either $\alpha = 0$ or $\alpha = 1$. Thus either $P_A(f) = f$ or $P_A(f) = \theta$. Let $\pi_n = \{A_i\}_{i=1}^{d}$. Then $I = \sum_{i=1}^{d} P_{A_i}$. Therefore $P_{A_i}(f) = f$ for some i. But then $N(f) = N\big(P_{A_i}(f)\big) \leqslant \varepsilon_n$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, $f = \theta$.

Finally we let $K = [F, -F]$. Therefore $K = \{\alpha f-(1-\alpha)g\colon f, g \in F\}$, since F is a compact convex set. Now θ is not an extreme point of K, since $F \subset K$ and if $f \in F$ with $f \neq \theta$, then $\frac{1}{2}f+\frac{1}{2}(-f) = \theta$. Suppose g is an extreme point in K. Then $g = \alpha f_1+(1-\alpha)(-f_2)$ for $f_1, f_2 \in F$. But then either $\alpha = 0$ or $\alpha = 1$. Thus $g \in F$ or $g \in -F$. Without loss of generality, we may suppose $g \in F$. Since $g \neq \theta$, g is not an extreme point of F. Thus K has no extreme points. This completes the proof.

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On the Jordan model of C_0 operators

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Abstract. The aim of Part I of this note is to find a Jordan quasisimilarity model for C_0 operators acting on Hilbert spaces of arbitrary dimensions. In § 1 the "non-separable" quasisimilarity invariants of a C_0 contraction are found. The main result of Part I is Theorem 1 (§ 2) which asserts the existence and the uniqueness of the Jordan model. In our proof we use the existence of the Jordan model, already known for C_0 operators acting on separable Hilbert spaces ([1]).

The second part of this note is a continuation of [3]. We apply the Jordan model of C_0 operators to the problem of classifying the representations of the convolution algebra $L^1(0, 1)$. In § 3 the canonical representations of $L^1(0,1)$ are defined and they are shown to be unitarily equivalent to some "obvious" representations. The main result of Part II is Theorem 2 (§ 4) where each representation of $L^1(0,1)$ into a Hilbert space is asserted to be quasisimilar to a unique representation which is the direct (orthogonal) sum of a canonical representation and a trivial representation.

In [2] the C_0 operators acting on a separable Hilbert space were shown to be quasisimilar to Jordan operators. The main result of Part I of this note is to extend this result to arbitrary C_0 operators.

The general result of Part I has been suggested by the particular case used in Part II. In the particular case of nilpotent operators, the problem of the Jordan model is already solved in [1]. The problem of classifying the representations of the convolution algebra $L^1(0,1)$ has been suggested by C. Foiaş.

Preliminaries. (a) Let us recall that C_0 is the class of those completely nonunitary (cnu) contractions T of a Hilbert space, for which there exists a function $u \in H^\infty$, $u \neq 0$, such that $u(T) = 0$. Among the functions u satisfying the relation $u(T) = 0$ there is an inner one which divides all the others. This function, determined up to a scalar multiplicative constant of modulus one, is called the *minimal function* of T and is denoted by m_T. The function m_T is constant if and only if T acts on the trivial space $\{0\}$.

For each nonconstant inner function m there exists an operator T of class C_0 for which $m_T = m$. Such an operator is $S(m)$ acting on $H(m)$

$= H^2 \ominus mH^2$, $S(m)u = P_{H(m)}(zu(z))$ or, equivalently,

$$S(m)^* u = \tfrac{1}{z}(u(z) - u(0)) \quad \text{for} \quad u \in H(m).$$

(b) For a cardinal number $\aleph$ we define $S(m)^{\aleph}$ as

(1) the operator 0 on the trivial space $\{0\}$, if $\aleph = 0$;

(2) the direct sum of $\aleph$ copies of $S(m)$, if $\aleph \neq 0$.

(c) We consider the set of the inner functions in H^∞ ordered by divisibility. We put $m_1 \leqslant m_2$ if m_1 is a divisor of m_2, and $m_1 < m_2$ if $m_1 \leqslant m_2$ and $m_1 \neq m_2$. If $m_1 < m_2$, we denote by (m_1, m_2) the set of all inner functions m such that $m_1 < m < m_2$; similarly, $[m_1, m_2) = (m_1, m_2) \cup \{m_1\}$, $(m_1, m_2] = (m_1, m_2) \cup \{m_2\}$, $[m_1, m_2] = (m_1, m_2] \cup \{m_1\}$.

If m_a is an inner function for each $a \in A$, we denote by $\bigwedge_{a \in A} m_a$ the greatest common divisor of the functions m_a and by $\bigvee_{a \in A} m_a$ the least common multiple of the functions m_a.

(d) For an order relation $\leqslant$, the opposite relation $\leqslant'$ is defined by $x \leqslant' y$ if and only if $y \leqslant x$. An ordered set $(M, \leqslant)$ is well ordered if each nonvoid subset of M contains the least element. We say that $(M, \leqslant)$ is a well anti-ordered set if $(M, \leqslant')$ is a well ordered set. Each set of cardinal numbers is well ordered.

(e) We will frequently use in the sequel the following assertions:

If $\aleph_1$ and $\aleph_2$ are cardinal numbers and $\aleph_1$ is transfinite, then $\aleph_1 + \aleph_2 = \max(\aleph_1, \aleph_2)$, $\aleph_0 \cdot \aleph_1 = \aleph_1$. Here $\aleph_0$ denotes, as usual, the first transfinite cardinal number. For the facts from the cardinal number theory, see e.g. [6].

(f) Let us consider an operator of the form

$$(*) \qquad T = \bigoplus_m S(m)^{\aleph_m},$$

where for each inner function m, $\aleph_m$ is a cardinal number (possibly $\aleph_m = 0$).

DEFINITION 1. The operator $(*)$ is a *Jordan operator* if:

(a) the set $A = \{m : \aleph_m \neq 0\}$ is well anti-ordered (by divisibility);

(b) if $m_1, m_2 \in A$, $m_1 < m_2$ and $\aleph_{m_2} \geqslant \aleph_0$, then $\aleph_{m_1} > \sum_{m \geqslant m_2} \aleph_m$;

(c) the set $A_0 = \{m : m \in A, \aleph_m \leqslant \aleph_0\}$ is a decreasing (possibly finite) sequence (this means that $A_0 = \{m_j\}_{j=1}^n$, where $n \leqslant \infty$ and $m_j > m_{j+1}$ for each j).

Remark 1. If the operator T acts on a separable space, then, after a permutation of the terms in the direct sum, we have

$$T = \bigoplus_{j=1}^{n} S(m_j), \quad n \leqslant \infty,$$

and for each j, m_{j+1} divides m_j, if T is a Jordan operator.

(g) Let us recall that a linear bounded operator X from a Hilbert space H_1 into another H_2 is a *quasiaffinity*, if it has a densely defined inverse. A bounded operator T_1 acting on H_1 is a quasiaffine transform of a bounded operator T_2 acting on H_2, if there exists a quasiaffinity X from H_1 into H_2 such that $T_2 X = X T_1$. We denote this relation by $T_1 \prec T_2$ or by $T_2 \succ T_1$. The relation $\prec$ is reflexive and transitive, and $T_1 \prec T_2$ yields $T_2^* \prec T_1^*$.

Two operators are *quasisimilar* if each one is a quasiaffine transform of the other. If T_a and S_a are quasisimilar for each $a \in A$, then $\bigoplus_{a \in A} T_a$ and $\bigoplus_{a \in A} S_a$ are quasisimilar.

PROPOSITION A (Theorem 1 in [2]). *Each C_0 operator acting on a separable Hilbert space is quasisimilar to a Jordan operator. Let T_1 and T_2 be Jordan operators acting on separable spaces. If $T_1 \prec T_2$ then $T_1 = T_2$.*

(h) We recall the definition of the *Nevanlinna class* N_T. For a cnu contraction T, K_T^∞ is the class of those $u \in H^\infty$ for which $u(T)$ has a densely defined inverse. Then we can define N_T as the class of the functions of the form $f = u/v$, $u \in H^\infty$, $v \in K_T^\infty$. For $f = u/v \in N_T$, $f(T)$ is defined as

$$v(T)^{-1} u(T).$$

For the properties of the *functional calculus* thus defined see [4], Chapter IV.

PART I

1. The multiplicity functions f and g. Let T be a C_0 operator acting on the Hilbert space H. We can decompose H into a direct sum $H = \bigoplus_{a \in A} H_a$, each H_a being a separable space, reducing for T. Aplying Proposition A to $T_a = T|_{H_a}$, we see that T is quasisimilar to an operator T' of the form

$$(*) \qquad T' = \bigoplus_m S(m)^{\aleph_m},$$

where for each divisor m of the minimal function m_T, $\aleph_m$ is a cardinal number (possibly $\aleph_m = 0$).

For a divisor m of m_T we define

$$(1.1) \qquad f(m) = \sum_{m'} \aleph_{m'},$$

where the sum is extended to all functions m' such that m' non $\leqslant m$.

Since $m(S(m')) = 0$ if and only if $m' \leqslant m$, it follows that $f(m)$ coincides with the number of the nonzero terms in the direct sum $m(T') = \bigoplus_{m'} m(S(m'))^{\aleph_{m'}}$. For m' non $\leqslant m$ we have $1 \leqslant \dim \overline{\text{Range}(m(S(m')))} \leqslant \aleph_0$;

thus, if we put

$$g(m) = \dim \overline{\operatorname{Range}\big(m(T')\big)}, \tag{1.2}$$

we have the relations

$$f(m) \leqslant g(m) \leqslant \aleph_0 f(m). \tag{1.3}$$

It follows that $f(m)$ is transfinite if and only if $g(m)$ is transfinite, and in this case we have $f(m) = g(m)$.

Now let us consider two operators T, T' acting on H, H', respectively, and a quasiaffinity X from H into H' such that $XT = T'X$. If T, T' are cnu contractions and $m \in H^\infty$, then $Xm(T) = m(T')X$. Therefore $\overline{X\big(\operatorname{Range} m(T)\big)} = \overline{m(T')(XH')}$ thus $\overline{X\big(\operatorname{Range} m(T)\big)} = \overline{\operatorname{Range} m(T')}$, $\dim \overline{\operatorname{Range} m(T)} = \dim \overline{\operatorname{Range} m(T')}$.

From this remark it follows that the function g defined by (1.2) is an invariant of the C_0 operator T with respect to quasiaffine transforms. In [1] it is shown that in the particular case of nilpotent operators, the function g provides a complete system of invariants of T, with respect to quasiaffine transforms. As we shall see later, this is not true for any C_0 operator.

DEFINITION 2. A divisor m of m_T is a *saltus point* for f if there is no $m' < m$ such that f is constant on $(m', m]$. The cardinal number

$$h(m) = \min\{f(m')\colon\ m' < m\} \tag{1.4}$$

is the *saltum* of f at m.

We shall use the notation A_{00} for the set of those saltus points m for f which satisfy $h(m) > \aleph_0$.

We now state the facts we need about f and g in the following proposition.

PROPOSITION 1.

(i) *f and g are cardinal number valued decreasing functions on the set of the divisors of m_T.*

(ii) $f(m_T) = g(m_T) = 0$, $g(1) = \dim H$.

(iii) *1 is not a saltus point for f.*

(iv) *A_{00} coincides with the set of those saltus points for g at which g has a saltum $> \aleph_0$.*

(v) *A_{00} and the cardinal numbers $h(m)$, $m \in A_{00}$, are invariant with respect to quasiaffine transformations.*

(vi) *If $f(m) \leqslant \aleph$ for $m \in (m_1, m_2)$ and $m_1 = \bigwedge\limits_{m \in (m_1, m_2)} m$, then $f(m_1) \leqslant \aleph$.*

(vii) $f(m_1 \wedge m_2) \leqslant f(m_1) + f(m_2)$.

(viii) *If $m \in A_{00}$, m' non $\leqslant m$ and m non $\leqslant m'$, then*

$$h(m) \leqslant \sum_{\substack{m_1 \leqslant m,\\ m_1 \text{ non} \leqslant m'}} \aleph_{m_1} \leqslant f(m').$$

(ix) *The set A_{00} is well anti-ordered (by divisibility) and at most countable.*

(x) *If $m_0 = \max(A_{00} \cup \{1\})$, then $f(m_0) \leqslant \aleph_0$.*

Proof. (i)–(iii) are obvious.

(iv) follows from the relation (1.3).

(v) obviously follows from (iv) and from the quasiaffine invariance of g.

(vi) Let $m^j \in (m_1, m_2)$, $j = 1, 2, \ldots$, be such that $m^j \geqslant m^{j+1}$ and $m_1 = \bigwedge\limits_{j=1}^{\infty} m^j$. Because an inner function divides m_1, if and only if it divides m^j for each j, we infer by (i) that

$$f(m_1) = \sup_{j \geqslant 1} f(m^j) \leqslant \aleph.$$

(vii) An inner function divides $m_1 \wedge m_2$ if and only if it divides m_1 and m_2; thus

$$f(m_1 \wedge m_2) = \sum_{m \text{non} \leqslant m_1 \wedge m_2} \aleph_m \leqslant \sum_{m \text{non} \leqslant m_1} \aleph_m + \sum_{m \text{non} \leqslant m_2} \chi_m = f(m_1) + f(m_2).$$

(viii). One and only one of the following two situations can occur: (a) there exists $m_0 < m$ such that m_0 non $\leqslant m'$; (b) $m(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^n$, $m'(z) = \left(\frac{z-a}{1-\bar{a}z}\right)^{n-1} m''(z)$, where $\frac{z-a}{1-\bar{a}z}$ does not divide $m''(z)$.

In case (a), if we denote by S the sum appearing in (viii), we have $h(m) \leqslant f(m_0) \leqslant S + f(m) = \max\big(S, f(m)\big)$. But $h(m) > f(m)$; we conclude that $h(m) \leqslant S$.

In case (b) we have $h(m) = f\left(\left(\frac{z-a}{1-\bar{a}z}\right)^{n-1}\right) = \aleph_m + f(m) = S + f(m)$ and the conclusion is obtained in the same way.

(ix) *A_{00} is well anti-ordered.* Let $B \subset A_{00}$ be a nonvoid set and denote

$$\aleph = \min\{h(m)\colon\ m \in B\} = h(m_0). \tag{1.5}$$

Let $m' \in B$, $m' \neq m_0$. The relation $m_0 < m'$ implies $h(m') \leqslant f(m_0) < h(m_0) = \aleph$, in contradiction with (1.5). If m' non $\leqslant m_0$, we obtain by (viii) $h(m') \leqslant f(m_0) < h(m_0)$, again in contradiction with (1.5). If follows that $m' \leqslant m_0$; thus m_0 is the greatest element of B.

A_{00} *is at most countable.* If m_T is a Blaschke product, the assertion is obvious. If m_T is a singular inner function:

$$m_T = \exp\left(-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right),$$

then each $m \in A_{00}$ has the same form, with μ replaced by a measure μ_m majorized by μ. With each $m \in A_{00}$ we associate the total mass a_m of μ_m. The assigement $m \mapsto a_m$ is increasing, and thus $\{a_m\}_{m\in A_{00}}$ is a well anti-ordered set of real numbers. But such a set is at most countable (indeed, for each $m \in A_{00}$ there exists $b_m < a_m$ such that the interval (b_m, a_m) contains no $a_{m'}$, $m' \in A_{00}$; thus $\{a_m\}_{m\in A_{00}}$ is equipotent to a set of pairwise disjoint open intervals).

Combining the two particular cases we obtain the general assertion (that is, for arbitrary m_T).

(x). From (vi) and (vii) we infer that the set of those inner functions $m \in [m_0, m_T]$ which satisfy the relation $f(m) \leqslant \aleph_0$ is an interval $[m_1, m_T]$. If $m_0 = m_1$, (x) is proved.

Suppose that $m_0 \neq m_1$; we claim $m_1 \in A_{00}$. Indeed, let $m' < m_1$. If $m' \geqslant m_0$, we have $f(m') > \aleph_0$ by the definition of the interval $[m_1, m_T]$. If $m' < m_0$, we have $f(m') \geqslant f(m_0) > \aleph_0$ by (i). If m' non $\leqslant m_0$ and m_0 non $\leqslant m'$, we infer by (viii) $\aleph_0 < f(m_0) < h(m_0) \leqslant f(m')$. Thus we have always $f(m') > \aleph_0 \geqslant f(m_1)$. But $m_1 \in A_{00}$ implies $m_1 \leqslant m_0$, a contradiction.

2. Quasisimilarity to the Jordan model. Due to the well anti-ordering of A_{00}, each element $m \in A_{00}$, with at most one exception, has a preceding element in A_{00}, denoted by m'. If m has no predecessor in A_{00}, we put $m' = 1$. Using Zorn's lemma, we can find for each $m \in A_{00}$ a maximal set S_m of sequences of inner functions with the following properties:

(a) if $s = \{m_j\}_{j=1}^{\infty} \in S_m$, we have $\bigvee_{j=1}^{\infty} m_j = m$ and m_j non $\leqslant m'$;

(b) each inner function m_1 appears at most $\aleph_{m_1}$ times in the sequences of S_m.

Lemma 1. card $S_m = h(m)$.

Proof. Let us suppose that card $S_m < h(m)$. We put $m_0 = \bigvee_{m_1} m_1$, where $\bigvee$ is taken for those inner functions m_1 such that $m_1 \leqslant m$, m_1 non $\leqslant m'$ and m_1 appears fewer than $\aleph_{m_1}$ times in the sequences of S_m. If $m_0 = m$, we can choose a new sequence $s = \{m_j\}_{j=1}^{\infty}$, which can be added to S_m without changing the conditions (a) and (b); thus the maximality of S_m is contradicted. It follows that $m_0 < m$; thus $f(m_0) \leqslant f(m) + \text{card } S_m \cdot \aleph_0 < f(m) + h(m) = h(m)$, in contradiction with the inequality $f(m_0) \geqslant h(m)$ which follows from the definition of $h(m)$.

Remark 2. For distinct elements $m_1, m_2 \in A_{00}$, the elements appearing in the sequences of S_{m_1} are distinct from the elements appearing in the sequences of S_{m_2}.

Proposition 2. *The C_0 operator*

$$(*) \qquad T = \bigoplus_m S(m)^{\aleph_m}$$

is quasisimilar to another one, of the same form, which satisfies $\aleph_m = h(m)$ for $m \in A_{00}$.

Proof. For a sequence $s = \{m_j\}_{j=1}^{\infty}$ of inner functions we put

$$T_s = \bigoplus_{j=1}^{\infty} S(m_j).$$

Then (after a permutation of the terms in the direct sum) T can be written as

$$T = T' \oplus \Big(\bigoplus_{m\in A_{00}} \big(\bigoplus_{s\in S_m} T_s\big)\Big).$$

From the condition (a), the minimal function of T_s is m if $s \in S_m$. Thus, by Proposition A, T_s is quasisimilar to an operator of the form $S(m) \oplus T_{s'}$. It follows that T is quasisimilar to

$$(2.1) \qquad T'' \oplus \Big(\bigoplus_{m\in A_{00}} \big(\bigoplus_{s\in S_m} S(m)\big)\Big).$$

The sum $\bigoplus_{s\in S_m} S(m)$ contains, by the preceding lemma, $h(m)$ terms; thus the operator (2.1) is of the form

$$(2.2) \qquad \bigoplus_m S(m)^{\aleph'_m}$$

with $\aleph'_m \geqslant h(m)$ for $m \in A_{00}$. Now, by Proposition 1(v), we have $\aleph'_m \leqslant h(m)$. The proposition is proved.

We use in the sequel the notation $m_0 = \max(A_{00} \cup \{1\})$.

We now want to eliminate from the sum (2.2) the terms $S(m)$ with $m \leqslant m_0$ and $m \notin A_{00}$.

Lemma 2. *Let T and S be acting on H, H', respectively. If S is unitarily equivalent to the restriction of T to an invariant subspace $H'' \subset H$, then*

$$\big(\bigoplus_1^{\infty} T\big) \oplus S \prec \bigoplus_1^{\infty} T.$$

Proof. Let $A\colon \bigoplus_1^{\infty} H \to \bigoplus_1^{\infty} H$ be defined by $A = \bigoplus_{j=1}^{\infty} \frac{1}{j} I_H$ and let $B\colon H' \to \bigoplus_1^{\infty} H$ be defined by $Bh = \bigoplus_{j=1}^{\infty} \frac{1}{j} Xh$, where $X\colon H' \to H''$ is a unitary operator satisfying the relation $XS = T|_{H''}X$. Then the operator

$Y\colon (\bigoplus_1^\infty H)\oplus H'\to\bigoplus_1^\infty H$ defined by $Y(h\oplus k)=Ah+Bk$ is a quasiaffinity (it is one-to-one because $\operatorname{Range} A\cap\operatorname{Range} B=\{0\}$ and has dense range because A has dense range) and satisfies the relation $Y\big((\bigoplus_1^\infty T)\oplus S\big)=(\bigoplus_1^\infty T)\,Y$.

COROLLARY 1. *Let $m_1\geqslant m_2$ be inner functions. Then $S(m_1)^{\aleph_0}$ and $S(m_1)^{\aleph_0}\oplus S(m_2)$ are quasisimilar.*

Proof. The conditions of Lemma 2 are satisfied for $T=S(m_1)$, $S=S(m_2)$ and for $T=S(m_1)^*$, $S=S(m_2)^*$ by [5], Lemma 1. Thus, by Lemma 2, we have

$$\big(\bigoplus_1^\infty S(m_1)\big)\oplus S(m_2)\prec\bigoplus_1^\infty S(m_1),$$

$$\big(\bigoplus_1^\infty S(m_1)\big)^*\oplus S(m_2)^*\prec\big(\bigoplus_1^\infty S(m_1)\big)^*,$$

and from the second relation it follows that

$$\big(\bigoplus_1^\infty S(m_1)\big)\oplus S(m_2)\succ\bigoplus_1^\infty S(m_1).$$

These relations prove the corollary.

Remark 3. If $\aleph_n$ is an increasing sequence of cardinal numbers then $\sup_n\aleph_n=\sum_n\aleph_n$.

Let us consider an operator of the form

$$(**)\qquad T=\bigoplus S(m)^{\aleph_m},$$

where the sum is extended to the divisors of the minimal function m_T and $\aleph_m=h(m)$ for $m\in A_{00}$.

We denote by A the set of all the direct summands $S(m)$ in $(**)$ such that $m\leqslant m_0$, $m\notin A_{00}$, each $S(m)$ being taken $\aleph_m$ times.

LEMMA 3. *With each $m\in A_{00}$ we can associate a subset $A_m\subset A$ with the following properties:*

(1) $A=\bigcup_{m\in A_{00}}A_m$;

(2) $A_m\cap A_{m'}=\emptyset$ *if* $m\neq m'$;

(3) *for* $m'\in A_m$ *we have* $m'\leqslant m$;

(4) $\operatorname{card}A_m\leqslant\aleph_m$ $(=h(m))$.

Proof. For each $m'\in A_{00}\cup\{1\}$ we denote by $A'_{m'}$ the set of all those $S(m_1)\in A$ with the property

$$(2.3)\qquad m'=\max\{m\colon m\in A_{00}\cup\{1\},\ m\ \text{non}\geqslant m_1\}.$$

Obviously we have

$$(2.4)\qquad \bigcup A'_{m'}=A,\qquad A_{m_0}=\emptyset,$$

and from the definition of f we infer

$$(2.5)\qquad f(m')\geqslant\operatorname{card}A'_{m'}.$$

If m' has an immediate successor m in A_{00}, we associate all the elements of $A'_{m'}$ to m. If m' has no immediate successor in A_{00}, let us put $m^1=\bigwedge m_1$, $\bigwedge$ being taken for $m_1\in A_{00}$ such that $m_1>m'$. If $m^1=m'$, we obviously have $A'_{m'}=\emptyset$. Let us suppose that $m^1>m'$. Then we can write $m^1=\bigwedge_{j=1}^\infty m_j$, where $m_j\in A_{00}$ and $m_j>m_{j+1}$ for each j.

Now, we have $m^1\notin A_{00}$ and, by the definition of m^1,

$$A_{00}\cap(m',m^1]=\emptyset.$$

It follows that f is constant on $[m',m^1]$, and so $f(m')=f(m^1)$. To compute $f(m^1)$ we observe that:

(i) $f(m^1)\geqslant h(m_j)$, by the definition of $h(m_j)$;

(ii) $f(m^1)\leqslant\sum_j f(m_j)=\sup_j f(m_j)$ by Remark 3.

It follows that $f(m')=f(m^1)=\sup_j h(m_j)$; thus by (2.5) and Remark 3 it follows that we can write

$$A'_{m'}=\bigcup_{j=1}^\infty A_j,\qquad \operatorname{card}A_j\leqslant h(m_j),$$

the sets A_j being pairwise disjoint. For each j, we associate with m the elements of A_j.

Let us denote by A_m, $m\in A_{00}$, the set of all the elements associated with m by one of the preceding procedures. Because the set $A_{00}\cup\{1\}$ is at most countable (Proposition 1 (ix)), it follows that

$$\operatorname{card}A_m\leqslant\aleph_0\cdot h(m)=h(m).$$

From (2.4) if follows that $A=\bigcup_{m\in A_{00}}A_m$. The properties (2) and (3) are obvious; thus the lemma is proved.

PROPOSITION 3. *The operator $(**)$ is quasisimilar to*

$$(\overset{**}{*})\qquad \big(\bigoplus_{m\in A_{00}}S(m)^{h(m)}\big)\oplus T'.$$

Here T' coincides with the sum of those terms $S(m)$ in $(**)$ for which m non $\leqslant m_0$ and it acts on a separable Hilbert space.

Proof. After a permutation of the direct summands, the operator (**) can be written as

$$\bigoplus_{m\in A_{00}} \Big(\Big(\bigoplus_{A_m} S(m')\Big) \oplus S(m)^{h(m)}\Big) \oplus T',$$

where A_m are the sets from Lemma 3. Because $h(m) \geqslant \operatorname{card} A_m$, we have $h(m) = \aleph_0 \cdot \operatorname{card} A_m + h(m)$; thus

$$(2.6)\qquad \Big(\bigoplus_{A_m} S(m')\Big) \oplus S(m)^{h(m)} = \bigoplus_{A_m} \big(S(m') \oplus S(m)^{\aleph_0}\big) \oplus S(m)^{h(m)}.$$

From Corollary 1 it follows that, for $m' \in A_m$, $S(m') \oplus S(m)^{\aleph_0}$ is quasisimilar to $S(m)^{\aleph_0}$; therefore the operator (2.6) is quasisimilar to

$$\Big(\bigoplus_{A_m} S(m)^{\aleph_0}\Big) \oplus S(m)^{h(m)} = S(m)^{\aleph_0 \cdot \operatorname{card} A_m + h(m)} = S(m)^{h(m)}.$$

The last assertion of our proposition is obviously equivalent to assertion (x) of Proposition 1.

LEMMA 4. *For two inner functions m_1 and m_2, the Jordan model of $S(m_1) \oplus S(m_2)$ is $S(m_1 \vee m_2) \oplus S(m_1 \wedge m_2)$.*

Proof. The minimal function of the C_0 operator $T = S(m_1) \oplus S(m_2)$ is $m_T = m_1 \vee m_2$. T is of multiplicity $\leqslant 2$, and its Jordan model is $S(n_1) \oplus S(n_2)$ with $n_1 = m_T = m_1 \vee m_2$ and $n_2 \leqslant n_1$. To find n_2 we use the relation $n_1 n_2 = m_1 m_2$ (relation (1.7) in [5]); thus we obtain $n_2 = \dfrac{m_1 m_2}{m_1 \vee m_2} = m_1 \wedge m_2$.

THEOREM 1.

(a) *Each C_0 operator is quasisimilar to a Jordan operator.*

(b) *If T and T' are Jordan operators and $T \prec\!\!\!\dashv T'$, then $T = T'$.*

Proof. (a) We have already seen that each operator of class C_0 is quasisimilar to an operator of the form (**). If $m_0 = 1$, our assertion follows from Proposition A.

Let us suppose that $m_0 \neq 1$. From Proposition A it follows that the operator T' appearing in (**) is quasisimilar to a Jordan operator

$$\bigoplus_{j=1}^{n} S(m_j), \qquad n \leqslant \infty,\ m_j \geqslant m_{j+1}.$$

Since $h(m_0) > \aleph_0$, the operator (**) is quasisimilar to

$$(2.7)\qquad \Big(\bigoplus_{m\in A_{00}} S(m)^{h(m)}\Big) \oplus S(m_0)^{\aleph_0} \oplus \Big(\bigoplus_{j=1}^{n} \big(S(m_j) \oplus S(m_0)\big)\Big).$$

By Lemma 4, $S(m_j) \oplus S(m_0)$ is quasisimilar to $S(m_j') \oplus S(m_j'')$, where $m_j'' \leqslant m_0 \leqslant m_j'$ and $m_j' \geqslant m_{j+1}'$ for each j. It follows that the operator (2.7) is quasisimilar to

$$(2.8)\qquad \Big(\bigoplus_{m\in A_{00}} S(m)^{h(m)}\Big) \oplus S(m_0)^{\aleph_0} \oplus \Big(\bigoplus_{j=1}^{n} S(m_j'')\Big) \oplus \Big(\bigoplus_{j=1}^{n} S(m_j')\Big)$$
$$= \Big(\bigoplus_{m\in A_{00}} S(m)^{h(m)}\Big) \oplus \Big(\bigoplus_{j=1}^{n} \big(S(m_j'') \oplus S(m_0)^{\aleph_0}\big)\Big) \oplus \Big(\bigoplus_{j=1}^{n} S(m_j')\Big).$$

Finally, because $m_j'' \leqslant m_0$, it follows by Lemma 2 that $S(m_j'') \oplus S(m_0)^{\aleph_0}$ is quasisimilar to $S(m_0)^{\aleph_0}$; thus the last operator in (2.8) is quasisimilar to

$$\Big(\bigoplus_{m\in A_{00}} S(m)^{h(m)}\Big) \oplus S(m_0)^{\aleph_0} \oplus \Big(\bigoplus_{j=1}^{n} S(m_j')\Big) = \Big(\bigoplus_{m\in A_{00}} S(m)^{h(m)}\Big) \oplus \Big(\bigoplus_{j=1}^{n} S(m_j')\Big).$$

The last operator is obviously a Jordan operator.

(b) Proposition 1(v) implies that the set A_{00} is the same for T and T' and moreover the saltum $h(m)$, $m \in A_{00}$, is the same for T and T'. Thus

$$T = \Big(\bigoplus_{m\in A_{00}} S(m)^{h(m)}\Big) \oplus \Big(\bigoplus_{j=1}^{n} S(m_j)\Big), \qquad T' = \Big(\bigoplus_{m\in A_{00}} S(m)^{h(m)}\Big) \oplus \Big(\bigoplus_{j=1}^{n'} S(m_j')\Big),$$

with $n \leqslant \infty$, $n' \leqslant \infty$, $m_j \geqslant m_{j+1}$, $m_j' \geqslant m_{j+1}'$, $m_j > m_0$, $m_j' > m_0$. From [5], pp. 105–106, it follows that

$$T_1 = T|\overline{\operatorname{Range} m_0(T)} = \bigoplus_{j=1}^{n} S(m_j/m_0), \qquad T_1' = T'|\overline{\operatorname{Range} m_0(T')} = \bigoplus_{j=1}^{n'} S(m_j'/m_0).$$

Since $T_1 \prec\!\!\!\dashv T_1'$, from the uniqueness assertion of Proposition A, we infer that $n = n'$ and $m_j/m_0 = m_j'/m_0$ for each j, such that $m_j = m_j'$ for $j = 1, 2, \ldots$ It follows that $T = T'$ and the theorem is proved.

Let us remark that the proof of Theorem 2 in [2] can be applied, with minor modifications, to arbitrary C_0 operators. Thus we have

COROLLARY 2. *If T is a C_0 operator and $X \in (T)''$, then there exists $f \in N_T$ such that $X = f(T)$.*

Here $(T)''$ denotes, as usual, the set of operators commuting with each operator which commutes with T.

PART II

3. Canonical representations of $L^1(0,1)$. We consider the space $L^1(0,1)$ as a Banach algebra with the convolution

$$(3.1)\qquad (f * g)(t) = \int_0^t f(s)\, g(t-s)\, ds, \qquad f, g \in L^1(0,1).$$

A representation of $L^1(0,1)$ into a Hilbert space H is a continuous linear map, $f \mapsto T_f$, which assigns to each $f \in L^1(0,1)$ a linear bounded operator acting on H, and satisfies the relation $T_{f*g} = T_f T_g$. A representation $f \mapsto T_f$ of $L^1(0,1)$ is complete if

$$\bigvee_{f \in L^1(0,1)} \text{Range } T_f = H.$$

Here "$\vee$" means the closed linear span. With each representation $f \mapsto T_f$ of $L^1(0,1)$ we associate the adjoint representation $f \mapsto T_f^{\sim} = (T_{\bar{f}})^*$ (here $\bar{f}(t) = \overline{f(t)}$).

Let us consider a strongly continuous semigroup of contractions $\{T(t)\}_{t \geq 0}$ acting on H, such that $T(1) = 0$. We can verify ([3]) that

$$f \mapsto C_f = \int_0^1 f(t) T(t)\, dt \tag{3.2}$$

is a complete representation of $L^1(0,1)$. Such a representation is called a *quasicanonical representation.*

Remark 4. The adjoint representation of (3.2) is still quasicanonical and is defined by the semigroup $\{T(t)^*\}_{t \geq 0}$.

Let us recall that a strongly continuous semigroup of contractions $\{T(t)\}_{t \geq 0}$ is uniquely determined by its cogenerator T ([4], III, § 8). T is a contraction without nonzero invariant vectors and

$$T(t) = e_t(T), \quad e_t(z) = \exp\left(t\frac{z+1}{z-1}\right), \quad t \geq 0. \tag{3.3}$$

Because the unitary part of T ([4], I, § 3) is the cogenerator of a semigroup of unitary operators, from $T(1) = 0$ we infer that T is completely nonunitary. Moreover, $T(1) = 0$ is equivalent to $e_1(T) = 0$; thus T is a C_0 operator and its minimal function is e_a, $a \leq 1$.

Definition 3. The quasicanonical representation (3.2) is *canonical* if the cogenerator T of the semigroup $\{T(t)\}_{t \geq 0}$ is a Jordan operator. A representation $f \mapsto T_f$ of $L^1(0,1)$ into H is *trivial* if $T_f = 0$, $f \in L^1(0,1)$.

Clearly, a trivial representation of $L^1(0,1)$ is characterised by $\dim H$.

Proposition 4. *Each canonical representation of $L^1(0,1)$ is unitarily equivalent to a representation of the form*

$$\Big(\bigoplus_{a \in A_{00}} C_{a,f}^{h(a)}\Big) \oplus \Big(\bigoplus_{j=1}^{n} C_{a_j,f}\Big), \tag{3.4}$$

where A_{00} is a well anti-ordered set of elements of $(0,1]$, h is a cardinal number valued decreasing function on A_{00} such that $h(a) > \aleph_0$ for $a \in A_{00}$, $0 \leq n \leq \infty$, $1 \geq a_1 \geq a_2 \geq \ldots$, $a_j > a$ for $a \in A_{00}$, and for each $a \in (0,1]$, $f \mapsto C_{a,f}$ is a representation of $L^1(0,1)$ into $L^2(0,1)$ defined by

$$C_{a,f}\, g(x) = \int_0^1 f(t) g(x - t/a)\, dt. \tag{3.5}$$

Proof. Let us consider the strongly continuous semigroup of contractions $\{T(t)\}_{t \geq 0}$, acting on $L^2(0,1)$ and defined by

$$T(t)f(x) = \begin{cases} f(x-t) & \text{if} \quad x \in (t,\infty) \cap (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

Then it is clear that

$$T(1) = 0, \quad T(t) \neq 0 \quad \text{for } t < 1. \tag{3.6}$$

If we denote by $\{T_a(t)\}_{t \geq 0}$ the semigroup defined by

$$T_a(t) = T(t/a),$$

we have for $a \in (0,1]$

$$C_{a,f} = \int_0^1 f(t) T_a(t)\, dt. \tag{3.7}$$

Now let us remark that for a C_0 operator for which $e_1(T) = 0$, the Jordan model takes the form

$$\Big(\bigoplus_{a \in A_{00}} S(e_a)^{h(a)}\Big) \oplus \Big(\bigoplus_{j=1}^{n} S(e_{a_j})\Big),$$

where A_{00} is a well anti-ordered set of elements of $(0,1]$, and h is a decreasing function on A_{00}, taking as values cardinal numbers $> \aleph_0$, $0 \leq n \leq \infty$, $1 \geq a_1 \geq a_2 \geq \ldots$, $a_j > a$ if $a \in A_{00}$. Thus, by (3.7) it follows that to prove the proposition it suffices to show that the cogenerator of the semigroup $\{T_a(t)\}_{t \geq 0}$ is unitarily equivalent to $S(e_a)$. It is clear that it is enough to prove the last assertion for $a = 1$.

If A is the generator of the semigroup $\{T(t)\}_{t \geq 0}$, then $A' = (iA)^{-1}$ is the Volterra operator:

$$A'f(x) = i\int_0^x f(t)\, dt.$$

Because the imaginary part of A' is of rank one, it follows that its Cayley transform

$$T = (A' - iI)(A' + iI)^{-1},$$

which coincides with the cogenerator of our semigroup, is an operator of class $C_0(1)$ ([4], IX, §§ 2, 4) unitarily equivalent to $S(e_1)$, by relation (3.6). The proposition is proved.

Remark 5. Another representation, which is unitarily equivalent to the canonical representation associated to the semigroup whose cogenerator is $S(e_a)$, is associated to the semigroup $\{T'_a(t)\}_{t\geqslant 0}$ acting on $L^2(0, a)$ and defined by

$$T'_a(t)f(x) = \begin{cases} f(x-t) & \text{if} \quad x\in(t,\infty)\cap(0,a), \\ 0 & \text{otherwise}. \end{cases}$$

For the corresponding representation $f\mapsto C'_{a,f}$, we have

$$C'_{a,f}g(x) = \int_0^x f(t)g(x-t)\,dt, \quad x\in(0,a).$$

4. Quasisimilarity models for the representations of $L^1(0,1)$. Let $f\mapsto T_f$, $f\mapsto T'_f$ be two representations of $L^1(0,1)$ into H, H', respectively. The first representation is a quasiaffine transform of the second if there exists a quasiaffinity $X\colon H\to H'$ such that

$$T'_f X = XT_f, \quad f\in L^1(0,1).$$

Two representations are quasisimilar if each one is a quasiaffine transform of the other.

Remark 6. The representation $f\mapsto T_f$ is a quasiaffine transform of $f\mapsto T'_f$, if and only if the adjoint representation $f\mapsto T'^{\sim}_f$ is a quasiaffine transform of $f\mapsto T^{\sim}_f$.

LEMMA 5. *Each quasicanonical representation of $L^1(0,1)$ is quasisimilar to a canonical representation.*

Proof. Let us consider the quasicanonical representation (3.2). Let T be the cogenerator of the semigroup $\{T(t)\}_{t\geqslant 0}$. By Theorem 1 there exists a Jordan operator T' and two quasiaffinities X, Y such that

$$T'X = XT, \quad TY = YT'.$$

If $\{T'(t)\}_{t\geqslant 0}$ is the strongly continuous semigroup of contractions whose cogenerator is T', we infer from (3.3)

$$T'(t)X = XT(t), \quad T(t)Y = YT'(t), \quad t\geqslant 0,$$

such that $XC_f = C'_f X$, $C_f Y = YC'_f$, where $f\mapsto C'_f$ is the canonical representation determined by the semigroup $\{T'(t)\}_{t\geqslant 0}$.

The main result in [3] can be reformulated as follows:

PROPOSITION B (Theorem 1 in [3]). *Each complete representation of $L^1(0,1)$ into a separable Hilbert space is a quasiaffine transform of a quasicanonical representation.*

Let $f\mapsto T_f$ be a representation of $L^1(0,1)$ into a Hilbert space H. Let k be a constant such that

$$\|T_f\| \leqslant k\|f\|_1, \quad f\in L^1(0,1). \tag{4.1}$$

Because $L^1(0,1)$ is a separable Banach space, H can be decomposed into a direct sum $H = \bigoplus_{a\in A} H_a$ of separable spaces, each space being reducing for T_f, $f\in L^1(0,1)$.

Let us apply the proof of Theorem 1 in [3] to each representation $f\mapsto T_f|_{H_a}$, $a\in A$. We obtain the existence of a family $\{H_a(t)\}_{t\geqslant 0}$ of operators acting on H_a, with the following properties:

(i) $\|H_a(t)\|\leqslant k$, $t\mapsto H_a(t)$ is strongly measurable;

(ii) $\{H_a(t)\}_{t\geqslant 0}$ is a commuting family;

(iii) $T_f|_{H_a} = \int_0^1 f(t)H_a(t)\,dt$;

(iv) the range of $H_a(t)$ is contained in

$$K_a = \bigvee_{f\in L^1(0,1)} T_f(H_a);$$

(v) there exists a strongly continuous semigroup of operators, $\{T_a(t)\}_{t\geqslant 0}$, acting on K_a, such that $T_a(1) = 0$ and the restriction $H_a(t)|_{K_a}$ coincides almost everywhere with $T_a(t)$. (The properties (ii) and (iv) are not explicitly proved in [3] but can be easily obtained by the methods used there.)

Let us consider the orthogonal decomposition

$$H_a = K_a \oplus (H_a \ominus K_a).$$

With respect to this decomposition $H_a(t)$ has almost everywhere a matrix of the form

$$\begin{bmatrix} T_a(t) & B_a(t) \\ 0 & 0 \end{bmatrix},$$

where $\{T_a(t)\}_{t\geqslant 0}$ is the semigroup given by (v). From the properties (i) and (v) we infer the existence of an operator of the form

$$\begin{bmatrix} I & X_a \\ 0 & 0 \end{bmatrix}, \quad \left\|\begin{bmatrix} I & X_a \\ 0 & 0 \end{bmatrix}\right\| \leqslant k$$

in the weak closure of the family $\{H_a(t)\}_{t\geqslant 0}$. This operator will commute with $H_a(t)$, thus we have almost everywhere the relation

$$T_a(t)X_a = B_a(t). \tag{4.2}$$

The operator

$$Z_a = \begin{bmatrix} I & -X_a \\ 0 & I \end{bmatrix}$$

is invertible, its inverse is

$$Z_a^{-1} = \begin{bmatrix} I & X_a \\ 0 & I \end{bmatrix}$$

and we have

$$\|Z_a\| \leqslant 1+k, \qquad \|Z_a^{-1}\| \leqslant 1+k. \tag{4.3}$$

From (4.2) it follows that

$$\begin{bmatrix} T_a(t) & B_a(t) \\ 0 & 0 \end{bmatrix} Z_a = Z_a \begin{bmatrix} T_a(t) & 0 \\ 0 & 0 \end{bmatrix}.$$

From (4.3) it follows that the operator

$$Z = \bigoplus_{a\in A} Z_a$$

is invertible. The strongly continuous semigroup of operators $\{T(t)\}_{t\geqslant 0}$ defined by

$$T(t) = \bigoplus_{a\in A} T_a(t)$$

is acting on

$$K = \bigoplus_{a\in A} K_a = \bigvee_{f\in L^1(0,1)} \text{Range } T_f.$$

From the preceding relations it follows that, with respect to the orthogonal decomposition $H = K \oplus (H \ominus K)$, we have the relation

$$Z^{-1}T_f Z = \begin{bmatrix} T_f' & 0 \\ 0 & 0 \end{bmatrix}$$

where $f \mapsto T_f' = \int_0^1 f(t)T(t)\,dt$ is a complete representation of $L^1(0,1)$.

From the last part of the proof of Theorem 1 in [3] it follows that the representation $f \mapsto T_f'$ is a quasiaffine transform of a quasicanonical representation.

Finally, an application of Lemma 5 gives the following result:

PROPOSITION 5.

(a) *Each representation of $L^1(0,1)$ into a Hilbert space is similar to the direct sum of a complete representation with a trivial representation.*

(b) *Each complete representation of $L^1(0,1)$ is a quasiaffine transform of a canonical representation.*

Let us suppose that $f \mapsto T_f$ is the direct sum of a canonical representation with a trivial representation. Thus, T_f acts on a direct sum $H_1 \oplus H$- and

$$T_f = \int_0^1 f(t) \begin{bmatrix} T(t) & 0 \\ 0 & 0 \end{bmatrix} dt.$$

The following relations are obvious:

$$H_1 = \bigvee_{f\in L^1(0,1)} \text{Range } T_f, \qquad H_2 = \bigcap_{f\in L^1(0,1)} \text{Ker } T_f. \tag{4.4}$$

If $f \mapsto T_f'$ is a second representation of the same form,

$$T_f' = \int_0^1 f(t) \begin{bmatrix} T'(t) & 0 \\ 0 & 0 \end{bmatrix} dt$$

with respect to the orthogonal decomposition $H_1' \oplus H_2'$, and X: $H_1 \oplus H_2 \to H_1' \oplus H_2'$ is a quasiaffinity such that $T_f' X = XT_f$, $f \in L^1(0,1)$, we infer from (4.4) that $XH_1 \subset H_1'$, $HX_2 \subset H_2'$, such that $X = X_1 \oplus X_2$, where X_1: $X_1 \to H_1'$, X_2: $H_2 \to H_2'$ are quasiaffinities and

$$\Big(\int_0^1 f(t)T'(t)\,dt\Big) X_1 = X_1 \Big(\int_0^1 f(t)T(t)\,dt\Big). \tag{4.5}$$

From (4.5) it follows that $T'(t)X_1 = X_1 T(t)$, thus $T'X_1 = X_1 T$, where T', T are the cogenerators of the semigroups $\{T'(t)\}_{t\geqslant 0}$, $\{T(t)\}_{t\geqslant 0}$, respectively. Because T' and T are Jordan operators, from Theorem 1 (b) it follows that $T' = T$. Finally, from the fact that X_2 is a quasiaffinity we infer $\dim H_2 = \dim H_2'$.

We are now able to prove:

THEOREM 2. (a) *Each representation of $L^1(0,1)$ into a Hilbert space is quasisimilar to the direct sum of a canonical representation with a trivial representation.*

(b) *Two representations, both being the direct sum of a canonical representation and a trivial representation, and one of them a quasiaffine transform of the other, coincide. (More precisely, they have the same canonical part and their trivial parts act on spaces of the same dimension.)*

Proof. (b) follows from the preceding considerations.

(a) If we apply Proposition 5 to the representation $f \mapsto T_f$ and its adjoint representation $f \mapsto T_f^{\sim}$ and if we use Remarks 4 and 6, we infer the existence of two representations of the form

canonical $\oplus$ trivial

such that the representation $f \mapsto T_f$ is a quasiaffine transform of the first and the second is a quasiaffine transform of $f \mapsto T_f$. An application of (b) shows that the three representations considered are pairwise quasisimilar. The theorem is proved.

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A property of determining sets for analytic functions

by

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Abstract. Any locally determining set at $0 \in K^n$ ($K = R$ or $K = C$) for analytic functions (1) contains a sequence convergent to $0 \in K^n$, which is itself locally determining at $0 \in K^n$.

1. In this note we prove the following theorem.

THEOREM. *Let $E \subset C^n$ be a locally determining set at $0 \in C^n$ for holomorphic functions. Then there is a sequence $\{a_\nu\} \subset E$ convergent to $0 \in C^n$ which is a locally determining set at $0 \in C^n$ for holomorphic functions.*

This theorem is an answer to Question 2 posed in [2]. Its proof is based on a lemma concerning locally complete sets in separable Hilbert spaces. The lemma seems to be interesting by itself.

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2. Let H be a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. A subset $A \subset H$ is called *complete* if and only if the equations $\langle x, a\rangle = 0$ for each $a \in A$ imply $x = 0$.

LEMMA. *Assume that A is a subset of a separable Hilbert space such that for every constant $0 < r < 1$ the set $A_r := \{x \in A: \|x\| < r\}$ is complete. Then there is a sequence $\{a_\nu\} \subset A$ convergent to $0 \in H$ which is complete.*

Proof. Let $\{e_i\}$ be an orthonormal base in H and let r_1 be a positive number. Then the set A_{r_1} is complete. Hence the closure of the linear subspace spanned by A_{r_1} is equal to H. Thus there exist scalars $\beta_1^{(1)}, \ldots, \beta_{s_1}^{(1)}$ and vectors $a_1^{(1)}, \ldots, a_{s_1}^{(1)} \in A_{r_1} - \{0\}$ such that

$$\|\beta_1^{(1)} a_1^{(1)} + \ldots + \beta_{s_1}^{(1)} a_{s_1}^{(1)} - e_1\| < 2^{-1}.$$

Put $r_2 = \min\{\|a_1^{(1)}\|, \ldots, \|a_{s_1}^{(1)}\|, 2^{-2}\}$. Then $a_k^{(1)} \notin A_{r_2}$ for $k = 1, \ldots, s_1$. As before, we can choose scalars $\beta_1^{(2)}, \ldots, \beta_{s_2}^{(2)}$ and vectors $a_1^{(2)}, \ldots, a_{s_2}^{(2)}$

(1) A subset E of K^n ($n \geqslant 1$) is called a *locally determining set at* $0 \in K^n$ for analytic functions, if for each connected neighbourhood U of $0 \in K^n$ the subset $E \cap U$ is determining for the analytic functions in U, i.e. if a function f is analytic in U and vanishing on $E \cap U$, the function f is identically zero ([1]).