Contents of volume LX, number 3

STUDIA MATHEMATICA

Managing editors: **Z. Ciesielski, W. Orlicz** (*Editor-in-Chief*), **A. Pełczyński, W. Żelazko**

The journal prints original papers in English, French, German and Russian, mainly on functional analysis, abstract methods of mathematical analysis and on the theory of probabilities. Usually 3 issues constitute a volume.

The papers submitted should be typed on one side only and accompanied by abstracts, normally not exceeding 200 words. The authors are requested to send two copies, one of them being the typed, not Xerox copy. Authors are advised to retain a copy of the paper submitted for publication.

Manuscripts and the correspondence concerning editorial work should be addressed to

STUDIA MATHEMATICA
ul. Śniadeckich 8
00-950 Warszawa, Poland

Correspondence concerning exchange should be addressed to

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ul. Śniadeckich 8,
00-950 Warszawa, Poland

The journal is available at your bookseller or at

ARS POLONA
Krakowskie Przedmieście 7
00-068 Warszawa, Poland

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

STUDIA MATHEMATICA, T. LX. (1977)

Approximation of analytic and continuous mappings by polynomials in Fréchet spaces

by

CZESŁAW MATYSZCZYK (Warsaw)

Abstract. In the paper it is shown that a complex or real Fréchet space X with the bounded approximation property has the strong polynomial approximation property (shortly SPAP, see Definition 0.1) if and only if there exists a continuous norm on X. Moreover, any separable real Banach space has the SPAP.

0. Introduction. The following theorem of Oka–Weil is well known:

If K is a compact polynomially convex subset of the space $C^n = C\times\ldots\times C$, then every complex valued function, analytic in a neighbourhood of K, is the uniform limit of a sequence of polynomials on K.

C. E. Rickart [19] has extended this theorem to the space $C^T = \prod_{t\in T} C_t$, where T is an arbitrary set and $C_t = C$ for every $t\in T$; the topology in C^T is product.

Ph. Noverraz [13] (see also, S. Dineen [6], R. Aron and M. Schottenloher [2]) has obtained an analogous theorem for complex Banach spaces with a Schauder basis (in papers [14] and [15] Noverraz gives a proof of the Oka–Weil theorem for spaces with "the strong approximation property").

From the Oka-Weil theorem it follows that the space $X = C^n$ has the property:

(∗) *For every open polynomially convex subset Q of X and for every function U analytic on Q there exists a sequence of polynomials convergent to U almost uniformly on Q, i.e., uniformly on each compact subset of Q.*

In author's paper [12] it is shown that every complex Banach space with the bounded approximation property has the property (∗), too. (A Fréchet space X is said to have the *bounded approximation property*, shortly BAP, if the identity operator on X is the pointwise limit of a sequence of finite-dimensional bounded linear operators [17].)

Since not every Fréchet space with the BAP has the property (∗), we introduce the following

Definition 0.1. Let X be a complex (resp. real) Fréchet space, i.e., a complete locally convex metric space.

(a) We say that X has the *strong polynomial approximation property*, shortly SPAP, if for every open polynomially convex (resp. open) subset Q of X, for every complex (resp. real) Fréchet space Y and for every mapping U analytic (resp. continuous) on Q with values in Y, there exists a sequence of polynomials convergent to U almost uniformly on Q.

(b) We say that X has the *polynomial approximation property*, shortly PAP, if for any Q, Y, U as above and for every compact subset K of Q there exists a sequence of polynomials convergent to U uniformly on K.

In the present paper it is shown that:

T1. *A complex or real Fréchet space X with the* BAP *has the* SPAP *if and only if there exists a continuous norm on X.*

T2. *Any separable real Banach space has the* SPAP.

T3. *Any complex Fréchet space with the Grothendieck approximation property has the* PAP (a Fréchet space X is said to have the *Grothendieck approximation property*, shortly GAP, if for every compact subset K of X there is a sequence of finite-dimensional bounded linear operators, convergent to the identity operator uniformly on K [8]).

T4. *If Q is an open polynomially convex subset of C^N, where N is the set of all positive integers, then the following conditions are equivalent:*

(i) *any complex valued function, analytic on Q, is the almost uniform limit of a sequence of polynomials* on Q,

(ii) *Q consists of a finite number of connected components.*

The problem of the approximation of continuous mappings by polynomials was investigated by S. Mazur around 1955. He obtained the following result (unpublished):

Let X, Y be two real Fréchet spaces, M a closed separable subset of X and let U be a continuous mapping from M to Y.

(a) If $M = \bigcup_{n=1}^{\infty} M_n$, where M_n are bounded, then U is the pointwise limit of a sequence of polynomials on M.

(b) If M is locally bounded, then U is the almost uniform limit of a sequence of polynomials.

S. Mazur has posed the problem: is there a real Fréchet (but not Banach) space X such that every continuous function on X is the limit of a sequence of polynomials on the whole space X? Theorem T1 (the real case) is the solution to this problem.

The paper is divided into two parts. In the first part we give some notation and some basic properties of polynomially convex subsets of Fréchet spaces. In the second part we give the main results of this paper.

The paper is a part of author's Ph. D. Th. written under the supervision of Professor W. Żelazko whom we would like to thank for his

guidance. We would like also to express our gratitude to Professor S. Mazur for introducing us to the subject and for making available for us his unpublished results. Finally we are indebted to Dr. E. Ligocka and Professor A. Pełczyński for valuable discussions.

1. Preliminaries. Throughout the paper $\boldsymbol{K}$ denotes either the field of real numbers $\boldsymbol{R}$ or the field of complex numbers $\boldsymbol{C}$. If the field is not explicitly indicated, the results are valid in both cases.

Let X, Y be two Fréchet spaces over $\boldsymbol{K}$. We denote by $\mathscr{P}[X, Y]$ the space of all continuous polynomials on X with values in Y, and by $\mathscr{N}[X, Y]$ the subspace of $\mathscr{P}[X, Y]$ defined as follows: $P \in \mathscr{N}[X, Y]$ if and only if there exist a positive integer n, elements $x_1^*, \ldots, x_n^*$ in the topologically conjugate space X^* and a polynomial $\hat{P} \in \mathscr{P}[\boldsymbol{K}^n, Y]$ such that $P(x) = \hat{P}(x_1^*(x), \ldots, x_n^*(x))$ for $x \in X$. If X has a Schauder basis, then, for a given basis $e = (e_i)$, we denote by $\mathscr{N}_e[X, Y]$ the space of all polynomials of the form $P(x) = \hat{P}(e_1^*(x), \ldots, e_n^*(x))$, where $\hat{P} \in \mathscr{P}[\boldsymbol{K}^n, Y]$, e_i^* are the coordinate functionals and $n = 1, 2, \ldots$ The space of all analytic (resp. continuous) mappings from an open subset Q of X to Y is denoted by $\mathscr{A}[Q, Y]$ (resp. $\mathscr{C}[Q, Y]$). The most important facts concerning polynomials and analytic mappings are gathered in papers [4], [5].

Assume now that the spaces X, Y are complex, $(\| \|_k)$ is a sequence of seminorms determining the topology on Y (if Y is a Banach space with a norm $\| \|$, we put $\| \|_k = \| \|$ for $k = 1, 2, \ldots$), Q is an open subset of X, $\mathscr{F}$ is a subfamily of $\mathscr{C}[Q, Y]$.

DEFINITION 1.1. The subset Q is said to be *$\mathscr{F}$-convex* if for any compact subset K of Q the set

$$K_{\mathscr{F}}^Q = \{x \in Q\colon \|F(x)\|_k \leqslant \sup_{y \in K} \|F(y)\|_k \text{ for each } F \in \mathscr{F} \text{ and } k = 1, 2, \ldots\}$$

is a compact subset of Q, too.

Obviously, if $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \mathscr{C}[Q, Y]$ and Q is $\mathscr{F}_1$-convex, then Q is $\mathscr{F}_2$-convex.

PROPOSITION 1.2. *Let $\mathscr{F}' = \{f \in \mathscr{C}[Q, C]\colon f = y^* \circ F$ for some $F \in \mathscr{F}$ and $y^* \in Y^*\}$. If for every $f \in \mathscr{F}'$ there exists $y \in Y$, $y \neq \theta$, such that $yf \in \mathscr{F}$ $((yf)(x) = yf(x))$, then $K_{\mathscr{F}}^Q = K_{\mathscr{F}'}^Q$ for any compact subset K of Q.*

Proof. We first show that $K_{\mathscr{F}}^Q$ is contained in $K_{\mathscr{F}'}^Q$. Let $x_0 \in K_{\mathscr{F}}^Q$. If $f \in \mathscr{F}'$, then there is $\theta \neq y \in Y$ such that $yf \in \mathscr{F}$, therefore $\|yf(x_0)\|_k \leqslant \sup_{x \in K} \|yf(x)\|_k$. Since $\|y\|_k \neq 0$ for some k and $\|yf(x)\|_k = \|y\|_k |f(x)|$, we can write $|f(x_0)| \leqslant \sup_{x \in K} |f(x)|$. Hence $x_0 \in K_{\mathscr{F}'}^Q$.

Let now $x_0 \in K_{\mathscr{F}'}^Q$. For every $F \in \mathscr{F}$ and for every k there is $y^* \in Y^*$ such that $|y^*(F(x_0))| = \|F(x_0)\|_k$ and $|y^*(x)| \leqslant \|x\|_k$ for $x \in Y$. From this we have $\|F(x_0)\|_k \leqslant \sup_{x \in K} |y^*(F(x))| \leqslant \sup_{x \in K} \|F(x)\|_k$, hence $x_0 \in K_{\mathscr{F}}^Q$.

Corollary 1.3. *The subset Q is $\mathscr{A}[Q, Y]$, $\mathscr{P}[X, Y]$ or $\mathscr{N}[X, Y]$-convex ($\mathscr{N}_e[X, Y]$-convex, if X has a basis) if and only if Q is $\mathscr{A}[Q, C]$, $\mathscr{P}[X, C]$ or $\mathscr{N}[X, C]$-convex ($\mathscr{N}_e[X, C]$-convex), respectively.*

Definition 1.4. $\mathscr{P}[X, C]$-convex (resp. $\mathscr{A}[Q, C]$-convex) subsets of X are called *polynomially* (resp. *analytically*) *convex.*

Lemma 1.5. *If $\mathscr{F}$ is a subfamily of $\mathscr{C}[Q, C]$ such that $z+x^*\in\mathscr{F}$ for any $z\in C$ and $x^*\in X^*$ ($(z+x^*)(x) = z+x^*(x)$, $x\in X$), then $K^Q_{\mathscr{F}}$ is contained in $\overline{\operatorname{conv} K}$ for each compact subset K of Q.*

Proof. Let $x_0\notin\overline{\operatorname{conv}K}$. Then there is $x^*\in X^*$ such that $x^*(x_0)\notin Z = \overline{x^*(\operatorname{conv}K)}$ (cf. [20]). Since Z is a convex compact subset of C, there exists a constant $c\in C$ such that $\sup_{z\in Z}|z+c| < |x^*(x_0)+c|$. But $\sup_{x\in K}|x^*(x)+c| \leqslant \sup_{z\in Z}|z+c|$, and this implies $x_0\notin K^Q_{\mathscr{F}}$.

Corollary 1.6. *Open convex subsets of a Fréchet space are polynomially and analytically convex.*

So-called polynomial polyhedrons in Fréchet spaces are also polynomially and analytically convex.

In the next section we use frequently the following

Lemma 1.7. *Let X, E be complex Fréchet spaces.*

(i) *If E is a subspace of X, then for every open $\mathscr{P}[X, C]$-convex subset Q of X the intersection $Q\cap E$ is $\mathscr{P}[E, C]$-convex.*

(ii) *If $E = L(X)$, where L is a bounded linear operator from X to E, then for every open $\mathscr{P}[E, C]$-convex subset D of E the set $L^{-1}(D)$ is $\mathscr{P}[X, C]$-convex.*

(iii) *If $L\colon X\to E$ is a topological isomorphism (i.e., a bicontinuous algebraic isomorphism) between X and E, then an open subset Q of X is $\mathscr{P}[X, C]$-convex if and only if the set $L(Q)$ is $\mathscr{P}[E, C]$-convex.*

Proof. (i) If Q satisfies the assumption and K is a compact subset of $Q\cap E$, then $K^Q_{\mathscr{P}[X,C]}$ is compact. But $K^{Q\cap E}_{\mathscr{P}[E,C]}$ is a subset of the last set and it is closed in this set, therefore $K^{Q\cap E}_{\mathscr{P}[E,C]}$ is compact.

(ii) Let K be a compact subset of $L^{-1}(D)$, let $K_1 = K^{L^{-1}(D)}_{\mathscr{P}[X,C]}$ and $K_2 = L(K)^D_{\mathscr{P}[E,C]}$. We must show that K_1 is compact. First we show that $L(K_1)$ is contained in K_2.

Let $y_0\in L(K_1)$, i.e., $y_0 = L(x_0)$ for some $x_0\in K_1$. If $P\in\mathscr{P}[E, C]$, then $P\circ L\in\mathscr{P}[X, C]$. Hence we have $|P(y_0)| = |P(L(x_0))| \leqslant \sup_{x\in K}|P(L(x))| = \sup_{y\in L(K)}|P(y)|$, and this implies $y_0\in K_2$. Now we can write $K_1\subset \overline{\operatorname{conv}K}\cap L^{-1}(K_2)$. Since the set on the right-hand side is a compact subset of X, K_1 is compact.

(iii) follows from (ii).

Corollary 1.8. *If Fréchet spaces X, E are topologically isomorphic and X has the SPAP, then E has this property, too.*

2. Main results. For a Fréchet space X over $\boldsymbol{K}$ with a Schauder basis, say $e = (e_i)$, we shall use the following notation:

$X_n = \operatorname{span}\{e_1, \ldots, e_n\}$, $n = 1, 2, \ldots$;

$I_n(z_1e_1+\ldots+z_ne_n) = (z_1, \ldots, z_n)$ for $z_1, \ldots, z_n\in\boldsymbol{K}$, $n = 1, 2, \ldots$ (I_n is a topological isomorphism between X_n and $\boldsymbol{K}^n$);

$\|\ \|_0$ is a Fréchet norm defining the topology on X such that

$$\|z_1e_1+\ldots+z_ne_n\|_0 \leqslant \|z_1e_1+\ldots+z_{n+p}e_{n+p}\|_0 \tag{2.1}$$

for $z_1, \ldots, z_{n+p}\in\boldsymbol{K}$, $n, p = 1, 2, \ldots$ ([1]);

$S_n(x) = \sum_{i=1}^{n} e_i^*(x)e_i$ for $x\in X$ and $n = 1, 2, \ldots$, where e_i^* are the coordinate functionals.

The operators $S_n\colon X\xrightarrow{\text{onto}} X_n$ are continuous linear projections and $S_n(x)$ converges to x for every $x\in X$. Thus, according to the Banach–Steinhaus principle, they are equicontinuous on X and the convergence is almost uniform.

Inequality (2.1) implies

$$\|S_n(x)\|_0 \leqslant \|x\|_0 \quad \text{for} \quad x\in X \text{ and } n = 1, 2, \ldots, \tag{2.2}$$

and this implies

$$\|e_n^*(x)e_n\|_0 \leqslant 2\|x\|_0 \quad \text{for} \quad x\in X \text{ and } n = 1, 2, \ldots \tag{2.3}$$

Definition 2.1. We shall say that a subset M of a Fréchet space X over $\boldsymbol{K}$, with a basis $e = (e_i)$, is *locally e^*-bounded*, if for every point x in M there exists a neighbourhood V_x of x such that the set $e_i^*(M\cap V_x)$ is bounded, as a subset of $\boldsymbol{K}$, for $i = 1, 2, \ldots$

It is evident that the whole space X is locally e^*-bounded if and only if for some neighbourhood V of θ the sets $e_i^*(V)$ are bounded.

The following theorem is fundamental in our study.

Theorem 2.2. *Let X, Y be two real (resp. complex) Fréchet spaces and let X have a Schauder basis, say, $e = (e_i)$, $i = 1, 2, \ldots$ Then for every open (resp. open polynomially convex) subset Q of X, for every locally e^*-bounded subset M of Q and for every mapping $U\in\mathscr{C}[Q, Y]$ (resp. $U\in\mathscr{A}[Q, Y]$) there exists a sequence of polynomials $P_m\in\mathscr{N}_e[X, Y]$ convergent almost uniformly to U on M.*

Proof. Denote by W the set of all linear combinations $z_1e_1+\ldots+z_ne_n$, $n = 1, 2, \ldots$, with rational (resp. complex rational) coefficients. Obviously, this set is dense in X. If Q is a non-empty open subset of X

and M is a non-empty locally e^*-bounded subset of Q, then we can find closed balls $B_j = \{x \in X: \|x - x_j\|_0 \leqslant r_j,\ r_j > 0\}$ such that:

$$x_j \in W \quad \text{and} \quad B_j \subset Q \quad \text{for } j = 1, 2, \ldots; \qquad M \subset \bigcup_{j=1}^{\infty} \operatorname{Int} B_j;$$

(2.4) *the set* $e_i^*(M \cap B_j)$ *is bounded for* $i, j = 1, 2, \ldots$

Now put $Q_m = \bigcup_{j=1}^{m} B_j$ for $m = 1, 2, \ldots$ We shall show that the sequence (Q_m) has the following properties:

(i) $M \subset \bigcup_{m=1}^{\infty} Q_m \subset Q$, Q_m is closed and $Q_m \subset Q_{m+1}$ for $m = 1, 2, \ldots$;

(ii) for any compact subset K of M there is an index m_0 such that $K \subset Q_m$ for all $m \geqslant m_0$;

(iii) for any m there is a positive integer $n_m \geqslant m$ such that $S_n(Q_m) = Q_m \cap X_n$ for all $n \geqslant n_m$.

Properties (i) and (ii) are obvious. To prove property (iii), it is enough to show that for any j there is n_j such that $S_n(B_j) = B_j \cap X_n$ for $n \geqslant n_j$, because $S_n(Q_m) = \bigcup_{j=1}^{m} S_n(B_j)$ for $n, m = 1, 2, \ldots$

For any fixed j the centre x_j of B_j belongs to some subspace X_{n_j}. If $n \geqslant n_j$ and $y_0 \in S_n(B_j)$, i.e., $y_0 = S_n(x_0)$ for some $x_0 \in B_j$, then $y_0 - x_j = S_n(x_0 - x_j)$ and, recalling (2.2), we have $y_0 \in B_j$. The inclusion $B_j \cap X_n \subset S_n(B_j)$ is evident.

Let now $U \in \mathscr{C}[Q, Y]$ ($U \in \mathscr{A}[Q, Y]$, here Q is polynomially convex) and $G_m = I_{n_m}(Q \cap X_{n_m})$, $m = 1, 2, \ldots$ Then for each m the set G_m is a non-empty open (resp. open polynomially convex, by Lemma 1.7) subset of $\boldsymbol{R}^{n_m}$ (resp. $\boldsymbol{C}^{n_m}$); moreover, the vector valued function $U(z_1 e_1 + \ldots + z_{n_m} e_{n_m})$ of the variables $z_1, \ldots, z_{n_m}$ is well defined and continuous (resp. analytic) on G_m.

Further, denote by K_m the closure of $I_{n_m}(S_{n_m}(Q_m \cap M))$, $m = 1, 2, \ldots$ It follows from (i), (iii) and (2.4) that K_m is a bounded, and consequently a compact subset of G_m. Therefore, by the theorem of Bernstein [18] (resp. of Oka–Weil(1)) for vector valued functions, for every m there exists a polynomial $\hat{P}_m \in \mathscr{P}[\boldsymbol{R}^{n_m}, Y]$ (resp. $\hat{P}_m \in \mathscr{P}[\boldsymbol{C}^{n_m}, Y]$) such that

$$\|U(z_1 e_1 + \ldots + z_{n_m} e_{n_m}) - \hat{P}_m(z_1, \ldots, z_{n_m})\| \leqslant m^{-1} \quad \text{for} \quad (z_1, \ldots, z_{n_m}) \in K_m.$$

(Here $\|\ \|$ denotes a Fréchet norm defining the topology on Y.) From the above inequality we obtain

$$\|U(S_{n_m}(x)) - \hat{P}_m(e_1^*(x), \ldots, e_{n_m}^*(x))\| \leqslant m^{-1} \quad \text{for} \quad x \in Q_m \cap M \text{ and } m = 1, 2, \ldots \tag{2.5}$$

(1) This theorem may be deduced from the integral formula of Weil for vector valued functions.

Now consider the sequence (P_m), where $P_m(x) = \hat{P}_m(e_1^*(x), \ldots, e_{n_m}^*(x))$, $x \in X$ and $m = 1, 2, \ldots$ It is clear that $P_m \in \mathscr{N}_e[X, Y]$ for $m = 1, 2, \ldots$ We shall prove that this sequence converges to U almost uniformly on M.

If K is a compact subset of M, then according to property (ii) there is m_0 such that $K \subset Q_m \cap M$ for $m \geqslant m_0$. By (2.5), $\|U(S_{n_m}(x)) - P_m(x)\| \leqslant m^{-1}$ for $x \in K$ and $m \geqslant m_0$. Since $\|U(x) - P_m(x)\| \leqslant \|U(x) - U(S_{n_m}(x))\| + \|U(S_{n_m}(x)) - P_m(x)\|$ for $x \in K$ and $m = 1, 2, \ldots$, in order to finish our proof it suffices to show that

$$U(S_{n_m}(x)) \to U(x) \text{ uniformly on } K. \tag{$*$}$$

Let $x_m \in K$ for $m = 1, 2, \ldots$ and let $x_m \to x_0$. From the uniform convergence of the sequence (S_n) on K and the continuity of U, it follows that $U(S_{n_m}(x_m)) \to U(x_0)$. This proves $(*)$, and the proof is complete.

Setting in the last theorem $M = Q$, we obtain

COROLLARY 2.3. *If a Fréchet space X with a Schauder basis is locally e^*-bounded for some basis e, then X has the* SPAP.

The space $\boldsymbol{K}^N$ of all sequences $x = (z_k)$, $z_k \in \boldsymbol{K}$ for $k = 1, 2, \ldots$, is not locally e^*-bounded. Here $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots), \ldots$, and the topology on $\boldsymbol{K}^N$ is defined by the Fréchet norm $\|x\| = \sum_{k=1}^{\infty} 2^{-k}|z_k|(1 + |z_k|)^{-1}$. We shall prove that this space has not the SPAP.

It is known that if $P \in \mathscr{P}[\boldsymbol{K}^N, \boldsymbol{K}]$, then there is a positive integer n and a polynomial $\hat{P} \in \mathscr{P}[\boldsymbol{K}^n, \boldsymbol{K}]$ such that $\hat{P}(x) = \hat{P}(z_1, z_2, \ldots, z_n)$, $x = (z_k) \in \boldsymbol{K}^N$.

Let $P_n \in \mathscr{P}[\boldsymbol{K}^N, \boldsymbol{K}]$, $n = 1, 2, \ldots$ According to this remark, for every n we can find j_n and $\hat{P}_n \in \mathscr{P}[\boldsymbol{K}^{j_n}, \boldsymbol{K}]$ such that $P_n(x) = \hat{P}_n(z_1, \ldots, z_{j_n})$ and $\hat{P}_n \notin \mathscr{P}[\boldsymbol{K}^{j_n - 1}, \boldsymbol{K}]$.

LEMMA 2.4. *If the sequence (j_n) is not bounded, then the sequence (P_n) is not convergent almost uniformly in any ball $B(x_0, R) = \{x \in \boldsymbol{K}^N: \|x - x_0\| < R\}$, $x_0 \in \boldsymbol{K}^N$, $R > 0$.*

Proof. It is enough to show that for every ball $B(x_0, R) \subset \boldsymbol{K}^N$ there exist a sequence (x_m) and a subsequence (P_{n_m}) of (P_n) such that $x_m \to x_0$, $x_m \in B(x_0, R)$ and $|P_{n_m}(x_m)| > m$ for $m = 1, 2, \ldots$ Put $x_0 = (z_k^0)$ and $B_m = B(x_0, Rm^{-1})$ for $m = 1, 2, \ldots$ Then for every m there are a real number $r_m > 0$ and a positive integer i_m such that the set $M_m = \{x = (z_k) \in \boldsymbol{K}^N: |z_k - z_k^0| < r_m \text{ for } k = 1, \ldots, i_m\}$ is contained in B_m. If (j_{n_m}) is a subsequence of (j_n) such that $j_{n_m} > i_m$ for $m = 1, 2, \ldots$, then for any m there is $x_m \in M_m$ such that $|P_{n_m}(x_m)| > m$, since the polynomial $P^{n_m}(x) = \hat{P}_{n_m}(z_1, \ldots, z_{j_{n_m}})$ is unbounded on M_m. It is evident that $x_m \to x_0$ as $m \to \infty$.

DEFINITION 2.5. Let F be a function defined on a subset Q of $\boldsymbol{K}^N$. We say that F is *globally finitely determined on* Q if there is a positive integer n such that for every $x \in Q$ this function is constant on the set $Q \cap S_n^{-1}(S_n(x))$, where $S_n(x) = (z_1, \ldots, z_n, 0, 0, \ldots)$ for $x = (z_k)$.

From Lemma 2.4, it easily follows

COROLLARY 2.6. *If a function $U: Q \to \boldsymbol{K}$, where Q is an open subset of $\boldsymbol{K}^N$, is the almost uniform limit of a sequence of polynomials, then U is globally finitely determined on Q. In particular, if U is analytic on Q, then U is locally finitely determined.*

THEOREM 2.7. *If a real (resp. complex) Fréchet space X contains a subspace E topologically isomorphic to $\boldsymbol{R}^N$ (resp. $\boldsymbol{C}^N$), then for every real (resp. complex) Fréchet space Y there exists an open (resp. open polynomially convex) subset Q of X and a mapping $U \in \mathscr{A}[Q, Y]$ which is not the almost uniform limit of any polynomial sequence; in particular, X has not the* SPAP.

Proof. Suppose that X is complex. Let G be an open polynomially convex subset of $\boldsymbol{C}^N$ consisting of infinitely many connected components G_n, $n = 1, 2, \ldots$ To show that such a subset G exists it suffices to choose such an open polynomially convex set D in $\boldsymbol{C}$ (D is polynomially convex if and only if $\overline{\boldsymbol{C}} - D$ is connected) and put $G = S^{-1}(D)$, where $S(x) = z_1$ as $x = (z_k)$. Then, by (ii) of Lemma 1.7, G is polynomially convex.

Setting $u(x) = z_n$ for $x = (z_k) \in G_n$, $n = 1, 2, \ldots$, we have $u \in \mathscr{A}[G, \boldsymbol{C}]$. Since u is not globally finitely determined on G, according to Corollary 2.6, u is not the almost uniform limit of any sequence of polynomials on G.

If a subspace E of X is topologically isomorphic to $\boldsymbol{C}^N$, then there exists a continuous linear projection $P: X \xrightarrow{\text{onto}} E$ (cf. C. Bessaga and A. Pełczyński [3]). Now put $Q = P^{-1}(I(G))$, where I is a topologically isomorphic mapping from $\boldsymbol{C}^N$ onto E, $\hat{U}(x) = u(I^{-1}(P(x)))$ for $x \in Q$, and $U(x) = y\hat{U}(x)$ for $x \in Q$; $y \neq \theta$ is a fixed point in Y. It can easily be shown that Q and U have the desired properties.

The above method holds equally well for real spaces. ■

Remark 2.8. If a real Fréchet space X contains a subspace E topologically isomorphic to $\boldsymbol{R}^N$, then for every real Fréchet space Y there is a mapping $F \in \mathscr{C}[X, Y]$ which is not even the pointwise limit of any polynomial sequence in any open subset of X. This follows from the fact that the functional $|\,|$ is not such a limit in any open subset of $\boldsymbol{R}^N$ (A. Pełczyński [18]).

THEOREM 2.9. *Let X be a complex (resp. real) Fréchet space with a Schauder basis, let $e = (e_i)$ be any such basis in X, (p_k) an arbitrary sequence of seminorms defining the topology on X and $k_i = \min\{k: p_k(e_i) \neq 0\}$, $i = 1, 2, \ldots$ Then the following properties are equivalent:*

(i) *X has the* SPAP;

(ii) *no subspace of X is topologically isomorphic to the space $\boldsymbol{C}^N$ (resp. $\boldsymbol{R}^N$);*

(iii) *there is a continuous norm on X;*

(iv) *the topology on X may be defined by a sequence of norms;*

(v) *X is locally e^*-bounded;*

(vi) *the sequence (k_i) is bounded.*

Proof. The implications (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) hold for arbitrary Fréchet spaces (cf. C. Bessaga and A. Pełczyński [3]).

(v) $\Rightarrow$ (i) follows from Theorem 2.2, and (i) $\Rightarrow$ (ii) follows from Theorem 2.7.

(iii) $\Rightarrow$ (vi). If $\|\;\|$ is a continuous norm on X, then there exists a positive real number M and a positive integer k_0 such that $\|x\| \leqslant M \sup_{k \leqslant k_0} p_k(x)$ for $x \in X$. Hence $k_i \leqslant k_0$ for $i = 1, 2, \ldots$

(vi) $\Rightarrow$ (v). If the sequence (k_i) is bounded, then there is a positive real number C such that $2^{-k_i} \geqslant C$ for $i = 1, 2, \ldots$ Let $|\!|x|\!| = \sum_{k=1}^{\infty} 2^{-k} p_k(x) \big(1 + p_k(x)\big)^{-1}$, $B = \{x \in X: |\!|x|\!| < R,\ 0 < R < C\}$, let r be a positive real number such that the set $\{x \in X: |\!|x|\!|_0 < 2r\}$ is contained in B, and write $V = \{x \in X: |\!|x|\!|_0 < r\}$. We shall show that $e_i^*(V)$ is bounded for $i = 1, 2, \ldots$

Suppose that there is i_0 such that $e_{i_0}^*(V)$ is unbounded. Then for every n there exists $x_n \in V$ such that $|e_{i_0}^*(x_n)| \geqslant n$. From this and from (2.3) we get the inequality

$$C > R > |\!|e_{i_0}^*(x_n)\, e_{i_0}|\!| \geqslant 2^{-k_{i_0}} n p_{k_{i_0}}(e_{i_0}) \big(1 + n p_{k_{i_0}}(e_{i_0})\big)^{-1}$$

for $n = 1, 2, \ldots$ Letting $n \to \infty$, we obtain $C > R \geqslant 2^{-k_{i_0}} \geqslant C$. ■

COROLLARY 2.10 *If $\hat{e} = (\hat{e}_i)$, $\hat{\hat{e}} = (\hat{\hat{e}}_i)$ are two Schauder bases in a Fréchet space X and if X is locally $\hat{e}^*$-bounded, then X is locally $\hat{\hat{e}}^*$-bounded.*

Denote by $\mathfrak{M}$ the class of all Fréchet spaces with a continuous norm. To extend Theorem 2.9 on Fréchet spaces belonging to $\mathfrak{M}$ which have BAP we need the following result of Pełczyński (announced in [17]) who communicated to us the proof presented below.

THEOREM 2.11. *If a Fréchet space X has* BAP, *then there exists a Fréchet space X_0 with a basis and an isomorphic embedding $I: X \to X_0$ such that $I(X)$ is a complemented subspace of X_0.*

Moreover, if $X \in \mathfrak{M}$, then X_0 may be chosen from $\mathfrak{M}$.

Proof. Pick a system of pseudonorms $(\|\cdot\|_k)$ defining the topology of X so that $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \ldots$ (if $X \in \mathfrak{M}$, one may also assume that all the $\|\cdot\|_k$'s are norms). Let (A_p) be a sequence of finite-dimensional continuous linear operators from X to X such that $A_p \neq \theta$ for $p = 1, 2, \ldots$ and

(2.6) $$\lim_n \Big\| x - \sum_{p=1}^{n} A_p(x) \Big\|_k = 0 \quad \text{for} \quad x \in X \text{ and } k = 1, 2, \ldots.$$

By the Banach–Steinhaus principle the operators $\sum_{p=1}^{n} A_p$, $n = 1, 2, \ldots$, are equicontinuous on X. Thus for every k there is a positive number M_k and a positive integer i_k such that

(2.7) $$\sup_n \Big\| \sum_{p=1}^{n} A_p(x) \Big\|_k \leqslant M_k \|x\|_{i_k} \quad \text{for} \quad x \in X.$$

This implies

(2.8) $$\sup_n \|A_n(x)\|_k \leqslant 2M_k \|x\|_{i_k} \quad \text{for} \quad x \in X \text{ and } k = 1, 2, \ldots$$

Put $E_p = A_p(X)$ and $m_p = \dim E_p$ for $p = 1, 2, \ldots$ For every p there are one-dimensional bounded linear operators $B_j^{(p)}$ from E_p to E_p, $j = 1, 2, \ldots, m_p$, such that

(2.9) $$\sum_{j=1}^{m_p} B_j^{(p)}(e) = e \quad \text{for} \quad e \in E_p.$$

Choose a positive number R_p such that

(2.10) $$\max_{1 \leqslant j \leqslant m_p} \|B_j^{(p)}(e)\|_k \leqslant R_p \|e\|_k \quad \text{for} \quad e \in E_p \text{ and } k = 1, 2, \ldots, p.$$

If a positive integer N_p satisfies the inequality

(2.11) $$m_p R_p N_p^{-1} \leqslant 1$$

and if $C_i^{(p)} = N_p^{-1} B_j^{(p)}$ for $i = rm_p + j$ $(r = 0, 1, \ldots, N_p - 1;\ j = 1, \ldots, m_p)$, then

(2.12) $$\sum_{i=1}^{m_p N_p} C_i^{(p)}(e) = e \quad \text{and} \quad \max_{1 \leqslant q \leqslant m_p N_p} \Big\| \sum_{i=1}^{q} C_i^{(p)}(e) \Big\|_k \leqslant 2\|e\|_k$$

for $e \in E_p$ and $k = 1, 2, \ldots, p$.

The last inequality follows from (2.9), (2.10), (2.11) and the fact that for every q, $1 \leqslant q \leqslant m_p N_p$,

$$\sum_{i=1}^{q} C_i^{(p)} = r \sum_{j=1}^{m_p} N_p^{-1} B_j^{(p)} + \sum_{j=1}^{w} N_p^{-1} B_j^{(p)}$$

for some r and w, $0 \leqslant r \leqslant N_p - 1$, $1 \leqslant w \leqslant m_p$.

Let $\hat{A}_s = C_i^{(p)} A_p$ for $s = m_0 N_0 + \ldots + m_{p-1} N_{p-1} + i$, where $m_0 N_0 = 0$, $p = 1, 2, \ldots$, $i = 1, 2, \ldots, m_p N_p$. We shall show that the operators $\sum_{s=1}^{n} \hat{A}_s$, $n = 1, 2, \ldots$, are equicontinuous. For every $n \geqslant m_1 N_1$ there are positive integers t and q, $1 \leqslant q \leqslant m_{t+1} N_{t+1}$, such that

$$\sum_{s=1}^{n} \hat{A}_s = \sum_{p=1}^{t} \sum_{i=1}^{m_p N_p} C_i^{(p)} A_p + \sum_{i=1}^{q} C_i^{(t+1)} A_{t+1}.$$

Let k be fixed. From (2.12), (2.7) and (2.8), it follows that

$$\Big\| \sum_{s=1}^{n} \hat{A}_s(x) \Big\|_k \leqslant \Big\| \sum_{p=1}^{t} A_p(x) \Big\|_k + 2\|A_{t+1}(x)\|_k \leqslant M_k \|x\|_{i_k} + 4\|x\|_{i_k}$$

for $x \in X$ and for almost all n. It hence follows that $\sum_{s=1}^{n} \hat{A}_s$ are equicontinuous. Consequently, for every k there are positive integers K_k and u_k such that

(2.13) $$\sup_n \Big\| \sum_{s=1}^{n} \hat{A}_s(x) \Big\|_k \leqslant K_k \|x\|_{u_k} \quad \text{for} \quad x \in X.$$

From (2.13), (2.12) and (2.6), it follows that

$$\lim_n \Big\| \sum_{s=1}^{n} \hat{A}_s(x) - x \Big\|_k = \lim_t \Big\| \sum_{p=1}^{t} \sum_{i=1}^{m_p N_p} C_i^{(p)} A_p(x) - x \Big\|_k = 0$$

for $x \in X$ and $k = 1, 2, \ldots$

Now denote by X_0 the Fréchet space consisting of all sequences $(y(s))$ such that $y(s) \in \hat{A}_s(X)$ for $s = 1, 2, \ldots$ and the series $\sum_{s=1}^{\infty} y(s)$ converges; the topology on X_0 is defined by the Fréchet norm $\| \ \| = \sum_{k=1}^{\infty} 2^{-k} ||| \ |||_k (1 + ||| \ |||_k)^{-1}$, where $|||(y(s))|||_k = \sup_n \| \sum_{s=1}^{n} y(s)\|_k$ for $(y(s)) \in X_0$. Obviously, if $\|\cdot\|_k$ are continuous norms, so are the $|||\cdot|||_k$. We shall show that X_0 has a basis.

Choose a sequence (y_s) such that $y_s \in \hat{A}_s(X)$ and $y_s \neq \theta$ for $s = 1, 2, \ldots$ Since $\dim \hat{A}_s(X) = 1$, then for every $y \in \hat{A}_s(X)$ there exists a scalar c such that $y = cy_s$. Let $\hat{y}_s(t) = \theta$ for $t \neq s$, $\hat{y}_s(s) = y_s$ and $e_s = (\hat{y}_s(t))$. Then it can easily be verified that the subspace span $\{e_s \colon s = 1, 2, \ldots\}$ is dense in X_0 and

$$\Big|\Big|\Big| \sum_{s=1}^{n} c_s e_s \Big|\Big|\Big|_k \leqslant \Big|\Big|\Big| \sum_{s=1}^{n+1} c_s e_s \Big|\Big|\Big|_k$$

for $n, k = 1, 2, \ldots$ and for arbitrary scalar sequences (c_s). Hence the analogous inequality holds for $\| \;\|$. Consequently, (e_s) is a Schauder basis in X_0 (cf. [1]).

Moreover, the mapping $I\colon X \to X_0$, defined by $I(x) = (\hat{A}_s(x))$ for $x \in X$, is a topological isomorphism between X and $I(X)$. The continuity of I and I^{-1} follows from the inequalities

$$|||I(x)|||_k \leqslant K_k \|x\|_{u_k}, \quad \|x\|_k \leqslant |||(\hat{A}_s(x))|||_k \text{ for } x \in X \text{ and } k = 1, 2, \ldots$$

The mapping $L\colon X_0 \to I(X)$, defined by $L((y(s))) = \left(\hat{A}_s\left(\sum_{t=1}^{\infty} y(t)\right)\right)$ for $(y(s)) \in X_0$, is a continuous linear projection of X_0 onto $I(X)$. The continuity of L follows from the inequality:

$$\left|\left|\left| \left(\hat{A}_s\left(\sum_{t=1}^{\infty} y(t)\right)\right) \right|\right|\right|_k \leqslant K_k |||(y(s))|||_{u_k} \quad \text{for} \quad (y(s)) \in X_0 \quad \text{and} \quad k = 1, 2, \ldots$$

THEOREM 2.12. *If a Fréchet space X over $\boldsymbol{R}$ (resp. over $\boldsymbol{C}$) has the* BAP, *then the following properties are equivalent*:

(i) *X has the* SPAP,

(ii) *no subspace of X is topologically isomorphic to the space $\boldsymbol{R}^N$ (resp. $\boldsymbol{C}^N$)*,

(iii) *there is a continuous norm on X*,

(iv) *the topology on X may be defined by a sequence of norms.*

Proof. It is enough to prove that (iii) implies (i) (see the proof of Theorem 2.9).

If X has the BAP and $X \in \mathfrak{M}$, then, by Theorem 2.11, there exists $X_0 \in \mathfrak{M}$ with a Schauder basis and a topological isomorphism $I\colon X \to X_0$ such that $I(X)$ is a continuous linear projection of X_0. Denote this projection by L.

Let Q be an open (resp. open polynomially convex) subset of X, Y a Fréchet space over $\boldsymbol{R}$ (resp. over $\boldsymbol{C}$) and $U \in \mathscr{C}[Q, Y]$ (resp. $U \in \mathscr{A}[Q, Y]$). Then the subset $D = L^{-1}(I(Q))$ is an open (resp. polynomially convex, by Lemma 1.7) subset of X_0 and $F = U \circ I^{-1} \circ L \in \mathscr{C}[D, Y]$ (resp. $F \in \mathscr{A}[D, Y]$). Thus, if $e = (e_i)$ is a Schauder basis in X_0, then, according to Theorem 2.9 and Theorem 2.2, there exists a sequence of polynomials $P_n \in \mathscr{N}_e[X_0, Y]$ convergent almost uniformly to F on D.

Putting $\hat{P}_n = P_n \circ I$ for $n = 1, 2, \ldots$, we obtain that $(\hat{P}_n)$ converges to U almost uniformly on Q and $\hat{P}_n \in \mathscr{N}[X, Y]$ for $n = 1, 2, \ldots$ ∎

THEOREM 2.13. *Any Fréchet space X with the Grothendieck approximation property* (GAP) *has the* PAP.

Proof. If X is real, this theorem follows immediately from the theorem of S. Mazur (see Introduction).

Let X be complex, let Q be an open polynomially convex subset of X, Y an arbitrary complex Fréchet space, $U \in \mathscr{A}[Q, Y]$ and let K be a compact subset of Q. There exists a sequence of finite-dimensional continuous linear operators B_n from X to X convergent to the identity operator uniformly on K (cf. Grothendieck [8]). Obviously, we can suppose that $B_n(K)$ is contained in Q for $n = 1, 2, \ldots$

Let $X_n = B_n(X)$ and $U_n = U|_{Q \cap X_n}$ for $n = 1, 2, \ldots$. By (i) of Lemma 1.7, $Q \cap X_n$ is an open polynomially convex subset of X_n; $B_n(K)$ is a compact subset of $Q \cap X_n$ and U_n is analytic on $Q \cap X_n$. Since $\dim X_n < \infty$, there is a polynomial $\hat{P}_n \in \mathscr{P}[X_n, Y]$ such that $\|U_n(y) - \hat{P}_n(y)\| \leqslant n^{-1}$ for all $y \in B_n(K)$, and hence $\|U(B_n(X)) - \hat{P}_n(B_n(X))\| \leqslant n^{-1}$ for all $x \in K$. (Here $\| \;\|$ denotes a Fréchet norm defining the topology on Y.)

Let now $P_n = \hat{P}_n \circ B_n$ for $n = 1, 2, \ldots$ Then $P_n \in \mathscr{N}[X, Y]$ for all n and $P_n(x) \to U(x)$ for $x \in K$. To show that the convergence is uniform on K, it suffices to repeat the last part of the proof of Theorem 2.2.

Remark. Conversely, if for every open polynomially convex subset Q of X, for every $U \in \mathscr{A}[Q, \boldsymbol{C}]$ and for every compact subset K of Q there is a sequence of polynomials $P_n \in \mathscr{N}[X, \boldsymbol{C}]$ convergent to U uniformly on K, then X has the GAP (see [2], Section 2).

COROLLARY 2.14. *Let X be a complex Fréchet space and Q an open subset of X. If X has the* GAP, *then the following conditions are equivalent*:

(i) *Q is $\mathscr{P}[X, \boldsymbol{C}]$-convex*,

(ii) *Q is $\mathscr{N}[X, \boldsymbol{C}]$-convex.*

If X has a Schauder basis $e = (e_i)$, then (i) *and* (ii) *are also equivalent to*

(iii) *Q is $\mathscr{N}_e[X, \boldsymbol{C}]$-convex.*

Proof. (i)$\Leftrightarrow$(ii). It suffices to show that $K^Q_{\mathscr{P}[X,\boldsymbol{C}]} = K^Q_{\mathscr{N}[X,\boldsymbol{C}]}$ for compact subsets K of Q. The relation $K^Q_{\mathscr{P}} \subset K^Q_{\mathscr{N}}$ is obvious. Let now $x_0 \in K^Q_{\mathscr{N}}$ and $P \in \mathscr{P}[X, \boldsymbol{C}]$. Since the set $Z = K \cup \{x_0\}$ is a compact subset of Q, there exists a sequence of polynomials $P_n \in \mathscr{N}[X, \boldsymbol{C}]$ convergent to P uniformly on Z (see the proof of Theorem 2.12). But $x_0 \in K^Q_{\mathscr{N}}$, therefore $|P_n(x_0)| \leqslant \sup_{x \in K} |P_n(x)|$ for $n = 1, 2, \ldots$ Letting $n \to \infty$, we obtain $|P(x_0)| \leqslant \sup_{x \in K} |P(x)|$. Consequently $K^Q_{\mathscr{N}} \subset K^Q_{\mathscr{P}}$.

(i)$\Leftrightarrow$(iii). Every compact subset K of X (with a basis) is locally e^*-bounded. Using Theorem 2.2 and arguments similar to those above, we obtain $K^Q_{\mathscr{N}_e[X,\boldsymbol{C}]} = K^Q_{\mathscr{P}[X,\boldsymbol{C}]}$.

COROLLARY 2.15. *An open subset Q of a complex Fréchet space X with the* GAP *is polynomially convex if and only if for every compact subset K of Q the polynomial convex hull $\hat{K} = K^X_{\mathscr{P}[X,\boldsymbol{C}]}$ is contained in Q.*

Proof. Let K be an arbitrary non-empty compact subset of Q. If Q contains $\hat{K}$, then $\hat{K} = \hat{K} \cap Q = K^{Q}_{\mathscr{P}[X,C]}$. But $\hat{K}$ is compact as a closed subset of the compact set $\overline{\mathrm{conv} K}$. Therefore Q is polynomially convex. Here the assumption that X has the GAP is not necessary.

Let now Q be polynomially convex. To prove that $\hat{K} \subset Q$, it is enough to show that the set $K_2 = (X \setminus Q) \cap \hat{K}$ is empty. Suppose that K_2 is not empty. Since K_2 and $K_1 = K^{Q}_{\mathscr{P}[X,C]}$ are compact, there are open subsets V_1 and V_2 such that $K_1 \subset V_1$, $K_2 \subset V_2$ and $V_1 \cap V_2 = \emptyset$.

E. Ligocka has proved in [11] that if Z is a compact subset of a complete locally convex topological vector space X over C and if $Z = \hat{Z}$, then for every open $V \supset Z$ there exists an open polynomially convex set G such that $Z \subset G \subset V$.

Let G be such a set for $Z = \hat{K}$ and for $V = V_1 \cup V_2$. If $U(x) = 0$ for $x \in V_1$ and $U(x) = 1$ for $x \in V_2$, then $U \in \mathscr{A}[V, C]$. By Theorem 2.13, there is a polynomial $P \in \mathscr{P}[X, C]$ such that $|U(x) - P(x)| < 2^{-1}$ for $x \in \hat{K}$. This implies the contradictious inequality: $2^{-1} < |P(y)| \leqslant \sup_{x \in K} |P(x)| \leqslant 2^{-1}$ for $y \in K_2$. ■

In an analogous way we can obtain

COROLLARY 2.16. *If X is a complex Fréchet space with the* GAP, *then*:

(i) *for every open polynomially convex subset Q of X the union of an arbitrary number of connected components of Q is polynomially convex,*

(ii) *for every connected compact subset K of X the polynomially convex hull $\hat{K}$ of K is connected, too.*

Now we return to the space C^N.

THEOREM 2.17. *If Q is an open polynomially convex subset of C^N, then the following properties are equivalent*:

(i) *for every $U \in \mathscr{A}[Q, C]$ there exists a polynomial sequence convergent to U almost uniformly on Q,*

(ii) *the subset Q consists of a finite number of connected components.*

Proof. Proving Theorem 2.7, we showed that (i) implies (ii).

(ii)$\Rightarrow$(i). Let $Q = D_1 \cup D_2 \cup \ldots \cup D_n$, where D_k are open connected subsets of C^N such that $D_i \cap D_k = \emptyset$ for $i \neq k$. By (i) of Corollary 2.16, D_k is polynomially convex; therefore $D_k = S_p^{-1}(S_p(D_k))$ for almost all p (here $S_p((z_1, z_2, \ldots)) = (z_1, \ldots, z_p, 0, 0, \ldots)$); moreover, if $F \in \mathscr{A}[D_k, C]$, then U is globally finitely determined on D_k (cf. A. Hirschowitz [9]). Hence, if $U \in \mathscr{A}[Q, C]$, then U is globally finitely determined on Q, i.e., there is a positive integer n_0 such that for every $x \in Q$ the function U is constant on $Q \cap S_{n_0}^{-1}(S_{n_0}(x))$.

Let n_0 be so large that $D_k = S_{n_0}^{-1}(S_{n_0}(D_k))$ for $k = 1, 2, \ldots, n$. Then $Q = S_{n_0}^{-1}(S_{n_0}(Q))$, and from this $S_{n_0}(Q) = Q \cap X_{n_0}$, where $X_{n_0} = S_{n_0}(C^N)$.

By (i) of Lemma 1.7, $S_{n_0}(Q)$ is polynomially convex subset of X_{n_0}. Since the restriction $\hat{U} = U|_{S_{n_0}(Q)}$ is analytic on $S_{n_0}(Q)$ and $\dim X_{n_0} < \infty$, there exists a sequence of polynomials $\hat{P}_m \in \mathscr{P}[X_{n_0}, C]$ convergent to $\hat{U}$ almost uniformly on $S_{n_0}(Q)$.

Setting $P_m = \hat{P}_m \circ S_{n_0}$ for $m = 1, 2, \ldots$, we obtain a sequence convergent to U almost uniformly on Q. ■

From the theorem of Mazur, it follows that for every separable real Banach space X, for every real Fréchet space Y and for every $F \in \mathscr{C}[X, Y]$ there exists a sequence of polynomials convergent to F almost uniformly on X. The following theorem is a generalization of this fact.

THEOREM 2.18. *Every separable real Banach space X has the* SPAP.

Proof. Without loss of generality we can suppose that X is a closed subspace of the space $\mathscr{C}\langle 0, 1\rangle$ of all continuous real valued functions on the interval $\langle 0, 1\rangle$.

Let Q be an open subset of X, Y a real Fréchet space and let $F \in \mathscr{C}[Q, Y]$. Obviously, for every $x \in Q$ there is $r_x > 0$ such that the ball $B_x = \{y \in X|: \|y - x\| < r_x\}$ is contained in Q ($\| \ \|$ denotes the norm in $\mathscr{C}\langle 0, 1\rangle$). Put $\hat{B}_x = \{y \in \mathscr{C}\langle 0, 1\rangle: \|y - x\| < r_x\}$, $M = \bigcup_{x \in Q} \hat{B}_x$ and $\mathrm{dist}(x, y) = \|x - y\|$; $x, y \in M$. Then M is a metric space and Q is a closed subset of M ($Q = M \cap X$).

By the theorem of Dugundji [7], there exists a continuous mapping $\hat{F}$ from M to Y such that $\hat{F}(x) = F(x)$ for $x \in Q$. On the other hand, M is an open subset of the space $\mathscr{C}\langle 0, 1\rangle$. Since $\mathscr{C}\langle 0, 1\rangle$ is a Banach space with a Schauder basis, then, according to Theorem 2.2, there exists a sequence of polynomials $\hat{P}_n \in \mathscr{N}[\mathscr{C}\langle 0, 1\rangle, Y]$ convergent to $\hat{F}$ almost uniformly on M.

Let $P_n = \hat{P}_n|_X$ for $n = 1, 2, \ldots$ Then $P_n \in \mathscr{N}[X, Y]$ and (P_n) converges to F almost uniformly on Q. ■

Note that not every Banach space has the SPAP. An example of such a space is $\mathscr{C}_0\langle 0, 1\rangle$, the space of all real valued functions $x = x(t)$, defined for $t \in \langle 0, 1\rangle$, such that for every $r > 0$ the set $\{t \in \langle 0, 1\rangle: x(t)| > r\}$ is finite; the norm on $\mathscr{C}_0\langle 0, 1\rangle$ is defined by $\|x\| = \sup_t |x(t)|$. A. Pełczyński [16] has proved that the functional $\| \ \|$ is not even the pointwise limit of any polynomial sequence in the ball $B(0, 1)$.

References

[1] A. Alexiewicz, *Functional analysis* (in Polish), Warsaw 1969.

[2] R. M. Aron and M. Schottenloher, *Compact holomorphic mappings on Banach spaces and the approximation property*, J. Functional Analysis 21 (1976), pp. 7–30.

[3] C. Bessaga and A. Pełczyński, *On a class of B_0-spaces*, Bull. Acad. Pol. Sci. 5 (1957) pp. 375–383.
[4] J. Bochnak and J. Siciak, *Polynomials and multilinear mappings in topological vector spaces*, Studia Math. 39 (1971), pp. 59–76.
[5] —, — *Analytic functions in topological vector spaces*, ibid., pp. 77–112.
[6] S. Dineen, *Runge domains in Banach spaces*, Proc. of Roy. Irish Acad. 71 (1971), pp. 85–89.
[7] J. Dugundji, *An extension of Tietze's theorem*, Pacific J. of Math. 1 (1951), pp. 353–367.
[8] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).
[9] A. Hirschowitz, *Remarque sur les ouverts d'holomorphie d'un produit dénombrable de droites*, Ann. Inst. Fourier, Grenoble 19, 1 (1969), pp. 219–229.
[10] J. Kurzweil, *On approximation in real Banach spaces by analytic operators*, Studia Math. 16 (1955), pp. 124–129.
[11] E. Ligocka, *A local factorization of analytic functions and its applications*, Studia Math. 47 (1973), pp. 239–252.
[12] C. Matyszczyk, *Approximation of analytic operators by polynomials in complex B_0-spaces with bounded approximation property*, Bull. Acad. Pol. Sci. 20 (1972), pp. 833–836.
[13] Ph. Noverraz, *Sur la convexité fonctionnelle dans les espaces de Banach à base*, C. R. Acad. Sci., Paris, Sér. A–B, 272 (1971), pp. A1564–1566.
[14] — *Pseudo-convexité, convexité polynomiale et domains d'holomorphie en dimension infinie*, North-Holland Math. Stud. 3, Amsterdam 1973.
[15] — *Sur la pseudo-convexité et la convexité polynomiale en dimension infinie*, Ann. Inst. Fourier 23, Grenoble 1973, 1, pp. 113–134.
[16] A. Pełczyński, *A property of multilinear operations*, Studia Math. 16 (1955), pp. 173–182.
[17] — *Any separable Banach space with the bounded approximation property is a complemented space of a Banach space with a basis*, ibid. 40 (1971), pp. 239–243.
[18] — *Some generalizations of Weierstrass' theorem*, Master Thesis, Warsaw University 1955 (in Polish).
[19] C. E. Rickart, *Analytic functions of an infinite number of complex variables*, Duke Math. J. 36 (1969), pp. 581–597.
[20] A. Robertson and W. Robertson, *Topological vector spaces*, Cambridge University Press, 1964.

MILITARY TECHNICAL ACADEMY, WARSAW
WOJSKOWA AKADEMIA TECHNICZNA, WARSZAWA

Received May 15, 1975 (1015(

STUDIA MATHEMATICA, T. LX. (1977)

A two-sided operational calculus

by

RAIMOND A. STRUBLE (Raleigh, N. C.)

Abstract. A two-sided operational calculus on the real line is constructed using the algebraic method introduced by Mikusiński. A field of two-sided operators is obtained which contains a large subspace of distributions, including the Laplace transformable distributions of Schwartz. The operator field is shown to be isomorphic with a field of meromorphic functions. The isomorphism is an extension of the classical (and distributional) Fourier transform, and is expressed by an integral of the classical form which is defined relative to a type I-convergence notion. A similar expression for the inverse Fourier transform is obtained, and the two are also expressed by sequential limits relative to a type II-convergence notion. The representation of distributions by these operators is discussed.

1. Introduction. As is well known, Mikusiński obtained ([7]) certain generalized functions by considering an algebraic field of fractions for the convolution ring of continuous functions on the half-line $[0, \infty)$. The result is a one-sided operational calculus which possesses all the advantages of rigor supplied by the Laplace transform method and none of the limitations imposed by the underlying analysis.

Recently, Boehme and Wygant introduced ([1]) a two-sided operational calculus on the unit circle which is equivalent to a periodic operational calculus on the real line $\boldsymbol{R}$. They constructed from the ring $\mathscr{C}$ of continuous 2π-periodic functions on $\boldsymbol{R}$, under convolution and addition, the ring $\mathscr{M}$ of fractions f/g, where g has all of its Fourier coefficients nonzero. The latter are the nondivisors of zero in $\mathscr{C}$. These fractions are called *operators* (following Mikusiński's example), and the ring $\mathscr{M}$ of operators is found to contain (isomorphically) the ring $\mathscr{P}'$ of 2π-periodic distributions, under convolution. They showed that this ring of operators is isomorphic with the (convolution) ring of formal trigonometric series (equivalently, the ring of doubly infinite series of complex numbers under coordinate addition and multiplication), and that every operator can be expressed as a Fourier series. For the latter, a convergence notion is introduced into $\mathscr{M}$ which is analogous to that given by Mikusiński, and is called *type* I.

In this paper we introduce still another example of a two-sided operational calculus on $\boldsymbol{R}$ which results in a *field* $\mathscr{M}_{\text{Exp}}$ of operators similar