r} z \right) = 0 \). Thus \( f = 0 \).

Remark 2. If \( E \subset C^\infty \) is a locally determining set at \( 0 \in C^\infty \) for holomorphic functions, then the set

\[
E_r := \left\{ g_r : z \in \frac{1}{r} E_r, \|g_r\| < \varepsilon \right\}
\]

where \( 0 < r < 1 \), \( 0 < \varepsilon < 1 \) and \( g_r := \left( g_r^k \right)^{k=1} \), is complete in \( \mathcal{H} \).

Indeed, the equations

\[
\|g_r\|^2 = \sum_{k=1}^{m} |g_r^k|^2 = \frac{1}{1-|z|^2} \frac{1}{1-|z|^2} \frac{1}{1-|z|^2} \frac{1}{1-|z|^2} = \frac{1}{|1-|z|^2|} - 1
\]

imply that \( g_r \to 0 \) in \( \mathcal{H} \) if and only if \( z \to 0 \) in \( C^\infty \). So, if \( f \in \mathcal{H} \) is such that \( f(z) = \langle f, g_r \rangle = 0 \) for every \( g_r \in E_r \), then there is a constant \( C \in (0, r) \) such that \( f(z) = 0 \) for every \( z \in \frac{1}{r} E_r \). Thus, according to Remark 1, we have \( f = 0 \) (because \( f(z) = \sum_{k=1}^{m} c_k z^k \) is holomorphic in \( P_1 \)).

Lemma and Remark 2 yield immediately the following

Remark 3. If \( E \) is a locally determining set at \( 0 \in C^\infty \) for holomorphic functions, then for every constant \( 0 < r < 1 \) there is a sequence \( \{z^n\} \subset \frac{1}{r} E_r \) convergent to \( 0 \in C^\infty \) with the property

\[
f \in \mathcal{H} \quad \text{and} \quad f(z) = \langle f, g_r \rangle = 0 \quad \text{for} \quad r \geq 1 \Rightarrow f = 0.
\]

5. Proof of Theorem. Let \( r_n = \frac{1}{k} \) and let \( \{\overrightarrow{r}_n\}_{n=1} \subset E_{r_n} \) be a sequence such that, if \( f \in \mathcal{H} \) and \( f \left( \frac{1}{r} \overrightarrow{r}_n \right) = 0 \) for \( r \geq 1 \), then \( f = 0 \).

Obviously, the origin is the only accumulation point of the countable set

\[
D := \left\{ \overrightarrow{r}_n^r : k, r \geq 1 \right\} \subset E_r
\]

We claim that \( D \) is a locally determining set at \( 0 \in C^\infty \) for holomorphic functions. Indeed, if a function \( f \) is holomorphic in a connected neighbourhood \( U \) of \( 0 \in C^\infty \) such that \( f \big|_{E_r} = 0 \), then there is a positive integer \( k \) such that the function \( f(c) = f(r_k c) \) is holomorphic in a neighbourhood of \( P_1 \), so \( f \in \mathcal{H} \).
Since \( \varphi \left( \frac{1}{v_b} e^{ib} \right) = f(e^{ib}) = 0 \) for \( \nu \geq 1 \), we have \( \varphi = 0 \), and consequently \( f = 0 \).

Remark 4. Our Theorem remains true for real analytic functions. The above proof may be repeated without any change.

References


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A countereexample to several questions about Banach spaces

by

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Abstract. There exists a separable Banach space \( X \) with nonseparable dual such that \( \mathcal{P} \) does not imbed in \( X \) and such that every normalised weakly null sequence in \( X \) has a subsequence equivalent to the usual basis of \( c_0 \). Weak sequential convergence and norm convergence in \( X^* \) coincide. Other properties of \( X \) and \( X^* \) are investigated.

1. Introduction. In this paper we construct a Banach space \( X \) which provides answers to many open questions about the isomorphic structure of Banach spaces. Our main result is

THEOREM 1. There exists a separable Banach space \( X \) satisfying the following:

(a) Every sequence in \( X \) which converges weakly but not in norm to zero has a \( c_0 \) subsequence.

(b) There exists a separable subspace \( F \) of \( X^* \) such that \( X^*/F \) is isometrically isomorphic to \( c_0(\Gamma) \), where \( \Gamma \) has cardinality \( \mathfrak{c} \).

(c) \( X \) is hereditarily \( c_0 \).

(d) There exists a subspace \( Y \) of \( X \) with \( X^* \) separable such that \( Y \) does not imbed in \( c_0 \).

(e) \( X^* \) has the Schur property; i.e., weak sequential convergence and norm convergence in \( X^* \) coincide. In particular, \( X^* \) is hereditarily \( F \).

(f) There exists a bounded set \( \Gamma \) in \( X^* \) of cardinality \( \mathfrak{c} \) such that no sequence in \( \Gamma \) is a weak Cauchy sequence. Yet, no subspace of the closed linear span of \( \Gamma \) is isomorphic to \( l^1(\mathcal{A}) \) for any uncountable set \( \mathcal{A} \).

Of course, (b) and (e) of Theorem 1 show that \( X \) is another example of a separable space with \( X^* \) nonseparable such that \( \mathcal{P} \) does not imbed in \( X \). The first of these examples was given by James in [7]. Later, Lindenstrauss and Stegall [11] gave a second example of such a space (which they called \( JP \)) and studied the duality properties of the above mentioned example of James (which they called \( JT \)). The space \( JT \) is hereditarily \( F \), while the space \( JP \) has many subspaces isomorphic to \( \mathcal{P} \) and many isomorphic to \( c_0 \). Both the spaces \( JT \) and \( JP \) are closely related to the non-reflexive space \( J^* \) isometric to \( J^* \) introduced by James in [6]. In the space \( X \) of Theorem 1, \( c_0 \) plays the role that \( J \) does in \( JT \) and \( JP \). (The influence of the papers [7] and [11] on this paper is considerable.