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A compact convex set with no extreme points

by

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Abstract. The purpose of the paper is to show the existence of a Fréchet space X containing a compact convex set K such that K contains no extreme points.

1. Introduction. The Krein–Milman theorem states that if K is a compact convex subset of a Hausdorff locally convex topological vector space X, then K is the closed convex hull of its extreme points ([2], p.70). In this paper we shall produce a Hausdorff topological vector space X containing a compact convex set K such that K has no extreme points. The question of the existence of such a compact convex set is mentioned in [1], p. 124, and [2], p. 70. The first step in the construction of the space X will be to construct some fairly pathological paranorms on finite-dimensional spaces. This will be done in Section 2. In Section 3 we shall inductively piece together the finite-dimensional spaces to obtain a linear metric space V. The space X will be obtained by taking the completion of V.

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2. Paranorms on finite-dimensional spaces. This section will deal almost exclusively with paranorms on finite-dimensional vector spaces. Throughout this paper all vector spaces will be over the reals and θ will always denote the zero element of the vector space. If V is a vector space, then a nonnegative real valued function N on V is called a *paranorm* if for every $x, y \in V$,

(1) $N(\theta) = 0$,

(2) $N(x) = N(-x)$,

(3) $N(x+y) \leqslant N(x)+N(y)$,

(4) $\lim\limits_{\alpha\to 0} N(\alpha x) = 0$.

A paranorm N is *total* if $N(x) \neq 0$ for every $x \in V$ such that $x \neq \theta$. N is *monotone* if, for every $x \in V$ and $\alpha \in [0,1]$, $N(\alpha x) \leqslant N(x)$ (equivalently, if $|\beta| \leqslant |\gamma|$, then $N(\beta x) \leqslant N(\gamma x)$). If N is a total paranorm on a vector space V and $d(x, y) = N(x-y)$ for every $x, y \in V$, then d is a *metric* on V. With the metric given by d, V is called a *linear metric space*. If d is a complete metric, then V is called a *Fréchet space*. If V is a linear metric space and $x_1, \ldots, x_n \in V$, then we shall let $[x_1, \ldots, x_n]$ denote the closed convex hull of $\{x_1, \ldots, x_n\}$. More generally if $E \subset V$, then $[E]$ will denote the closed convex hull of E. If V is finite dimensional and N is a paranorm on V, then we shall say that N is *norm bounded* if there exists a norm $\|\cdot\|$ on V such that $N(x) \leqslant \|x\|$ for every $x \in V$. Since V is finite dimensional, an equivalent formulation of this is that if $\|\cdot\|$ is any norm on V, then there exists a constant $c > 0$ such that $N(x) \leqslant c\|x\|$ for every $x \in V$. If V is a finite-dimensional vector space with paranorm N and basis $B = \{v_1, \ldots, v_m\}$ with $v = \sum_{i=1}^{m} v_i$, then for $h, \varepsilon > 0$, v is called an *ε-needle point* of height h with respect to N and the basis B if

(1) N is monotone, total, and norm bounded.

(2) If $x \in [\theta, mv_1, \ldots, mv_m]$, then there exists $a \in [0,1]$ such that $N(x-av) < \varepsilon$.

(3) $N(mv_i) < \varepsilon$ for $1 \leqslant i \leqslant m$.

(4) If $a \in [0,1]$, then $N(av) = ah$.

The purpose of this section is to show that for any $h, \varepsilon > 0$ an ε-needle point of height h occurs in spaces of suitably high dimension. This will be crucial in the construction of the compact convex set with no extreme points. We now obtain some results which will be useful in constructing paranorms on finite-dimensional spaces.

PROPOSITION 2.1. *Let V be a finite-dimensional space and let φ: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ such that*

(i) $0 = \varphi(0) = \lim_{a\to 0} \varphi(a)$,

(ii) *φ is monotone increasing, and*

(iii) *there exist $\varepsilon > 0$ and $c > 0$ such that if $0 \leqslant a < \varepsilon$, then $\varphi(a) \geqslant ca$.*

If K is a compact subset of V that spans V and if for every $x \in V$ we let

$$N(x) = \inf\Big\{\sum_{i=1}^{m} \varphi(|a_i|):\ a_i \in \boldsymbol{R},\ x_i \in K,\ \sum_{i=1}^{m} a_i x_i = x\Big\} \tag{2.1}$$

then N is a monotone, total paranorm on V.

Proof. If is clear that $N(\theta) = 0$ and that if $x \in V$, then $N(x) = N(-x)$. Suppose $x, y \in V$ and $\varepsilon > 0$. Then there exists $a_i, \beta_j \in \boldsymbol{R}$, $x_i \in K$, $y_j \in K$ for $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$ such that $\sum_{i=1}^{p} a_i x_i = x$, $\sum_{j=1}^{q} \beta_j y_j = y$, $\sum_{i=1}^{p} \varphi(|a_i|) - N(x)$

$< \varepsilon/2$, and $\sum_{j=1}^{q} \varphi(|\beta_j|) - N(y) < \varepsilon/2$. Then

$$N(x)+N(y) > \sum_{i=1}^{p} \varphi(|a_i|) + \sum_{j=1}^{q} \varphi(|\beta_j|) - \varepsilon \geqslant N(x+y) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $N(x)+N(y) \geqslant N(x+y)$. Now suppose that $x \in V$. Let $a_i \in \boldsymbol{R}$ and $x_i \in K$ for $1 \leqslant i \leqslant m$ such that $\sum_{i=1}^{m} a_i x_i = x$. Then for $a \in \boldsymbol{R}$ $N(ax) \leqslant \sum_{i=1}^{m} \varphi(|aa_i|)$. Since $\lim_{a\to 0} \sum_{i=1}^{m} \varphi(|aa_i|) = 0$, $\lim_{a\to 0} N(ax) = 0$. Thus N is a paranorm on V.

Now suppose $x \in \boldsymbol{R}^n$ with $x \neq \theta$. Since K is compact, there exists $\delta > 0$ such that if $\sum_{i=1}^{m} a_i x_i = x$ for $a_1, \ldots, a_m \in \boldsymbol{R}$ and $x_1, \ldots, x_m \in K$, then $\sum_{i=1}^{m} |a_i| \geqslant \delta$. Now let $a_1, \ldots, a_m \in \boldsymbol{R}$, $x_1, \ldots, x_m \in K$ such that $\sum_{i=1}^{m} a_i x_i = x$. If $|a_j| \geqslant \varepsilon$ for some j, then $\sum_{i=1}^{m} \varphi(|a_i|) \geqslant \varphi(\varepsilon)$. If $|a_i| \leqslant \varepsilon$ for $1 \leqslant i \leqslant m$, then $\sum_{i=1}^{m} \varphi(|a_i|) \geqslant c \sum_{i=1}^{m} |a_i| \geqslant c\delta$. In either case $\sum_{i=1}^{m} \varphi(|a_i|) \geqslant \min\{c\delta, \varphi(\varepsilon)\}$. Thus $N(x) \geqslant \min\{c\delta, \varphi(\varepsilon)\} > 0$. It is clear from the monotonicity of φ that N is monotone.

If K and φ satisfy the conditions of Proposition 2.1, then N is called the *paranorm generated by* (K, φ) if N is defined by (2.1).

PROPOSITION 2.2. *Let V be a finite-dimensional vector space over the reals and let K be a compact subset which spans V. If $x_0 \in V$ and*

$$\min\Big\{n:\ \sum_{i=1}^{n} a_i x_i = x_0,\ x_i \in K,\ |a_i| \leqslant 1\Big\} = M > 1,$$

then there exists a total, monotone paranorm N on V such that $N(x) \leqslant 1$ for every $x \in K$ and $N(x_0) \geqslant M-1$.

Proof. If $r \geqslant 1$, define $\varphi_r(a) = \max\{a^r, a^{1/r}\}$ for every $a \geqslant 0$. Let N_r be the paranorm generated by (K, φ_r). Note that for every $x \in K$, $N_r(x) \leqslant 1$. Observe also that if $x \in V$, then $N_r(x)$ is a monotone increasing function of r for $r \geqslant 1$. We let A_r denote all finite sequences of pairs $\langle(a_i, x_i)\rangle_{i=1}^{m}$, where $a_1, \ldots, a_m \in \boldsymbol{R}$ with $|a_1| \geqslant \ldots \geqslant |a_m|$, $x_1, \ldots, x_m \in K$, $\sum_{i=1}^{m} a_i x_i = x_0$, and $\sum_{i=1}^{m} \varphi_r(|a_i|) \leqslant M-1$. The result will follow if we can show that for some $r \geqslant 1$, $A_r = \emptyset$. Observe that if $1 \leqslant p \leqslant r$, then $A_p \supset A_r$. By the compactness of K, it is easily demonstrated that there exists $\varepsilon > 0$ such that if $|a_i| \leqslant 1+\varepsilon$ for $1 \leqslant i \leqslant m \leqslant M-1$ and $x_1, \ldots, x_m \in K$, then $\sum_{i=1}^{m} a_i x_i \neq x_0$. Since $\lim_{r\to\infty} (1+\varepsilon)^r = \infty$, there exists $r_1 > 1$ such that if $a \geqslant 1+\varepsilon$, then $\varphi_r(a) \geqslant M-1$. Thus if $r \geqslant r_1$ and $\langle (a_i, x_i)\rangle_{i=1}^m \in A_r$, then $|a_i| \leqslant 1+\varepsilon$ and $m \geqslant M$. Now let $E = \{\sum_{i=1}^{M-1} a_i x_i \colon |a_i| \leqslant 1+\varepsilon\}$. Then E is a compact set and $x_0 \notin E$. Thus there exists a neighborhood U of θ in V such that if $y \in E$ and $x \in U$, then $x+y \neq x_0$. Since K is compact, there exists $\delta > 0$ such that if $a_1, \ldots, a_m \in \boldsymbol{R}$, $x_1, \ldots, x_m \in K$ with $\sum_{i=1}^m a_i x_i \notin U$, then $\sum_{i=1}^m |a_i| \geqslant \delta$. Now choose a positive integer β such that $\beta\delta > M$. Now let $\eta > 0$ such that if $0 \leqslant a \leqslant \eta$, then $a^{1/2} \geqslant \beta a$. Since $\lim_{r\to\infty} \varphi_r(\eta) = 1$, we may choose $r_2 \geqslant \max\{2, r_1\}$ such that if $r \geqslant r_2$ and $\langle (a_i, x_i)\rangle_{i=1}^m \in A_r$ then $|a_i| < \eta$ for $M \leqslant i \leqslant m$. We claim that for $r \geqslant r_2$, $A_r = \emptyset$. Suppose $\langle (a_i, x_i)\rangle_{i=1}^m \in A_r$. Then $\sum_{i=1}^{M-1} a_i x_i \in E$. Thus $\sum_{i=M}^m a_i x_i \notin U$ and $\sum_{i=M}^m |a_i| \geqslant \delta$. But since $|a_i| \leqslant \eta$, $\sum_{i=M}^m \varphi_r(|a_i|) \geqslant \sum_{i=M}^m \beta|a_i| \geqslant \beta\delta \geqslant M$.

Proposition 2.3. *Let V be a finite-dimensional space. If for $1 \leqslant i \leqslant m$, V_i is a subspace of V with a paranorm N_i such that span $\{V_1, \ldots, V_m\} = V$ and for every $x \in V$ we let*

$$N(x) = \inf\Big\{\sum_{i=1}^m N_i(x_i) \colon x_i \in V_i,\ \sum_{i=1}^m x_i = x\Big\}, \tag{2.2}$$

then N is a paranorm on V.

Furthermore:

(1) *If each N_i is monotone, then N is monotone.*

(2) *If each N_i is total on V_i, then N is total on V.*

(3) *If each N_i is norm bounded in V_i, then N is norm bounded in V.*

Proof. It is clear that $N(\theta) = 0$, and $N(x) = N(-x)$ for every $x \in V$. Suppose $x, y \in V$, $\varepsilon > 0$, and $x_i, y_i \in V_i$ such that $\sum_{i=1}^m x_i = x$ and $\sum_{i=1}^m y_i = y$ with $\sum_{i=1}^m N_i(x_i) - N(x) < \varepsilon/2$ and $\sum_{i=1}^m N_i(y_i) - N(y) < \varepsilon/2$. Then $N(x)+N(y) > \sum_{i=1}^m N_i(x_i) + N_i(y_i) - \varepsilon \geqslant \sum_{i=1}^m N_i(x_i+y_i) - \varepsilon \geqslant N(x+y) - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $N(x)+N(y) \geqslant N(x+y)$. Now suppose $x \in V$ and $\sum_{i=1}^m x_i = x$ with each $x_i \in V_i$. If $a \in \boldsymbol{R}$, then $N(ax) \leqslant \sum_{i=1}^m N_i(ax_i)$. Since $\lim_{a\to 0} \sum_{i=1}^m N_i(ax_i) = 0$, $\lim_{a\to 0} N(ax) = 0$. Thus N is a paranorm.

It is easy to verify (1). Suppose that each N_i is total on V_i. Let $x \in V$ such that $x \neq \theta$. Let $\|\cdot\|$ be any norm on V and let $0 < \varepsilon < \|x\|/m$. Now define $S = \{y \in V \colon \|y\| \geqslant \varepsilon\}$ and let $\varepsilon_i = \inf\{N_i(y) \colon y \in V_i \cap S\}$ for $1 \leqslant i \leqslant m$. Since each N_i is total, each $\varepsilon_i > 0$. Let $\delta = \min\{\varepsilon_1, \ldots, \varepsilon_m\}$.

If $x_i \in V_i$ and $\sum_{i=1}^m x_i = x$, then $x_j \in S$ for some $1 \leqslant j \leqslant m$. But then $\sum_{i=1}^m N_i(x_i) \geqslant \delta$. Thus $N(x) \geqslant \delta > 0$.

Now suppose each N_i is norm bounded, i.e. for $1 \leqslant i \leqslant m$ let N_i' be a norm on V_i such that for every $x \in V_i$, $N_i(x) \leqslant N_i'(x)$. Now for $x \in V$ define $N'(x) = \inf\{\sum_{i=1}^m N_i'(x_i) \colon \sum_{i=1}^m x_i = x,\ x_i \in V_i\}$. It is easily show that if $a \in \boldsymbol{R}$ and $x \in V$, then $N'(ax) = |a| N'(x)$. Thus N' is a norm. It is also clear that if $x \in V$, then $N(x) \leqslant N'(x)$.

If $N_1, \ldots, N_m$ satisfy the conditions of Proposition 2.3, then the paranorm N defined by (2.2) will be denoted $\inf\{N_1, \ldots, N_m\}$. Notice that N is the largest paranorm on V dominated by each N_i in V_i.

Lemma 2.4. *If N is a monotone paranorm on a vector space V, $x \in V$, and $a \in [0,1]$, then $aN(x) \leqslant 2N(ax)$.*

Proof. Let n be the least integer such that $na \geqslant 1$. Then $1 \leqslant na \leqslant 2$. Therefore $aN(x) \leqslant aN(nax) \leqslant anN(ax) \leqslant 2N(ax)$.

Proposition 2.5. *If N_0 is a monotone, total paranorm on a finite-dimensional vector space V, $x_0 \in V$ with $N_0(x_0) = 1$, and $h > 0$, then there exists a monotone, norm bounded, total paranorm N on V such that*

(1) *For every $x \in V$, $N(x) \leqslant 4hN_0(x)$.*

(2) *For every $a \in [0,1]$, $N(ax_0) = ah$.*

Proof. Let N' be any norm on V and let $Q_m = \inf\{2N_0, mN'\}$. Then $\lim_{m\to\infty} Q_m(x_0) = 2$. Thus for some m, $Q_m(x_0) \geqslant 1$. Thus if $Q = Q_m$, then Q is a monotone total paranorm such that $Q(x) \leqslant 2N_0(x)$ for every $x \in V$, Q is norm bounded (by mN'), and $Q(x_0) \geqslant 1$. Now let $a \in [0,1]$. By Lemma 2.4, $Q(ax_0) \geqslant a/2$. Now define a norm P on Rx_0 by $P(ax_0) = |a|$ for every $a \in \boldsymbol{R}$ and let $N_1 = \inf\{P, 2Q\}$. If $\beta \in [0,1]$ and for $a \in \boldsymbol{R}$, $y \in V$, $ax_0 + y = \beta x_0$, then $y = (\beta - a)x_0$. If $|\beta - a| \geqslant 1$, then $2Q((\beta-a)x_0) \geqslant 1$. If $|\beta - a| \leqslant 1$, then $|a| + 2Q((\beta-a)x_0) \geqslant |a| + |\beta - a| \geqslant |\beta|$. Hence $N_1(\beta x_0) \geqslant \beta$. Since N_1 is dominated by P on Rx_0, we have $N_1(\beta x_0) = \beta$. N_1 is also a monotone, norm bounded, total paranorm by Proposition 2.3. Also if $x \in V$, then $N_1(x) \leqslant 2Q(x) \leqslant 4N_0(x)$. Now for $x \in V$, define $N(x) = hN_1(x)$. It is clear that N satisfies the conditions of the proposition.

Proposition 2.6. *Let h, $\varepsilon > 0$. Then there exists an integer m such that if V is an m-dimensional vector space with basis $B = \{v_1, \ldots, v_m\}$ and $v = \sum_{i=1}^m v_i$, then there exists a paranorm N on V such that v is an ε-needle point of height h with respect to N and the basis B.*

Proof. For each integer n, we shall construct a paranorm on $\boldsymbol{R}^n$. For $1 \leqslant i \leqslant n$ we let e_i denote the coordinate vectors of $\boldsymbol{R}^n$ and we let

$e = \sum_{i=1}^{n} e_i$. We will show that for $m = n$ with n suitably large, e is an ε-needle point of height h with respect to the paranorm constructed and the basis $\{e_1, \ldots, e_n\}$. This will prove the proposition since the basis used is irrelevent.

For $a \in (0,1)$, define φ_a on nonnegative real numbers x by $\varphi_a(x) = x^a$. Then φ_a is continuous, concave, monotone increasing, and subadditive. It follows that if $x, y \geqslant 0$, then $\varphi_a(x)+\varphi_a(y) \leqslant 2\varphi_a((x+y)/2) = 2^{1-a}\varphi_a(x+y)$. Thus for every $\delta > 0$, there exists $\delta_0 > 0$ such that if $1-\delta_0 < a < 1$, then

$$\varphi_a(x+y) \leqslant \varphi_a(x)+\varphi_a(y) \leqslant (1+\delta)\varphi_a(x+y). \tag{2.3}$$

Now fix an integer n. Let $E = [\theta, ne_1, \ldots, ne_n]$ and $P = \{\sum_{i=1}^{n} a_i e_i : a_i \geqslant 0\}$. If $x = \sum_{i=1}^{n} a_i e_i \in P$, we define

$$\gamma_a(x) = \frac{1}{n}\sum_{i=1}^{n} \varphi_a(a_i) \quad \text{for} \quad a \in (0,1).$$

Observe that $\gamma_a(x) \leqslant 1$ if $x \in E$. If φ_a satisfies (2.3), then for $x, y \in P$,

$$x = \sum_{i=1}^{n} a_i e_i, \quad \text{and} \quad y = \sum_{i=1}^{n} \beta_i e_i,$$

$$\gamma_a(x+y) = \frac{1}{n}\sum_{i=1}^{n} \varphi_a(a_i+\beta_i) \leqslant \frac{1}{n}\sum_{i=1}^{n} [\varphi_a(a_i)+\varphi_a(\beta_i)]$$

$$= \gamma_a(x)+\gamma_a(y) \leqslant \frac{1}{n}\sum_{i=1}^{n} (1+\delta)\varphi_a(a_i+\beta_i) = (1+\delta)\gamma_a(x+y).$$

Now let M be a positive integer such that $(M-1)\varepsilon > 8h$. Now if $x, y \in P$ and $\gamma_a(x), \gamma_a(y) \leqslant M+1$, then it easily follows from the above that for every $\delta > 0$, there exists $\delta_0 > 0$ such that if $1-\delta_0 < a < 1$ and $\gamma_a(x), \gamma_a(y) \leqslant M+1$ for $x, y \in P$, then $\delta > \gamma_a(x)+\gamma_a(y)-\gamma_a(x+y) \geqslant 0$. Hence it can be easily shown by induction that if $x_1, \ldots, x_{M+1} \in E$, then

$$M\delta \geqslant \sum_{i=1}^{M+1} \gamma_a(x_i) - \gamma_a\Big(\sum_{i=1}^{M+1} x_i\Big) \geqslant 0.$$

Suppose further that $a_i \in [0,1]$ for $1 \leqslant i \leqslant M+1$. Then

$$\Big|\gamma_a\Big(\sum_{i=1}^{M+1} a_i x_i\Big) - \sum_{i=1}^{M+1} a_i\gamma_a(x_i)\Big| \leqslant \Big|\gamma_a\Big(\sum_{i=1}^{M+1} a_i x_i\Big) - \sum_{i=1}^{M+1} \gamma_a(a_i x_i)\Big| +$$

$$+\Big|\sum_{i=1}^{M+1} \gamma_a(a_i x_i) - a_i\gamma_a(x_i)\Big| \leqslant M\delta + \sum_{i=1}^{M+1} |\varphi_a(a_i)-a_i|\gamma_a(x_i)$$

$$\leqslant (M+1)\big(\delta + \max\{|\varphi_a(a_i)-a_i| : 1 \leqslant i \leqslant M+1\}\big).$$

Since $\varphi_a(x)$ converges to x uniformly for x in $[0,1]$, we can obtain $\delta_1 > 0$ such that $\delta_1 < \delta_0$ and if $1-\delta_1 < a < 1$, then

$$\Big|\gamma_a\Big(\sum_{i=1}^{M+1} a_i x_i\Big) - \sum_{i=1}^{M+1} a_i\gamma(x_i)\Big| < 2\delta(M+1). \tag{2.4}$$

Now for $x \in E$ define $\psi_a(x) = \gamma_a(x)e$. Thus $\gamma_a(\psi_a(x)) = \varphi_a(\gamma_a(x))$. Once again since $\varphi_a(x)$ converges to x uniformly for $x \in [0,1]$, we can find $\delta_3 < \delta_2$ such that $\delta_3 > 0$ and if $1-\delta_3 < a < 1$, then

$$\Big|\sum_{i=1}^{M+1} a_i\gamma_a(x_i) - \sum_{i=1}^{M+1} a_i\gamma_a(\psi_a(x_i))\Big| < \delta(M+1). \tag{2.5}$$

Notice that the inequalities (2.4) and (2.5) are dependent only on the pregiven M and δ. Now let $\delta > 0$ be such that $2(M+1)\delta < 1/6$ and $1-\delta_3 < a < 1$. Suppose that for some $a_i, \beta_j \in [0,1]$, $x_i, y_j \in E$ with $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$ with $r+s = M$ we have

$$\sum_{i=1}^{r} a_i(x_i-\psi_a(x_i)) - \sum_{j=1}^{s} \beta_j(y_j-\psi_a(y_j)) = e.$$

Then

$$\sum_{i=1}^{r} a_i x_i + \sum_{j=1}^{s} \beta_j\psi_a(y_j) = \sum_{i=1}^{r} a_i\psi_a(x_i) + \sum_{j=1}^{s} \beta_j y_j + e.$$

But then

$$\gamma_a\Big(\sum_{i=1} a_i x_i + \sum_{j=1}^{s} \beta_j\psi_a(y_j)\Big) \leqslant \sum_{i=1}^{r} a_i\gamma_a(x_i) + \sum_{j=1}^{s} \beta_j\gamma_a(\psi_a(y_j)) + \tfrac{1}{6}$$

$$\leqslant \sum_{i=1}^{r} a_i\gamma_a(\psi_a(x_i)) + \sum_{i=1}^{s} \beta_j\gamma_a(y_j) + \gamma_a(e) + \tfrac{1}{2} - 1$$

$$\leqslant \gamma_a\Big(\sum_{i=1}^{r} a_i\psi_a(x_i) + \sum_{j=1}^{s} \beta_j y_j\Big) + \gamma_a(e) + \tfrac{2}{3} - 1$$

$$\leqslant \gamma_a\Big(\sum_{i=1}^{r} a_i\psi_a(x_i) + \sum_{j=1}^{s} \beta_j y_j + e\Big) + \tfrac{5}{6} - 1.$$

This is a contradiction. Let I denote the identity map on $\boldsymbol{R}^n$. Then $I-\psi_a$ is continuous. Since E is compact, $(I-\psi_a)(E) = K$ is compact. By the above argument, $\min\{m : \sum_{i=1}^{m} a_i x_i = e,\ |a_i| \leqslant 1,\ x_i \in K\} \geqslant M$. Notice that if $\beta \in [0,1]$, then $(I-\psi_a)(\beta e) = (\beta-\beta^a)e \neq \theta$. Thus $\boldsymbol{R}\cdot e \subset \operatorname{span} K$. It is then easy to see that $E \subset \operatorname{span} K$. Therefore K spans $\boldsymbol{R}^n$.

By Proposition 2.2, there exists a monotone, total paranorm N_0 on $\boldsymbol{R}^n$ such that $N_0(x) \leqslant 1$ for every $x \in K$ and $N_0(e) \geqslant M-1$. Now by Proposition 2.5, there exists a monotone, norm bounded, total paranorm N on $\boldsymbol{R}^n$ such that $N \leqslant (4h/N_0(e))N_0$ and $N(\beta e) = \beta h$ for every $\beta \in [0,1]$.

Since all of the above is independent of n, we may choose $m = n$ so that $n^{a-1}h < \varepsilon/2$. Then for $x \in E$,

$$N\big(x - \gamma_a(x)e\big) = N\big(x - \psi_a(x)\big) \leqslant \big(4h/(M-1)\big)N_0\big(x - \psi_a(x)\big)$$
$$\leqslant 4h/(M-1) < \varepsilon/2.$$

Also for $1 \leqslant i \leqslant m$

$$N(ne_i) \leqslant N\big(ne_i - \psi_a(ne_i)\big) + N\big(\psi_a(ne_i)\big) \leqslant \varepsilon/2 + N(n^{a-1}e) < \varepsilon.$$

3. The compact convex set. In this section we shall construct the compact convex set K with no extreme points and the containing topological vector space X which will, in fact, be a Fréchet space. Initially we shall construct a linear metric space which will be a union of finite-dimensional spaces. The construction will be by induction.

Let Y be all functions on $[0, 1]$ which are finite linear combinations of characteristic functions of the form $\chi_{[a,b)}$ where $a, b \in [0, 1]$. If $A \subset [0, 1)$ is a finite disjoint union of intervals of the form $[a, b)$, then define P_A: $Y \to Y$ by $P_A(f) = \chi_A f$ for all $f \in Y$. We let $D = \{f \in Y: f \geqslant 0,$ and $\int f d\mu \leqslant 1\}$, where μ denotes Lebesgue measure on $[0, 1)$. For $f \in Y$, $\|f\|$ will denote the usual $L'\big([0, 1)\big)$ norm, i.e. $\|f\| = \int |f| d\mu$.

Now let $\langle\varepsilon_n\rangle$ be a monotone decreasing sequence of positive numbers such that $\varepsilon_1 = 4$ and $\lim\limits_{n\to\infty} \varepsilon_n = 0$. We shall construct a sequence $\langle\pi_n\rangle$ of partitions of $[0, 1]$ such that each π_n consists of intervals of the form $[a, b)$ with each of equal length and π_{n+1} will refine π_n for every n. We let $V_n = \operatorname{span}\{\chi_A: A \in \pi_n\}$. Thus $V_n \subset V_{n+1}$ for every n. Let $E_n = V_n \cap D$. Each vector space V_n will have a paranorm N_n. We will show that there exists a sequence of positive integers $\langle M_n\rangle$ such that the sequence $\langle(\pi_n, V_n, N_n, M_n)\rangle$ satisfies the following conditions:

(1) Each paranorm N_n is monotone, norm bounded, and total.

(2) If $n \leqslant m$, $f \in V_n$, then $N_m(f) \leqslant N_n(f)$ and $N_n(f) = N_m(f)$ if $N_n(f) \leqslant 4$.

(3) If $A \in \pi_n$, then $\sup\{N_m(f): f \in P_A(E_m)\} < \varepsilon_n$ for every $m \geqslant n$.

(4) For every n, there exists M_n many points $\{g_{ni}\} \subset E_n$ such that if $m \geqslant n$ and B_{ni} denotes the open ε_n-ball in V_m centered at g_{ni}, then $\bigcup\limits_{i=1}^{M_n} B_{ni} \supset E_m$.

(5) If $A \in \pi_n$, then there exist constants $K(A) \geqslant 1$ and $0 < \varepsilon(A) < 1$ such that if $f \in V_m$ for $m \geqslant n$, and $N_m(f) < \varepsilon(A)$, then $N_m\big(P_A(f)\big) \leqslant K(A)N_m(f)$.

PROPOSITION 3.1. *If $\langle\varepsilon_n\rangle$ is a monotone decreasing sequence of positive numbers with $\varepsilon_1 = 4$ and $\lim\limits_{n\to\infty}\varepsilon_n = 0$, then there exists a sequence $\langle(\pi_n, V_n, N_n, M_n)\rangle$ satisfying conditons (1)–(5).*

Proof. The construction is by induction on n. Let π_1 consist of the interval $[0, 1)$, let $N_1(f) = \|f\|$ for $f \in V_1$, and let $M_1 = 1$. $P_{[0,1]}$ is the identity map so we may take $K\big([0, 1)\big) = 1$. Thus (π_1, V_1, N_1, M_1) satisfies (1)–(5).

Now suppose that the finite sequence $\langle\pi_i, V_i, N_i, M_i\rangle$ satisfies conditions (1)–(5) for $1 \leqslant i \leqslant n$. For each $1 \leqslant i \leqslant n$, there exist M_i many points $\{g_{ij}\}_{j=1}^{M_i}$ such that if $f \in E_n$, then for some $1 \leqslant j \leqslant M_i$, $N_n(f - g_{ij}) < \varepsilon_i$. Since E_n is compact, there exists $\delta_1 > 0$ such that for every $f \in E_n$ and $1 \leqslant i \leqslant n$, there exists $1 \leqslant j \leqslant M_i$ such that $N_n(f - g_{ij}) < \varepsilon_i - \delta_1$. For each $A \in \pi_i$ with $1 \leqslant i \leqslant n$, $\sup\{N_n(f): f \in P_A(E_n)\} < \varepsilon_i$. Because $P_A(E_n)$ is compact and because $\bigcup\limits_{i=1}^{n} \pi_i$ is finite, there exists $\delta_2 > 0$ such that if $A \in \pi_i$ with $1 \leqslant i \leqslant n$, then $\sup\{N_n(f): f \in P_A(E_n)\} < \varepsilon_i - \delta_2$.

Lastly since $\sup\{N_n(f): f \in E_n\} < 2$, there exists $\delta_3 > 0$ such that $\sup\{N_n(f): f \in E_n\} < 2 - \delta_3$. Now let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Now let d denote the dimension of V_n and let $\varepsilon = \min\{\delta/2d, \varepsilon_{n+1}/2\}$.

Let $\pi_n = \{A_i: 1 \leqslant i \leqslant d\}$. Each A_i is a half-open interval of length $1/d$. Since N_n is norm bounded, there exists a constant $c > 0$ such that for every $f \in V_n$, $N_n(f) \leqslant c\|f\|$. Thus there exists $h \geqslant 4$ such that if $a \in [0, 1]$, then for every $1 \leqslant i \leqslant d$, $N_n\big(a \cdot d\chi_{A_i}\big) \leqslant ah$. We now choose an integer m satisfying the conditions of Proposition 2.6 for h, $\varepsilon > 0$. Subdivide each A_i into m many intervals $\{A_{ij}, 1 \leqslant j \leqslant m\}$ each of length $1/md$. Let $B_i = \{d\chi_{A_{ij}}\}$ and let $W_i = \operatorname{span} B_i$. By Proposition 2.6, there exists a paranorm N_{ni} on W_i such that $d\chi_{A_i}$ is an ε-needle point of height h with respect to the basis B_i. Let $\beta \geqslant 0$ such that $N_{ni}(\beta d\chi_{A_i}) < 4$. It follows that $\beta < 1$ since if $\beta \geqslant 1$, then $N_{ni}(\beta d\chi_{A_i}) \geqslant N_{ni}(d\chi_{A_i}) = h \geqslant 4$. Thus $N_{ni}(\beta d\chi_{A_i}) = \beta h \geqslant N_n(\beta d\chi_{A_i})$. Hence if $N_{ni}(\beta d\chi_{A_i}) < 4$, then $N_n(\beta d\chi_{A_i}) \leqslant N_{ni}(\beta d\chi_{A_i})$.

Now define $\pi_{n+1} = \{A_{ij}: 1 \leqslant i \leqslant d,\ 1 \leqslant j \leqslant m\}$, and $V_{n+1} = \operatorname{span}\{\chi_A: A \in \pi_{n+1}\}$. Define N_{n+1} on V_{n+1} by $N_{n+1} = \inf\{N_n, N_{n1}, \ldots, N_{nd}\}$.

(1) follows by Proposition 2.1. It is clear that if $f \in V_n$, then $N_{n+1}(f) \leqslant N_n(f)$. Suppose further that $N_n(f) < 4$. To see that $N_{n+1}(f) \geqslant N_n(f)$, let $g \in V_n$, $f_i \in W_i$ such that $g + \sum\limits_{i=1}^{d} f_i = 1$ and we may suppose that $N_{ni}(f_i) < 4$. But now $\sum\limits_{i=1}^{d} f_i = f - g \in V_n$. Thus $f - g = \sum\limits_{i=1}^{d} a_i\chi_{A_i}$ for $a_i \in \boldsymbol{R}$, $1 \leqslant i \leqslant d$. Since the spaces W_i are in direct sum, the functions f_i are uniquely determined. Hence $f_i = a_i\chi_{A_i}$. But as we have already demonstrated $N_{ni}(f_i) \geqslant N_n(f_i)$. Thus

$$N_n(g) + \sum_{i=1}^{d} N_{ni}(f_i) \geqslant N_n(g) + \sum_{i=1}^{d} N_n(f_i) \geqslant N_n(f).$$

Therefore $N_{n+1}(f) = N_n(f)$ and we have verified condition (2).

Suppose that $f\in W_i\cap D = W_i\cap E_{n+1} = [\theta, md\chi_{A_{i1}}, \ldots, md\chi_{A_{im}}]$. Since $d\chi_{A_i}$ is an ε-needle point of height h with respect to N_{ni} and the basis B_i, there exists $\alpha\in[0,1]$ such that $N_{ni}(f-\alpha d\chi_{A_i})<\varepsilon$. Hence $N_{n+1}(f-\alpha d\chi_{A_i}) < \varepsilon$. Also we have $N_{n+1}(md\chi_{A_{ij}}) \leqslant N_{ni}(md\chi_{A_{ij}}) < \varepsilon$ if $A_{ij}\in B_i$. Suppose $f\in E_{n+1}$. Then for $1\leqslant i\leqslant d$, there exist $f_i\in W_i\cap D$, $\alpha_i\geqslant 0$ with $\sum_{i=1}^{d} \alpha_i\leqslant 1$ such that $\sum_{i=1}^{d} \alpha_i f_i = f$. Now let $\beta_i\in[0,1]$ such that $N_{n+1}(f_i-\beta_i d\chi_{A_i}) < \varepsilon$. Thus if $g=\sum_{i=1}^{d} \alpha_i\beta_i d\chi_{A_i}$, then $g\in E_n$ and $N_{n+1}(f-g)\leqslant \sum_{i=1}^{d} N_{n+1}\big(\alpha_i(f_i-\beta_i d\chi_{A_i})\big)\leqslant \sum_{i=1}^{d} N_{n+1}(f_i-\beta d\chi_{A_i}) < d\varepsilon < \delta/2$. For $1\leqslant i\leqslant n$, there exists $1\leqslant j\leqslant M_i$ such that $N_n(g-g_{ij}) < \varepsilon_i-\delta_1$. Thus $N_{n+1}(f-g_{ij})\leqslant N_{n+1}(f-g)+N_{n+1}(g-g_{ij}) < \delta/2+\varepsilon_i-\delta_1<\varepsilon_i$. Hence E_{n+1} satisfies condition (4). Since E_{n+1} is compact in V_{n+1}, there exist M_{n+1} many points $\{g_{(n+1)i}\}$ such that if $f\in E_{n+1}$, then $N_{n+1}(f-g_{(n+1)i})<\varepsilon_{n+1}$ for some $1\leqslant i\leqslant M_{n+1}$.

Now let $A\in\pi_K$ for K such that $1\leqslant K\leqslant n$ and let $f\in P_A(E_{n+1})$. As we have observed earlier there exist $f_i\in W_i\cap D$ and $\alpha_i\geqslant 0$ with $\sum_{i=1}^{d}\alpha_i f_i = f$. We have $\alpha_i=0$ if $A_i\subset A^c$. For each i such that $A_i\subset A$ there exists $\beta_i\in[0,1]$ such that $N_{n+1}(f_i-\beta_i d\chi_{A_i})<\varepsilon$ (take $\beta_i=0$ if $A_i\subset A^c$). Hence $\sum_{i=1}^{d} \alpha_i\beta_i d\chi_{A_i}\in P_A(E_n)$. Also $N_{n+1}\big(f-\sum_{i=1}^{d}\alpha_i\beta_i d\chi_{A_i}\big) < \delta/2$. Thus $N_{n+1}(f) < \delta/2+\sup\{N_n(g): g\in P_A(E_m)\} < \varepsilon_k-\delta+\delta/2$. Hence $\sup\{N_{n+1}(f): f\in P_A(E_{n+1})\}\leqslant \varepsilon_k-\delta+\delta/2<\varepsilon_k$. If $A\in\pi_{n+1}$ and $f\in P_A(E_{n+1})$, then $f=\alpha md\chi_A$ for $\alpha\in[0,1]$. But then $N_{n+1}(f)\leqslant N_{n+1}(md\chi_A)<\varepsilon$. Therefore $\sup\{N_{n+1}(f): f\in P_A(E_{n+1})\}\leqslant\varepsilon<\varepsilon_{n+1}$. Thus condition (3) holds.

Now suppose $A\in\pi_j$ for $1\leqslant j\leqslant n$ and $f\in V_{n+1}$ with $N_{n+1}(f)<\varepsilon(A)<1$. Take $\varepsilon'>0$. Let $g\in V_n$ and $f_i\in W_i$, $1\leqslant i\leqslant d$, be such that $f=g+\sum_{i=1}^{d} f_i$ and $N_{n+1}(f)\geqslant N_n(g)+\sum_{i=1}^{d} N_{ni}(f_i)-\varepsilon'$. Let $\pi_n=\{A_i: 1\leqslant i\leqslant d\}$. Then $P_A(f_i)=f_i$ if $A_i\subset A$ or $P_A(f_i)=0$ if $A_i\subset A^c$. Since $N_n(g)<\varepsilon(A)$ for sufficiently small ε',

$$\begin{aligned} N_{n+1}\big(P_A(f)\big) &\leqslant N_n\big(P_A(g)\big)+\sum_{i=1}^{d} N_{ni}\big(P_A(f_i)\big)\leqslant K(A)N_n(g)+\sum_{i=1}^{d} N_{ni}(f_i)\\ &\leqslant K(A)\Big(N_n(g)+\sum_{i=1}^{d} N_{ni}(f_i)\Big)\leqslant K(A)\big(N_{n+1}(f)+\varepsilon'\big). \end{aligned}$$

Since ε' is arbitrary, $N_{n+1}\big(P_A(f)\big)\leqslant K(A)N_{n+1}(f)$. Now let $A\in\pi_{n+1}$. Since N_{n+1} is norm bounded, there exists a constant $c>0$ such that $N_{n+1}(f)\leqslant c\|f\|$ for every $f\in V_{n+1}$. Since V_{n+1} is finite dimensional, the N_{n+1} topology and the norm topology coincide. Thus there exists $0<\varepsilon(A)<1$ such that if $f\in V_{n+1}$ and $N_{n+1}(f)\leqslant\varepsilon(A)$, then $\|f\|\leqslant 1$ and if $N_{n+1}(f)<\varepsilon(A)$,

then there exists $\beta>1$ such that $N_{n+1}(\beta f)=\varepsilon(A)$. Now let $f\in V_{n+1}$ with $N_{n+1}(f)=\varepsilon(A)$ and let $\alpha\in[0,1]$. Then

$$N_{n+1}\big(P_A(\alpha f)\big)\leqslant c\|\alpha P_A(f)\|\leqslant c\alpha\|f\|\leqslant c\alpha = \big(c\alpha/\varepsilon(A)\big)N_{n+1}(f).$$

By Lemma 2.4, $\alpha N_{n+1}(f)\leqslant 2N_{n+1}(\alpha f)$. Thus if we take $K(A)=\max\{2c/\varepsilon(A),1\}$ and $g\in V_{n+1}$ with $N_{n+1}(g)<\varepsilon(A)$, then

$$N_{n+1}\big(P_A(g)\big)\leqslant K(A)N_{n+1}(g).$$

This completes the proof.

THEOREM. *There exists a Hausdorff topological vector space X which contains a compact convex set K such that K contains no extreme points.*

Proof. Let $\langle\varepsilon_n\rangle$ be a monotone decreasing sequence of positive numbers such that $\varepsilon_1=4$ and $\lim_{n\to\infty}\varepsilon_n=0$ and let $\langle(\pi_n,V_n,N_n,M_n)\rangle$ be the sequence constructed in Proposition 3.1. Let $V=\bigcup_{n=1}^{\infty}V_n$. Then V is a subspace of Y. For any $f\in V_i\subset V$, the sequence $\langle N_n(f)\rangle_{n=i}^{\infty}$ is monotone decreasing and is therefore convergent. Define N_0 on V by $N_0(f)=\lim_{n\to\infty}N_n(f)$. If $f\in V_i$ and $N_i(f)<4$, then $N_0(f)=N_i(f)$. Now let $E=\bigcup_{n=1}^{\infty}E_n$ and let d be the metric given by N_0.

By construction, E can be covered by M_n many open ε_n-balls. Thus E is totally bounded. Now let X be the completion of V and let N be the total paranorm obtained on X. We shall consider V and E as subsets of X. Let F be the closure of E. Since E is totally bounded, F is compact and since E is convex, F is convex. Suppose $A\in\pi_n$ for some n. If $f\in V$ and $N_0(f)<\varepsilon(A)$, then $N_0\big(P_A(f)\big)\leqslant K(A)N_0(f)$. Hence P_A can be extended to a continuous linear operator on X which will also be denoted P_A. Also P_A is a projection on X, since it is a projection when it is restricted to V. It is easily seen that $P_A(E)\subset E$. If we let I denote the identity operator, then $(I-P_A)(E)\subset E$ also. Therefore $P_A(F)\subset F$ and $(I-P_A)(F)\subset F$. It is easily verifed that

$$E=\{\alpha f+(1-\alpha)g:\ f\in P_A(E),\ g\in(I-P_A)(E),\ \alpha\in[0,1]\}.$$

Therefore

$$F=[E]=[P_A(E)\cup(I-P_A)(E)]\subset[P_A(F)\cup(I-P_A)(F)]\subset F,$$

i. e. $F=[P_A(F)\cup(I-P_A)(F)]$. Therefore $F=\{\alpha f+(1-\alpha)g:\ f\in P_A(F),\ g\in(I-P_A)(F)\}$ since $P_A(F)$ and $(I-P_A)(F)$ are compact convex sets. It is also easily seen that $\sup\{N(f): f\in P_A(F)\}\leqslant\varepsilon_n$.

We claim that θ is the only possible extreme point of F. Let f be an extreme point of F. If $A\in\pi_n$, then there exist $\alpha\in[0,1]$ and $f_1 f_2\in F$ such that $f=\alpha P_A(f_1)+(1-\alpha)(I-P_A)(f_2)$. Since $P_A(f_1),(I-P_A)(f_2)\in F$, either $\alpha = 0$ or $\alpha = 1$. Thus either $P_A(f) = f$ or $P_A(f) = \theta$. Let $\pi_n = \{A_i\}_{i=1}^{d}$. Then $I = \sum_{i=1}^{d} P_{A_i}$. Therefore $P_{A_i}(f) = f$ for some i. But then $N(f) = N(P_{A_i}(f)) \leqslant \varepsilon_n$. Since $\lim_{n\to\infty} \varepsilon_n = 0$, $f = \theta$.

Finally we let $K = [F, -F]$. Therefore $K = \{\alpha f-(1-\alpha)g\colon f, g \in F\}$, since F is a compact convex set. Now θ is not an extreme point of K, since $F \subset K$ and if $f \in F$ with $f \neq \theta$, then $\frac{1}{2}f+\frac{1}{2}(-f) = \theta$. Suppose g is an extreme point in K. Then $g = \alpha f_1+(1-\alpha)(-f_2)$ for $f_1, f_2 \in F$. But then either $\alpha = 0$ or $\alpha = 1$. Thus $g \in F$ or $g \in -F$. Without loss of generality, we may suppose $g \in F$. Since $g \neq \theta$, g is not an extreme point of F. Thus K has no extreme points. This completes the proof.

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On the Jordan model of C_0 operators

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Abstract. The aim of Part I of this note is to find a Jordan quasisimilarity model for C_0 operators acting on Hilbert spaces of arbitrary dimensions. In § 1 the "non-separable" quasisimilarity invariants of a C_0 contraction are found. The main result of Part I is Theorem 1 (§ 2) which asserts the existence and the uniqueness of the Jordan model. In our proof we use the existence of the Jordan model, already known for C_0 operators acting on separable Hilbert spaces ([1]).

The second part of this note is a continuation of [3]. We apply the Jordan model of C_0 operators to the problem of classifying the representations of the convolution algebra $L^1(0,1)$. In § 3 the canonical representations of $L^1(0,1)$ are defined and they are shown to be unitarily equivalent to some "obvious" representations. The main result of Part II is Theorem 2 (§ 4) where each representation of $L^1(0,1)$ into a Hilbert space is asserted to be quasisimilar to a unique representation which is the direct (orthogonal) sum of a canonical representation and a trivial representation.

In [2] the C_0 operators acting on a separable Hilbert space were shown to be quasisimilar to Jordan operators. The main result of Part I of this note is to extend this result to arbitrary C_0 operators.

The general result of Part I has been suggested by the particular case used in Part II. In the particular case of nilpotent operators, the problem of the Jordan model is already solved in [1]. The problem of classifying the representations of the convolution algebra $L^1(0,1)$ has been suggested by C. Foiaş.

Preliminaries. (a) Let us recall that C_0 is the class of those completely nonunitary (cnu) contractions T of a Hilbert space, for which there exists a function $u \in H^\infty$, $u \neq 0$, such that $u(T) = 0$. Among the functions u satisfying the relation $u(T) = 0$ there is an inner one which divides all the others. This function, determined up to a scalar multiplicative constant of modulus one, is called the *minimal function* of T and is denoted by m_T. The function m_T is constant if and only if T acts on the trivial space $\{0\}$.

For each nonconstant inner function m there exists an operator T of class C_0 for which $m_T = m$. Such an operator is $S(m)$ acting on $H(m)$