[4] L. Nachbin, *The Haar integral*, Princeton 1965.
[5] J. C. Oxtoby, *Invariant measures in groups which are not locally compact*, Trans. Amer. Math. Soc. 60 (1946), pp. 215–237.
[6] C. A. Rogers, *Hausdorff measures*, Cambridge Univ. Press, Cambridge 1970.
[7] S. Saks, *Theory of the integral*, Reproduction of second ed., New York 1964.
[8] R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Annals of Math. 92 (1970), pp. 1–56.

MATHEMATICS DEPARTMENT
UNIVERSITY OF COLORADO, BOULDER, COLORADO 80309

Received March 5, 1975 (970)

STUDIA MATHEMATICA, T. LX. (1977)

On a class of Banach spaces

by

M. VALDIVIA (Valencia)

Abstract. A Banach space E with E^{**}/E separable is the direct sum of a reflexive subspace and a separable one.

The Banach spaces we use here are defined over the field K of the real or complex numbers. If $\langle E, F\rangle$ is a dual pair of vector spaces, with the bilinear form $\langle x, y\rangle$, $x\in E$, $y\in F$, we represent by $\sigma(E, F)$ the locally convex topology on E such that the origin of E has as neighbourhood sub-basis $\{U_y\colon y\in F\}$, being $U_y = \{x\in E\colon |\langle x, y\rangle| \leqslant 1\}$. If E is a Banach space, we consider it as a subspace of its second conjugate E^{**} by means of the canonical injection. If F is a subspace of E, we denote by $F^{\perp}$ the subspace of E^* orthogonal to F and by $F^{\perp\perp}$ the subspace of E^{**} orthogonal to $F^{\perp}$. We say that E is *weakly compactly generated space*, or WCG space, if there is in E a weakly compact fundamental set.

THEOREM. *Let E be a Banach space such that E^{**}/E is separable. Then E is a direct sum of a reflexive subspace and a separable subspace (clearly, every separable subspace of E has its second dual separable).*

We shall need the following lemmas:

LEMMA 1. *Let F be a closed subspace of a Banach space X. Assume that every $x^{**}\in X^{**}$ that belongs to the $\sigma(X^{**}, X^*)$-closure of a countable bounded subset of X is of the form $x^{**} = x+f^{\perp\perp}$ with $x\in X$ and $f^{\perp\perp}\in F^{\perp\perp}$. Then the space X/F is reflexive.*

Proof. Let $(\bar{x}_n)$ be a bounded sequence in X/F. If φ is the canonical mapping of X onto X/F, let (x_n) be a bounded sequence in X such that $\varphi(x_n) = \bar{x}_n$, $n = 1, 2, \ldots$ If x_0^{**} is an accumulation point of (x_n) in $X^{**}[\sigma(X^{**}, X^*)]$, we set

$$x_0^{**} = x_0+f_0^{\perp\perp}, \quad x_0\in X, \quad f_0^{\perp\perp}\in F^{\perp\perp}.$$

If u is an element of $F^{\perp}$, the sequence of elements of K, $(u(x_n))$, has an accumulation point $x_0^{**}(u)$. On the other hand,

$$u(x_n) = u(\bar{x}_n), \quad x_0^{**}(u) = (x_0+f^{\perp\perp})(u) = u(x_0),$$

and therefore the sequence $(u(x_n))$ has as accumulation point $u(x_0) = u(\varphi(x_0))$, hence $\varphi(x_0)$ is a weakly adherent point of $(\bar{x}_n)$ in X/F. Thus X/F is reflexive by using the theorem of Eberlein–Šmulian, ([2], p. 58). ■

Lemma 2. *If Y is a closed subspace of a Banach space X such that X/Y is separable, then there is a closed separable subspace $F \subseteq X$ such that $Y+F = X$.*

Proof. Let φ be the canonical mapping from X onto X/Y. We take a dense countable set $\{\bar{x}_1, \bar{x}_3, \ldots, \bar{x}_n, \ldots\}$ in X/Y. For every positive integer n, let x_n be an element of X such that

$$\varphi(x_n) = \bar{x}_n, \qquad \|x_n\| \leqslant \|\bar{x}_n\| + \frac{1}{n^2}.$$

Let F be the closed linear hull in X of $\{x_1, x_2, \ldots, x_n, \ldots\}$. Given any $\bar{x}$ of X/Y we obtain a strictly increasing sequence (n_p) of natural numbers so that

$$\|\bar{x} - (\bar{x}_{n_1} + \bar{x}_{n_2} + \ldots + \bar{x}_{n_p})\| \leqslant \frac{1}{p^2}, \qquad p = 1, 2, \ldots$$

Since

$$\begin{aligned}
\|x_{n_{p+1}}\| &\leqslant \|\bar{x}_{n_{p+1}}\| + \frac{1}{n_{p+1}^2} \\
&\leqslant \|\bar{x} - (\bar{x}_{n_1} + \bar{x}_{n_2} + \ldots + \bar{x}_{n_{p+1}}) - [\bar{x} - (\bar{x}_{n_1} + \bar{x}_{n_2} + \ldots + \bar{x}_{n_p})]\| + \frac{1}{n_{p+1}^2} \\
&\leqslant \|\bar{x} - (\bar{x}_{n_1} + \bar{x}_{n_2} + \ldots + \bar{x}_{n_{p+1}})\| + \|\bar{x} - (\bar{x}_{n_1} + \bar{x}_{n_2} + \ldots + \bar{x}_{n_p})\| + \frac{1}{n_{p+1}^2} \\
&\leqslant \frac{1}{(p+1)^2} + \frac{1}{p^2} + \frac{1}{n_{p+1}^2} < \frac{3}{p^2}, \qquad p = 1, 2, \ldots,
\end{aligned}$$

we have that the series $\sum_{p=1}^{\infty} x_{n_p}$ converges in X to an element x and therefore $x \in F$ and $\varphi(x) = \bar{x}$, and thus $Y+F = X$. ■

Lemma 3. *Under the assumption of the theorem there is a separable subspace F of E^* such that E^*/F is reflexive and F is $\sigma(E^*, E)$-closed.*

Proof. Let $E^\perp = \{e^{***} \in E^{***}\colon e^{***}(e) = 0, \ \forall e \in E\}$. By Lemma 2, there is a closed separable subspace H of E^{**} such that $E^{**} = E+H$. Thus

$$\sigma(E^\perp, E^{**}) = \sigma(E^\perp, E+H) = \sigma(E^\perp, H).$$

Therefore the weak-star topology of the unit ball B of $E^\perp$ is separable and metrizable. Hence there is a countable subset $A_1 \subset E^*$ such that every $b \in B$, that is in the weak-star closure of a countable subset of E^*, belongs to the weak-star closure of A_1. We set $F = \overline{\operatorname{span} A_1}$ and use Lemma 1 for $X = E^*$. If G is the $\sigma(E^{***}, E^{**})$-closure of F in E^{***}, then $G+E^* = E^{***}$, since E^*/F is reflexive. On the other hand, $E^\perp$ is $\sigma(E^{***}, E^{**})$-separable and therefore it is possible to take F so that $G \supset E^\perp$. Then the orthogonal subspace of F in E^{**} is contained in E, hence F is $\sigma(E^*_E, E)$-closed. ■

Proof of Theorem. Let F be a subspace of E^* constructed in Lemma 3. Since, by Lemma 3, the annihilator $F_\perp$ is reflexive, $F_\perp \subset$ E, it is sufficient to show that $E/F_\perp$ is separable. Indeed, this implies that E is a WCG space, hence if we write (Lemma 2) $E = F_\perp + G_1$ with G_1 separable, we can, by the Amir–Lindenstrauss theorem [1] (see also [2], p. 74), replace G_1 by a larger separable subspace G which is complemented in E. Since E/G is a factor space of $E/F_\perp$, it is reflexive.

To prove that $E/F_\perp$ is separable, observe that the conjugate space of $E/F_\perp$ coincides with F, which is separable, and therefore $E/F_\perp$ is separable. ■

The author wants to thank the Referee for his kindness.

References

[1] D. Amir and J. Lindestrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. 88 (1968), pp. 35–46.
[2] M. Day, *Normed Linear Spaces*, Springer, Berlin-Heidelberg-New York 1973.

FACULTAD DE CIENCIAS, VALENCIA, SPAIN

Received May 19, 1975
New version November 25, 1975 (1017)