[6] M. Milman, *Hausdorff–Young inequalities for the Laplace transform*, preprint.
[7] — *Integral transforms of weak type between rearrangement invariant spaces*, Bull. Amer. Math. Soc. 81 (1975), pp. 761–762.
[8] M. Milman and R. C. Sharpley, *Convolution operators on $\Lambda_a(X)$ spaces*, preprint.
[9] R. O'Neil, *Convolution operators and $L(p, q)$ spaces*, Duke Math. J. 30 (1963), pp. 129–142.
[10] — *Fractional integration in Orlicz spaces, I*, Trans. Amer. Math. Soc. 115 (1695), pp. 300–328.
[11] — *Integral transforms and tensor products on Orlicz spaces and $L(p, q)$ spaces*, J. Analyse Math. 21 (1968), pp. 1–276.
[12] E. M. Semenov, *Imbedding theorems for Banach spaces of measurable functions*, Soviet Math. Dokl. 5 (1964), pp. 831–834.
[13] R. C. Sharpley, *Spaces $\Lambda_a(X)$ and interpolation*, J. Functional Analysis 11 (1972), pp. 479–513.
[14] — *Interpolation of n pairs and counterexamples employing indices*, J. Approximation Theory 13 (1975), pp. 117–127.
[15] — *Fractional integration in Orlicz spaces*, Proc. Amer. Math. Soc., 59 (1976), pp. 99–106.
[16] G. Sparr, *Interpolation of several Banach spaces*, preprint.
[17] A. Torchinsky, *Interpolation of operations and Orlicz classes*, to appear in Studia Math.
[18] A. Yoshikawa, *Sur la théorie d'espaces d'interpolation — les espaces de moyenne de plusieurs espaces de Banach*, Journ. Fac. Sci. Univ. Tokyo, Sec. I A 16 (1970), pp. 407–468.
[19] M. Zafran, *A multilinear interpolation theorem*, to appear in Studia Math.

UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, S.C. USA

Received October 24, 1975 (1080)

STUDIA MATHEMATICA, T. LX. (1977)

Equivalence of Haar and Franklin bases in L_p spaces*

by

Z. CIESIELSKI (Sopot), P. SIMON (Budapest) and P. SJÖLIN (Stockholm)

Abstract. The main result of this paper states that the Haar and Franklin orthonormal sets do form equivalent bases in $L_p\langle 0, 1\rangle$ for each p, $1 < p < \infty$, i.e. the spaces of coefficients for the two bases are identical. The proof depends on the unconditionality of Haar and Franklin bases. The original proof of S. V. Bockariev of the unconditionality of the Franklin basis is rather complicated and a simplified version is presented in this paper. As a consequence of our main result we obtain the L_p version of the maximal inequality for the Fourier partial sums of the uniformly bounded orthonormal system of polygonals introduced earlier by one of the authors.

1. Introduction. In his recent paper S. V. Bockariev [1] (see also [2]) proved that the Franklin system is an unconditional basis in $L_p(I)$, $I = \langle 0, 1\rangle$, $1 < p < \infty$. His ingenious proof regulies only, appart from the properties of the Franklin functions established by Z. Ciesielski in [4], a modification of the A. Zygmund lemma on decomposition of functions and the weak type L_1 estimate for the Hardy–Littlewood maximal function. It appears that with the help of some technics known in the theory of singular integrals the original proof of Bockariev can be modified considerably. Such a simplified version of the proof of unconditionality of the Franklin basis is presented below, and the additional tools used in it are the Whitney's decomposition lemma of open sets into dyadic cubes, the Marcinkiewicz integral and distance function.

The unconditionality of the Franklin basis (S. V. Bockariev) and of the Haar basis (J. Marcinkiewicz) in L_p, $1 < p < \infty$, is the starting point in the proof of our main result, i.e. the equivalence of the Haar and Franklin bases in L_p, $1 < p < \infty$. The essential step leading to the desired result is an application of the Fefferman–Stein inequality (Theorem E) to both Haar and Franklin systems. To do this, it is necessary to compare maximal functions of the function belonging to one system with the corresponding functions from the other system. However, this can be done on the basis of the estimate for Franklin functions obtained by Z. Ciesielski in [4].

* It has been proved recently by P. Sjölin that the Haar and Franklin bases are not equivalent in L_1.

There is a natural way of obtaining the Walsh orthonormal system from the Haar orthonormal system. The same procedure was applied by Z. Ciesielski in [5] to obtain a uniformly bounded orthonormal complete system of polygonals (polygonal lines, i.e. splines of degree 1). It was an open problem whether this set is a system of convergence a.e. in L_p, $1 < p < \infty$. For $p = 2$ the positive answer was obtained by F. Schipp [11], and for arbitrary p, $1 < p < \infty$, the positive answer to this question is a consequence of the maximal inequality (Theorem 5.1) which is an analogue of the maximal inequality for the Walsh system established by P. Sjölin [12] (Theorem C). The proof of the maximal inequality in Theorem 5.1 as well as the proof of F. Schipp depend on the Sjölin's inequality. Moreover, our proof uses in essential way the equivalence of Haar and Franklin bases — more precisely Theorem 4.2.

It should be mentioned in this place that the results of this paper can be extended to the spline systems of higher orders $\{f_j^{(m)}, j = -m, -m+1, \ldots\}$, $m \geq -1$, as defined e.g. in [6].

2. Preliminaries. In this section notation is being introduced and some necessary known results are quoted.

The $L_p(I)$ and $C(I)$, spaces over $I = \langle 0, 1\rangle$, are considered as real Banach spaces of real-valued functions with the usual norms $\|\ \|_p$ and $\|\ \|$, respectively. Moreover, (f, g) is defined as the integral $\int_I fg$.

The Haar $\{\chi_n, n = 1, 2, \ldots\}$ and Franklin $\{f_n, n = 0, 1, \ldots\}$ orthonormal sets are defined as in [3], the Walsh $\{\omega_n, n = 1, 2, \ldots\}$ and the $\{c_n, n = 0, 1, \ldots\}$ orthonormal sets are defined as in [5].

The construction of $\{c_n\}$ can be described briefly as follows. We know that $\chi_1 = \omega_1$ and there is for each $\mu \geq 0$ an orthogonal matrix $2^\mu \times 2^\mu$ transforming $\{\chi_{2^\mu+1}, \ldots, \chi_{2^{\mu+1}}\}$ onto $\{\omega_{2^\mu+1}, \ldots, \omega_{2^{\mu+1}}\}$. Similarly, we put $c_0 = f_0$, $c_1 = f_1$, and define $\{c_{2^\mu+1}, \ldots, c_{2^{\mu+1}}\}$ as a result of application of the same orthogonal matrix to the vector $\{f_{2^\mu+1}, \ldots, f_{2^{\mu+1}}\}$. It was shown in [5] that the system $\{c_n, n = 0, 1, \ldots\}$ is uniformly bounded.

The nth Fourier partial sums with respect to $\{\chi_n\}$ and $\{\omega_n\}$, and the $(n+1)$st Fourier partial sums with respect to $\{f_n\}$, $\{c_n\}$ are denoted by H_n, W_n and F_n, C_n, respectively.

For given n, $n \geq 1$, the *dyadic partition* $0 = s_{n,0} < \ldots < s_{n,n} = 1$ is defined as follows: if $n = 1$, then

$$0 = s_{n,0} < s_{n,n} = 1,$$

and if $n = 2^\mu + \nu$, $1 \leq \nu \leq 2^\mu$, $\mu \geq 0$, then

$$s_{n,i} = \begin{cases} i/2^{\mu+1} & \text{for } i = 0, \ldots, 2\nu-1, \\ (i-\nu)/2^\mu & \text{for } i = 2\nu, \ldots, n. \end{cases} \tag{2.1}$$

Now, let $I_{n,i} = \langle s_{n,i-1}, s_{n,i}\rangle$ for $1 \leq i < n$ and $I_{n,n} = \langle s_{n,n-1}, s_{n,n}\rangle$.

The Haar function corresponding to $n = 2^\mu + \nu$, $1 \leq \nu \leq 2^\mu$, $\mu \geq 0$, has the following properties

$$\|\chi_n\| = \sqrt{2^\mu}, \quad |\chi_n(t)| = \begin{cases} \sqrt{2^\mu}, & t \in I_{n,2\nu-1} \cup I_{n,2\nu}, \\ 0, & t \notin I_{n,2\nu-1} \cup I_{n,2\nu}. \end{cases} \tag{2.2}$$

For the Franklin function f_n, $n = 2^\mu + \nu$, the following estimate was established by Z. Ciesielski (cf. [4], Theorem 1)

$$C^{-1}\sqrt{2^\mu}\, q^{|k-(2\nu-1)|} \leq (-1)^{k+1} f_n(s_{n,k}) \leq C\sqrt{2^\mu}\, q^{|k-(2\nu-1)|}, \tag{2.3}$$

where $k = 0, \ldots, n$; $q = 2 - \sqrt{3}$ and C is a positive constant.

J. Marcinkiewicz [8], using the decomposition theorem of R.E.A.C. Paley [9] on Walsh system, proved that the Haar system is an unconditional basis in $L_p(I)$, $1 < p < \infty$. His result, on the ground of Khinchine's inequality, can be formulated as follows:

THEOREM A (J. Marcinkiewicz). *Let $f \in L_p(I)$ and let $1 < p < \infty$. Then there is a constant $C(p)$, depending on p only, such that*

$$\frac{1}{C(p)} \|f\|_p \leq \Big\| \Big(\sum_{n=1}^{\infty} (f, \chi_n)^2 \chi_n^2 \Big)^{\frac{1}{2}} \Big\|_p C(p) \|f\|_p.$$

The analogous result for the Franklin system was proved recently only.

THEOREM B (S. V. Bockariev [1], [2]). *Let $f \in L_p(I)$ and let $1 < p < \infty$. Then, for some constant $C(p)$ depending on p only, we have*

$$\frac{1}{C(p)} \|f\|_p \leq \Big\| \Big(\sum_{n=0}^{\infty} (f, f_n)^2 f_n^2 \Big)^{\frac{1}{2}} \Big\|_p \leq C(p) \|f\|_p.$$

This result follows from Theorem 3.1, rather simple proof of which is presented below.

In the remarkable paper of R.E.A.C. Paley [9] it was shown that the Walsh system is a basis in $L_p(I)$, $1 < p < \infty$. After the Carleson's work on the convergence of Fourier series a much stronger result was obtained by P. Sjölin. It implies that $\{\omega_n\}$ is an a.e. convergence system in $L_p(I)$, $1 < p < \infty$.

THEOREM C (P. Sjölin [12]). *If $f \in L_p(I)$, $1 < p < \infty$, then there is a constant $C(p)$, depending on p only, such that*

$$\|\sup_n |W_n(f)|\|_p \leq C(p) \|f\|_p.$$

A similar result for the Franklin system, although easier to prove, will be needed later too.

THEOREM D (Z. Ciesielski [6]). *Let $f \in L_p(I)$, and let $1 < p < \infty$. Then there is a constant $C(p)$ such that*

$$\|\sup_n |F_n(f)|\|_p \leqslant C(p)\|f\|_p .$$

It has been proved recently by S. Ropela [10] that $\{c_n\}$ is a basis in $L_p(I)$, $1 < p < \infty$.

Finally we need to state an inequality of C. Fefferman and E. Stein [7]. To do this, let us compare the two definitions of maximal functions on $R = (-\infty, \infty)$ and on I. If $F \in L_1^{\mathrm{loc}}(R)$, then let us define

$$(M_R F)(t) = \sup_{r>0} \frac{1}{2r} \int_{t-r}^{t+r} |F(s)|\,ds, \qquad t \in R,$$

and, if $f \in L_1(I)$, then let

$$(Mf)(t) = \sup_{\omega} \frac{1}{|\omega|} \int_{\omega} |f(s)|\,ds, \qquad t \in I,$$

where the sup is taken over all intervals ω contained in I and containing t.

For given f on I let us denote by $\tilde{f}$ the extension of f to R by defining $\tilde{f}(t) = 0$ for $t \in R \setminus I$. One finds easily that $M_R f(t) \leqslant Mf(t) \leqslant 2M_R f(t)$ for $t \in I$. In view of this, the general inequality of Feffermann and Stein gives, in particular,

THEOREM E (C. Fefferman and E. Stein [7]). *Let $1 < p < \infty$ and let $g_1, g_2, \ldots$ be the sequence of functions in $L_p(I)$ with the property that*

$$\Big(\sum_n |g_n|^2\Big)^{1/2} \in L_p(I).$$

Then, there is a constant $C(p)$ depending on p only such that

$$\Big\|\Big(\sum_n (Mg_n)^2\Big)^{1/2}\Big\|_p \leqslant C(p)\Big\|\Big(\sum_n g_n^2\Big)^{1/2}\Big\|_p .$$

3. The unconditionality of the Franklin basis. For a given $g \in L_1(I)$ and for a given sequence $\varepsilon = (\varepsilon_k)_0^\infty$, $\varepsilon_k = \pm 1$, we set

$$T_\varepsilon g(t) = \Big|\sum_{n=0}^{\infty} \varepsilon_n (g, f_n) f_n(t)\Big| .$$

The following Theorem 3.1 is due to S. V. Bockariev and it claims that the operator T_ε^1 is of weak type $(1, 1)$. On the other hand, T_ε is of type $(2, 2)$. Consequently, by Marcinkiewicz's interpolation theorem, T_ε is of type (p, p), $1 < p \leqslant 2$. This and the duality argument imply that T_ε is of type (p, p), $1 < p < \infty$, and therefore Theorem B is implied by Theorem 3.1.

In the proof of Theorem 3.1 the following lemma will be needed.

LEMMA 3.1. *There is an absolute constant C such that*

$$\sum_{|t-s| > \frac{1}{2^\mu}} \ \sum_{2^\mu < n \leqslant 2^{\mu+1}} |f_n(t) f_n'(s)| \leqslant C \frac{1}{|t-s|^2}, \qquad t, s \in I.$$

Proof. It follows from (2.3) that

$$|f_n(t)| \leqslant C \cdot 2^{\frac{\mu}{2}} q^{|\langle t\rangle - \nu|}, \qquad |f_n'(s)| \leqslant C \cdot 2^{\frac{3}{2}\mu} q^{|\langle s\rangle - \nu|},$$

where $n = 2^\mu + \nu$, $1 \leqslant \nu \leqslant 2^\mu$, and $\langle t\rangle$ is the unique integer such that $\langle t\rangle - 1 \leqslant 2^\mu t < \langle t\rangle$. Now, it is easily seen that to each r, $q < r < 1$, there is a constant $C(r)$ such that

$$\sum_{k=1}^{\infty} q^{|i-k|+|j-k|} \leqslant C(r) r^{|i-j|},$$

whence we infer, putting for instance $r = q^{1/2}$, that

$$\sum_{2^\mu < n \leqslant 2^{\mu+1}} |f_n(t) f_n'(s)| \leqslant C \cdot 2^{2\mu} \cdot \sum_{\nu=1}^{2^\mu} q^{|\langle t\rangle - \nu| + |\nu - \langle s\rangle|}$$
$$\leqslant C \cdot 2^{2\mu} r^{|\langle t\rangle - \langle s\rangle|} \leqslant C \cdot 2^{2\mu} r^{2^\mu |t-s|}.$$

However,

$$\sum_{|t-s| > \frac{1}{2^\mu}} 2^{2\mu} r^{2^\mu |t-s|} \leqslant \frac{C}{|t-s|^2}$$

and this completes the proof.

THEOREM 3.1 (S. V. Bockariev [2]). *If $g \in L_1(I)$ and $\varepsilon = (\varepsilon_k)_0^\infty$ is given, then there exists a constant C independent of g and ε such that*

$$\big|\{t \in I:\ |T_\varepsilon g(t)| > y\}\big| \leqslant \frac{C}{y} \|g\|_1, \qquad y > 0. \tag{3.1}$$

Proof. For fixed $y > 0$ let us define $Q = \{t \in I:\ Mg(t) > y\}$ and $P = I \setminus Q$. From the weak L_1 estimate for the Hardy–Littlewood maximal function (see [13], p. 5) it follows that

$$|Q| \leqslant \frac{5}{y} \|g\|_1 . \tag{3.2}$$

Now, if $0 < y \leqslant 5\|g\|_1$, then (3.1) holds with $C = 5$. Therefore, we may assume that $y > 5\|g\|_1$, so that the set P is non-empty. In this case let $(Q_i)_1^\infty$, be a Whithney's decomposition of Q (see [13], p. 167–168), i.e. each $Q_i = \langle \alpha_i, \beta_i\rangle$ is a dyadic interval of the form $\langle 2^{-\mu}(\nu - 1), 2^{-\mu}\nu\rangle$, and

$$Q = \bigcup_{i=1}^{\infty} Q_i, \quad \operatorname{int} Q_i \cap \operatorname{int} Q_j = \emptyset \quad \text{for } i \neq j, \tag{3.3}$$

$$|Q_i| \leqslant \operatorname{dist}(Q_i, P) \leqslant 4|Q_i|. \tag{3.4}$$

For each Q_i there is therefore a point $p_i \in P$ such that $\operatorname{dist}(Q_i, p_i) \leqslant 4|Q_i|$. Since, by the definition of P, $M_g(p_i) \leqslant y$ it follows that

$$\int_{Q_i} |g(t)|\,dt \leqslant 5y|Q_i|. \tag{3.5}$$

Next step is to decompose (as in the original Bočkariev's proof) the function g into a bounded function g_1 and a nice function g_2. Let us denote by T_i the orthogonal projection of $L_2(Q_i)$ onto the two-dimensional subspace generated by the functions 1 and t. Now let us set

$$g_1(t) = \begin{cases} g(t), & t \in P, \\ T_i g(t), & t \in \operatorname{int} Q_i; \end{cases} \qquad g_2 = g - g_1. \tag{3.6}$$

Now, the first two Legendre polynomials in $L_2(I)$ (i.e. the first two Franklin functions) are given as follows

$$f_0(t) = 1, \quad f_1(t) = \sqrt{3}(2t-1), \quad t \in I.$$

Thus, the corresponding basis in the range of T_i is

$$f_{0,i}(t) = \frac{1}{|Q_i|^{1/2}}, \quad f_{1,i}(t) = \frac{1}{|Q_i|^{1/2}} f_1\left(\frac{t-a_i}{|Q_i|}\right), \quad t \in Q_i,$$

and therefore

$$T_i g(t) = \int_{Q_i} g(s)[f_{0,i}(s) f_{0,i}(t) + f_{1,i}(s) f_{1,i}(t)]\,ds,$$

whence we infer

$$|T_i g(t)| \leqslant \frac{4}{|Q_i|} \int_{Q_i} |g(s)|\,ds, \quad t \in Q_i.$$

This, (3.5), and (3.6) give

$$|g_1(t)| \leqslant 20y \quad \text{a.e. in } I. \tag{3.7}$$

The properties of g_2 are described as follows. First of all, $g_2(t) = 0$ for $t \in P$, then, g_2 on each Q_i is orthogonal to 1 and t, and therefore

$$\int_{Q_i} g_2(t)\,dt = 0, \quad \int_{Q_i} G_2(t)\,dt = 0, \tag{3.8}$$

where $G_2(t) = \int_0^t g_2(s)\,ds$. This implies

$$G_2(t) = \int_{a_i}^t g_2(s)\,ds \quad \text{for} \quad t \in Q_i. \tag{3.9}$$

Now, the combination of (3.9) and (3.7) gives

$$|G_2(t)| \leqslant \int_{a_i}^t (|g(s)| + 20y)\,ds, \quad t \in Q_i,$$

whence by (3.5)

$$\int_{Q_i} |G_2(t)|\,dt \leqslant 25y|Q_i|^2. \tag{3.10}$$

We are ready now to start proving (3.1).

The estimate for $T_s g_1$. The operator T_s is an isometry on $L_2(I)$, and therefore (3.7) and (3.2) imply

$$|\{t: |T_s g_1(t)| > y\}| \leqslant \frac{\|T_s g_1\|_2^2}{y^2} = \frac{\|g_1\|_2^2}{y^2} = \frac{1}{y^2}\left\{\int_P |g_1(t)|^2\,dt + \int_Q |g_1(t)|^2\,dt\right\}$$
$$\leqslant \frac{20}{y}\left\{\int_P |g_1(t)|\,dt + 20y|Q|\right\} \leqslant \frac{2020}{y}\|g\|_1.$$

The estimate for $T_s g_2$. Let

$$g_2 = \sum_{n=0}^{\infty} b_n f_n, \quad b_n = (g_2, f_n).$$

The properties of g_2 imply that $b_0 = b_1 = 0$. Now, for $n = 2^\mu + \nu \geqslant 2$, (3.8) gives

$$b_n = \int_I g_2(s) f_n(s)\,ds = \int_Q g_2(s) f_n(s)\,ds = \sum_{i=1}^{\infty} \int_{Q_i} g_2(s) f_n(s)\,ds$$
$$= -\sum_{i=1}^{\infty} \int_{Q_i} G_2(s) f_n'(s)\,ds = -\sum_{|Q_i| \geqslant 2^{-\mu}} \int_{Q_i} G_2(s) f_n'(s)\,ds.$$

In getting the last equality we have used the fact that Q_i are dyadic and that f_n' is constant on Q_i if $|Q_i| \leqslant 2^{-(\mu+1)}$. Now, if $t \in P$ and $s \in Q_i$, then (3.4) implies $|t-s| \geqslant |Q_i|$. Thus, Lemma 3.1 and (3.10), for $t \in P$, give

$$|T_s g_2(t)| \leqslant \sum_{\mu=0}^{\infty} \sum_{2^\mu < n \leqslant 2^{\mu+1}} |b_n f_n(t)|$$
$$\leqslant \sum_{\mu=0}^{\infty} \sum_{2^\mu < n \leqslant 2^{\mu+1}} \sum_{|Q_i| \geqslant 2^{-\mu}} \int_{Q_i} |G_2(s)|\,|f_n(t) f_n'(s)|\,ds$$
$$= \sum_{i=1}^{\infty} \int_{Q_i} |G_2(s)| \Big(\sum_{|Q_i| \geqslant 2^{-\mu}} \sum_{2^\mu < n \leqslant 2^{\mu+1}} |f_n(t) f_n'(s)| \Big)\,ds$$
$$\leqslant \sum_{i=1}^{\infty} \int_{Q_i} |G_2(s)| \Big(\sum_{|t-s| \geqslant 2^{-\mu}} \sum_{2^\mu < n \leqslant 2^{\mu+1}} |f_n(t) f_n'(s)| \Big)\,ds

$$\leqslant C\cdot\sum_{i=1}^{\infty}\int_{Q_i}|G_2(s)|\,|t-s|^{-2}ds$$
$$\leqslant C\cdot\sum_{i=1}^{\infty}\int_{Q_i}|G_2(s)|\,ds\,[\operatorname{dist}(t,Q_i)]^{-2}$$
$$\leqslant C\cdot 15\cdot y\sum_{i=1}^{\infty}|Q_i|^2[\operatorname{dist}(t,Q_i)]^{-2}.$$

Notice that (3.4) implies $|Q_i|\leqslant \operatorname{dist}(s,P)$ for $s\in Q_i$ and $\operatorname{dist}(t,Q_i)\geqslant \frac{1}{2}|t-s|$ for $s\in Q_i$, $t\in P$. Thus, for each i we have

$$|Q_i|^2[\operatorname{dist}(t,Q_i)]^{-2}\leqslant 4\int_{Q_i}\frac{\operatorname{dist}(s,P)}{|t-s|^2}\,ds,$$

and consequently,

$$|T_s g_2(t)|\leqslant C\cdot y\int_{Q}\frac{\operatorname{dist}(s,P)}{|t-s|^2}\,ds,\qquad t\in P.$$

It now follows from a property of Marcinkiewicz's integral (see [13], p. 14–15) that for some constant C

$$\int_P|Tg_2(t)|\,dt\leqslant C\cdot y\cdot|Q|.$$

This and (3.2) are used to complete the proof as follows

$$|\{t:\ |T_s g_2(t)|>y\}|\leqslant |Q|+|\{t\in P:\ |T_s g_2(t)|>y\}|$$
$$\leqslant |Q|+\frac{1}{y}\int_P|T_s g_2(t)|\,dt\leqslant (1+C)\,|Q|\leqslant 5\,(1+C)\,\frac{\|g\|_1}{y}.$$

4. The equivalence of Haar and Franklin bases. To prove our main result, i.e. Theorem 4.1, we need

LEMMA 4.1. *Let $2^\lambda<n\leqslant 2^{\lambda+1}$ and let $\lambda\geqslant 1$. Then there is a constant C such that for $t\in I$:*

(4.1) $\qquad |\chi_n(t)|\leqslant C\cdot Mf_{n-1}(t)\quad$ *for* $\quad 2^\lambda+1<n,$

(4.2) $\qquad |\hat{\chi}_n(t)|\leqslant C\cdot Mf_{n-1}(t)\quad$ *for* $\quad n=2^\lambda+1,$

(4.3) $\qquad |f_{n-1}(t)|\leqslant C\cdot M\chi_n(t)\quad$ *for* $\quad 2^\lambda+1<n,$

(4.4) $\qquad |f_{n-1}(t)|\leqslant C\cdot M\hat{\chi}_n(t)\quad$ *for* $\quad n=2^\lambda+1,$

where $\hat{f}(t)=f(1-t)$.

Proof. Let us consider at first the case of $2^\lambda+1<n$. Then $n-1=2^\lambda+l$, $n=2^\lambda+k$ and $1\leqslant l=k-1<k\leqslant 2^\lambda$. In this case it is sufficient to check (4.1) according to (2.2) for $\frac{k-1}{2^\lambda}<t<\frac{k}{2^\lambda}$ only. However, since f_{n-1} is a polygonal line, (2.3) gives for some constant $C>0$

$$2^\lambda\int_{\frac{l-1}{2^\lambda}}^{\frac{l}{2^\lambda}}|f_{n-1}(s)|\,ds\geqslant C\cdot 2^{\frac{1}{2}\mu},$$

and this implies (4.1). Now (2.3) gives

$$|f_{n-1}(t)|\leqslant C\cdot 2^{\frac{\lambda}{2}}q^{2^\lambda|t-k2^{-\lambda}|},$$

and since

$$2^{\lambda/2}q^{2^\lambda|t-k2^{-\lambda}|}\leqslant C'\cdot 2^{\lambda/2}\frac{1}{1+2^\lambda|t-l2^{-\lambda}|}\leqslant C''\cdot M\chi_n(t)$$

we get (4.3).

Now, let $n=2^\lambda+1$, i.e. $n-1=2^{\lambda-1}+2^{\lambda-1}$. Thus it suffices to check (4.2) for $1-2^{-\lambda}<t<1$. However, in this interval $|\hat{\chi}_n(t)|=2^{\lambda/2}$ and by (2.3)

$$2^\lambda\int_{\frac{2^\lambda-1}{2}}^{1}|f_{n-1}(s)|\,ds\geqslant C\cdot 2^{\lambda/2}$$

which implies (4.2). Finally, (4.4) is being obtained with the help of (2.3) as follows

$$M\chi_n(t)\geqslant C\cdot 2^{\lambda/2}\frac{1}{1+2^\lambda|t-1|}\geqslant C'\cdot 2^{\frac{\lambda-1}{2}}q^{2^{\lambda-1}|t-1|}\geqslant C''\cdot|f_{n-1}(t)|,$$

and this completes the proof.

THEOREM 4.1. *The series*

$$\sum_{n=1}^{\infty}a_nf_{n-1}$$

converges in $L_p(I)$, $1<p<\infty$, if and only if

$$\sum_{n=1}^{\infty}a_n\chi_n$$

converges in $L_p(I)$. In other words, the bases $\{f_n\}$ and $\{\chi_n\}$ are equivalent in $L_p(I)$, $1<p<\infty$.

Proof. For the proof, let $\Lambda_1=\{1,2\}$, $\Lambda_2=\{n:\ 2^\lambda+1<n\leqslant 2^{\lambda+1},\ \lambda\geqslant 1\}$ and $\Lambda_3=\{n:\ n=2^\lambda+1,\ \lambda\geqslant 1\}$. Clearly, the three sets are disjoint and their sum gives all the positive integers. To prove our theorem we need to show

(4.5) $$\Big\|\Big(\sum_{\Lambda_i}a_n^2f_{n-1}^2\Big)^{1/2}\Big\|_p\sim\Big\|\Big(\sum_{\Lambda_i}a_n^2\chi_n^2\Big)^{1/2}\Big\|_p,\qquad i=1,2,3,$$

where $\sim$ means equivalence of the norms.

The case of $i = 1$ is trivial. For $i = 2$ (4.5) is obtained from Theorem E and inequalities (4.1) and (4.3). To obtain (4.5) for $i = 3$, we use Theorem E, inequalities (4.2) and (4.4) and the fact that the transformation $t \to 1-t$ is measure preserving. The relations (4.5) imply

$$\Big\|\Big(\sum_{n=1}^{\infty} a_n^2 f_{n-1}^2\Big)^{1/2}\Big\|_p \sim \Big\|\Big(\sum_{n=1}^{\infty} a_n^2 \chi_n^2\Big)^{1/2}\Big\|_p. \tag{4.6}$$

To complete the proof it remains to combine (4.6) with Theorems A and B.

THEOREM 4.2. *The series*

$$\sum_{n=1}^{\infty} a_n f_n \quad \text{and} \quad \sum_{N=1}^{\infty} a_n \chi_n$$

are equiconvergent in $L_p(I)$, $1 < p < \infty$, *and their norms are equivalent.*

The proof is similar to that one of Theorem 4.1.

Using Theorem 4.2 we can establish easily

THEOREM 4.3. *The series*

$$\sum_{n=1}^{\infty} b_n w_n \quad \text{and} \quad \sum_{n=1}^{\infty} b_n c_n$$

are equiconvergent in $L_p(I)$ $1 < p < \infty$, *and their norms are equivalent.*

5. The maximal inequality for the $\{c_n\}$ system. The aim of this section is to prove, for the $\{c_n\}$ orthonormal set, the analogue of Theorem C. To do this we need the following

LEMMA 5.1. *Let r_m be the m-th Rademacher function and let for $\mu \geqslant 0$*

$$G_\mu(t, s) = r_{\mu+1}(s) 2^{\frac{\mu}{2}} \begin{cases} f_{2^\mu+\nu}(t), & \dfrac{\nu-1}{2^\mu} \leqslant s < \dfrac{\nu}{2^\mu},\ 1 \leqslant \nu < 2^\mu, \\ f_{2^\mu+\nu}(t), & \dfrac{\nu-1}{2^\mu} \leqslant s \leqslant \dfrac{\nu}{2^\mu},\ \nu = 2^\mu. \end{cases}$$

Then there is a constant C such that

$$\Big|\int_I G_\mu(t, s) h(s)\, ds\Big| \leqslant C \cdot Mh(t), \quad t \in I,\ h \in L_1(I).$$

Proof. The estimate (2.3) implies

$$|G_\mu(t, s)| \leqslant C \cdot 2^\mu q^{2^\mu |t-s|}, \quad 0 < q < 1.$$

Now the proof can be continued in the same way as in [6] (cf. Theorem 3.3), and therefore the details are omitted.

Let us recall the definition of c_n. Let $n = 2^\mu + \nu$, $1 \leqslant \nu \leqslant 2^\mu$; then

$$\begin{aligned} c_{2^\mu+\nu}(t) &= 2^{-\mu/2} \sum_{j=1}^{2^\mu} w_\nu\Big(\frac{2j-1}{2^{\mu+1}}\Big) f_{2^\mu+j}(t) \\ &= \int_I w_\nu(s) r_{\mu+1}(s) G_\mu(t, s)\, ds = \int_I w_{2^\mu+\nu}(s) G_\mu(t, s)\, ds. \end{aligned} \tag{5.1}$$

It is now time to prove

THEOREM 5.1. *Let $1 < p < \infty$. Then there is a constant $C(p)$ such that*

$$\|\sup_n |C_n(f)|\|_p \leqslant C(p) \|f\|_p, \quad f \in L_p(I).$$

Proof. We use the fact that $C_{2^\mu}(f) = F_{2^\mu}(f)$ and write

$$\sup_n |C_n(f)| \leqslant \sup_{\mu \geqslant 0} |F_{2^\mu}(f)| + \sup_{\mu \geqslant 0} \sup_{1 \leqslant \nu \leqslant 2^\mu} |C_{2^\mu+\nu}(f) - C_{2^\mu}(f)|.$$

According to Theorem D, we get

$$\|\sup_{\mu \geqslant 0} |F_{2^\mu}(f)|\|_p \leqslant C(p) \|f\|_p. \tag{5.2}$$

Now let

$$f = \sum_{n=0}^{\infty} a_n c_n,$$

and let

$$f_{\mu,\nu} = \sum_{2^\mu < n \leqslant 2^\mu+\nu} a_n c_n,$$

$$h_{\mu,\nu} = \sum_{2^\mu < n \leqslant 2^\mu+\nu} a_n w_n.$$

It follows from (5.1) that

$$f_{\mu,\nu}(t) = \int_I h_{\mu,\nu}(s) G_\mu(t, s)\, ds,$$

whence, by Lemma 5.1,

$$\begin{aligned} |f_{\mu,\nu}(t)| &\leqslant C \cdot M h_{\mu,\nu}(t), \\ |f^*(t)| &\leqslant C \cdot M h^*(t), \end{aligned} \tag{5.3}$$

where

$$f^* = \sup_{\mu \geqslant 0} \sup_{1 \leqslant \nu \leqslant 2^\mu} |f_{\mu,\nu}|,$$

$$h^* = \sup_{\mu \geqslant 0} \sup_{1 \leqslant \nu \leqslant 2^\mu} |h_{\mu,\nu}|.$$

If we define

$$h = \sum_{n=1}^{\infty} a_n w_n,$$

then, by Theorem 4.3,

(5.4) $$\|h\|_p \leqslant C(p)\|f\|_p.$$

On the other hand,

(5.5) $$h^* \leqslant 2 \sup_{n\geqslant 1} |W_n(h)|.$$

It follows now from (5.3) by the Hardy–Littlewood maximal theorem that

(5.6) $$\|f^*\|_p \leqslant C(p)\|h^*\|_p,$$

and from (5.4)–(5.5), by Theorem C, that

(5.7) $$\|h^*\|_p \leqslant C(p)\|f\|_p.$$

Thus the combination of (5.7) and (5.6) gives

$$\|f^*\|_p \leqslant C(p)\|f\|_p.$$

This jointly with (5.2) gives the required result.

THEOREM 5.2. *Let $1 < p < \infty$ and let $f \in L_p(I)$. Then the series*

(5.8) $$\sum_{n=0}^{\infty} (f, c_n) c_n(t)$$

converges a.e. on I.

The standard proof of this result, in view of Theorem 5.1, is omitted.

We conclude this section with the remark that a recent result of S. V. Bockariev on uniformly bounded orthonormal systems implies the existence of $f \in L_1(I)$ such that the series (5.8) diverges on a set of positive Lebesgue measure.

6. Estimates for the constants and some of their consequences. It has been shown by G. Watari [14] that the operators W_n are uniformly in n of weak type (1, 1), i.e. for $n \geqslant 1$ with some constant C

(6.1) $$|\{t: |W_n f(t)| > y\}| \leqslant \frac{C}{y}\|f\|_1, \quad y > 0.$$

On the other hand, the W_n's are uniformly in n of type (2, 2). This, (6.1) and the Marcinkiewicz interpolation theorem (see [15], Vol. II, (4.6)) imply that there is an absolute constant C such that

(6.2) $$\|W_n(f)\|_p \leqslant C\frac{p^2}{p-1}\|f\|_p, \quad 1 < p < \infty.$$

For the system $\{c_n\}$ we have a similar inequality

(6.3) $$\|C_n(f)\|_p \leqslant C\frac{p^2}{p-1}\|f\|_p, \quad 1 < p < \infty.$$

To see this we follow step by step the proof of S. Ropela [10] that $\{c_n\}$ is a basis in $L_p(I)$. First of all, we notice that there is C such that (cf. [3])

(6.4) $$\|F_n(f)\|_p \leqslant C\|f\|_p, \quad 1 \leqslant p \leqslant \infty.$$

Since $F_{2^\mu} = C_{2^\mu}$, it follows from (6.4) that for $2^\mu < n \leqslant 2^{\mu+1}$

(6.5) $$\|C_n(f)\|_p \leqslant C\|f\|_p + \|(C_n - C_{2^\mu})(f)\|_p.$$

The argument in [10] gives for some constant C

(6.6) $$\|(C_n - C_{2^\mu})(f)\|_p \leqslant C\cdot W(p)\cdot \|(C_{2^{\mu+1}} - C_{2^\mu})f\|_p,$$

where $W(p) = \sup\{\|W_n(f)\|_p: \|f\|_p \leqslant 1,\ n > 1\}$. Thus, combining (6.5), (6.6), (6.4) and (6.2), we obtain (6.3).

THEOREM 6.1. *For the system $\{c_n\}$ the following statements hold.*

If $f \in L \log^+ L$, then for some constant C

$$\|C_n(f)\|_1 \leqslant C \int_I |f(t) \log^+|f(t)|\,dt + C,$$

and, moreover, $\|C_n(f) - f\|_1 = o(1)$ as $n \to \infty$.

There are absolute positive constants λ and C such that

$$\int_I \exp[\lambda |C_n f(t)|]\,dt \leqslant C \quad \text{for} \quad \|f\|_\infty \leqslant 1.$$

This result follows from (6.3) by a general Theorem (4.41) in [15], Vol. II.

To obtain some results concerning

$$C_*(f) = \sup_n |C_n(f)|$$

in the limitting cases of $p = 1$ and $p = \infty$, we need to take more careful look at the proofs of Theorems 5.1 and 4.3. Moreover, rather delicate results on

$$W_*(f) = \sup_n |W_n(f)|$$

and arguments from [12] will be needed.

The following operators

$$T(f) = \sum_{n=1}^{\infty} (f, \chi_n) f_n,$$

$$Q(f) = \sum_{n=1}^{\infty} (f, f_n) \chi_n,$$

are well defined for Haar and Franklin polynomials, respectively.

By a *Franklin–Haar polynomial* we understand a finite linear combination of Franklin–Haar functions.

THEOREM 6.2. *There is a constant C such that*

$$|\{t:\ |Tf(t)|>y\}|\leqslant\frac{C}{y}\|f\|_1,\quad y>0, \tag{6.7}$$

and

$$|\{t:\ |Qg(t)|>y\}|\leqslant\frac{C}{y}\|g\|_1,\quad y>0, \tag{6.8}$$

hold for arbitrary Haar and Franklin polynomials f and g, respectively.

Proof. Inequality (6.8) can be proved exactly in the same way as Theorem 3.1. The proof of (6.7) is even simpler, it requires only instead of inequality (3.10) the following one

$$2^{-\mu}\int_{Q_i}|g_2(t)|\,dt\leqslant|Q_i|\int_{Q_i}|g_2(t)|\,dt\leqslant Cy\,|Q_i|^2,$$

where $|Q_i|\geqslant 2^{-\mu}$, and no integration by parts in the formula for coefficients is needed.

Now it follows from (6.7) and (6.8) that $TH_n(f)$ and $QF_n(f)$ converge in measure for each $f\in L_1(I)$, and the limits are used as definitions of $T(f)$ and $Q(f)$, correspondingly. It is therefore clear that with such definitions (6.7) and (6.8) hold for all $f,g\in L_1(I)$. Thus, the operators T and Q defined on $L_1(I)$ are of the weak type $(1,1)$. According to Bessel's inequality, both of them are of type $(2,2)$. Thus, the Marcinkiewicz's interpolation theorem (cf. [15], Vol. II, (4.6)) gives

$$\|Tf\|_p\leqslant\frac{C}{p-1}\|f\|_p,\qquad \|Qf\|_p\leqslant\frac{C}{p-1}\|f\|_p,\qquad 1<p\leqslant 2.$$

Thus, $T\colon L_p\to L_p$, $1<p\leqslant 2$, is bounded. Consequently, its conjugate $T^*\colon L_q\to L_q$, where $p^{-1}+q^{-1}=1$, is bounded too and $\|T\|_p=\|T^*\|_q$. However, $T^*=Q$, whence we infer

$$\|Q\|_p\leqslant C\frac{p^2}{p-1},\qquad 1<p<\infty. \tag{6.9}$$

Similarly, $Q^*=T$, $\|Q^*\|_q=\|Q\|_p$, i.e. $\|T\|_q=\|Q\|_p$, whence we get

$$\|T\|_p\leqslant C\frac{p^2}{p-1},\qquad 1<p<\infty. \tag{6.10}$$

Notice that $QTf=f$ for $f\in L_p$, and therefore $\|Q\|_p\|T\|_p\geqslant 1$. This, (6.9), and (6.10) give then

$$(C\cdot pq)^{-1}\|f\|_p\leqslant\|Tf\|_p\leqslant C\cdot pq\,\|f\|_p, \tag{6.11}$$

where $p^{-1}+q^{-1}=1$, $1<p<\infty$, and, clearly, this implies Theorem 4.3.

THEOREM 6.3. *There is a constant C such that*

$$\|C_*f\|_1\leqslant C\int_I|f|(\log^+|f|)^4+C \tag{6.12}$$

and

$$\lim_{n\to\infty}C_nf(t)=f(t)\ \textit{a.e.}\quad \textit{if } f\in L(\log L)^3. \tag{6.13}$$

Moreover, there are positive constants λ and C such that

$$\int_I\exp[\lambda(C_*f)^{1/3}]\leqslant C\quad \textit{if } \|f\|_\infty\leqslant 1. \tag{6.14}$$

Proof. To show (6.12), we use the inequality (cf. the proof of Theorem 5.1)

$$C_*f\leqslant C\{Mf+MW_*Qf\}, \tag{6.15}$$

where C is a constant and M is the Hardy–Littlewood maximal operator. It is proved in the P. Sjölin's paper [12], p. 567, Lemma 3.3, that

$$\|W_*\chi_F\|_p\leqslant C\frac{1}{(p-1)^2}|F|^{1/p},\qquad 1\leqslant p\leqslant 2,$$

where χ_F is the characteristic function of the set F. It follows that

$$\|MW_*\chi_F\|_p\leqslant C\frac{1}{(p-1)^3}|F|^{1/p},\qquad 1<p\leqslant 2,$$

whence, in the same way as in the proof on pp. 568–569 of [12], we get

$$\|MW_*f\|_1\leqslant C\int_I|f|(\log^+|f|)^3+C.$$

Therefore (cf. [15], Vol. II, p. 119)

$$\|MW_*Qf\|_1\leqslant C\int_I|Qf|(\log^+|Qf|)^3+C\leqslant C\int_I|f|(\log^+|f|)^4+C,$$

and this in combination with (6.15) gives (6.12).

To prove (6.13), we use Theorem (B) of [12], i.e.

$$\|W_*f\|_1\leqslant C\int_I|f|(\log^+|f|)^2+C$$

which implies

$$\begin{aligned}|\{t:\ MW_*Qf(t)>y\}|&\leqslant C\frac{1}{y}\|W_*Qf\|_1\leqslant\frac{1}{y}\Big[C\int_I|Qf|(\log^+|Qf|)^2+C\Big]\\&\leqslant\frac{1}{y}\Big[C\int_I|f|(\log^+|f|)^3+C\Big],\qquad y>0.\end{aligned} \tag{6.16}$$

Now (6.15) and (6.16) give

$$|\{t: C_* f(t) > y\}| \leqslant \frac{1}{y}\Big[C\int_I |f|(\log^+|f|)^3 + C\Big], \quad y > 0,$$

from which (6.13) follows.

It remains to prove (6.14). It is known that, for $p \geqslant 2$, $\|Mf\|_p \leqslant C\|f\|_p$ and it is shown in [12], p. 569, that $\|W_* f\|_p \leqslant Cp^2\|f\|_p$ for $p \geqslant 2$. Moreover, according to (6.9), we have $\|Qf\|_p \leqslant Cp\|f\|_p$ for $p \geqslant 2$. The combination of these three inequalities and (6.15) give $\|C_* f\|_p \leqslant Cp^3\|f\|_p$ for $p \geqslant 2$. The inequality (6.14) follows now from a general result (cf. [15], Vol. II, Theorem (4.41)).

References

[1] S. V. Bockariev, *Existence of basis in the space of analytic functions in the disc, and some properties of the Franklin system*, Mat. Sbornik 95 (137), (1974), pp. 3–18 (in Russian).

[2] — *Some inequalities for the Franklin series*, Analysis Mathematica 1 (1975), pp. 249–257.

[3] Z. Ciesielski, *Properties of the orthonormal Franklin system*, Studia Math. 23 (1963), pp. 141–157.

[4] — *Properties of the orthonormal Franklin system II*, ibid. 27 (1966), pp. 289–323.

[5] — *A bounded orthonormal system of polygonals*, ibid. 31 (1968), pp. 339–346.

[6] — *Constructive function theory and spline systems*, ibid. 43 (1975), pp. 277–302.

[7] C. L. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), pp. 107–115.

[8] J. Marcinkiewicz, *Quelques théorèmes sur les séries orthogonales*, Ann. Soc. Polon. Math. 16 (1937), pp. 85–96.

[9] R. E. A. C. Paley, *A remarkable series of orthogonal functions*, Proc. London Math. Soc. 34 (1932), pp. 241–279.

[10] S. Ropela, *Properties of bounded orthonormal spline bases*, in *Approximation theory* 1975, Banach Center Publications, Vol. 3 (to appear).

[11] F. Schipp, *On a.e. convergence of expansion with respect to a bounded orthonormal system of polygonals*, Studia Math. 58 (1976), pp. 287–290.

[12] P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh-Fourier series*, Arkiv Mathematik 7 (1968), pp. 551–570.

[13] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton, N. J., 1970.

[14] Ch. Watari, *Mean convergence of Walsh Fourier series*, Tôhoku Math. J. 16 (1964), pp. 183–188.

[15] A. Zygmund, *Trigonometric series*, Cambridge 1959.

MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES, WARSAW
DEPARTMENT OF COMPUTER MATHEMATICS, EOTVOS LÓRÁND UNIVERSITY, BUDAPEST
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF STOCKHOLM, STOCKHOLM

Received November 11, 1975 (1088)

STUDIA MATHEMATICA, T. LX. (1977)

Quelques propriétés des opérateurs uniformément convexifiants

par

B. BEAUZAMY (Palaiseau, France)

Résumé. Soient E et F deux espaces de Banach entre lesquels existe une injection continue i. En utilisant les constructions de Brunel–Sucheston, nous montrons que i n'est pas uniformément convexifiable si et seulement si l'on peut construire deux espaces de Banach E_1 et F_1, munis de bases E.S.A. (e_n) et (f_n), et une injection continue i_1, de E_1 dans F_1, qui envoie (e_n) sur (f_n) et qui est finiment représentable dans i. Nous en déduisons en particulier que si i n'est pas uniformément convexifiante, on peut trouver des carrés homothétiques dans E et F (et aussi dans n'importe quel espace intermédiaire entre E et F). Nous étudions aussi les rapports avec les ultrapuissances et l'interpolation.

1. Carrés dans les espaces intermediaires. Soient E et F deux espaces de Banach, et T un opérateur linéaire continu de E dans F. Nous dirons que E et F *ont des carrés liés par* T si, pour tout $\varepsilon > 0$, on peut trouver dans E deux points u, v, avec

$$(1)\qquad \|u\|_E \leqslant 1, \quad \|v\|_E \leqslant 1, \quad \left\|\frac{u \pm v}{2}\right\|_E \geqslant 1-\varepsilon, \quad \left\|\frac{Tu \pm Tv}{2}\right\|_F \geqslant (1-\varepsilon)\sup(\|Tu\|_F, \|Tv\|_F).$$

Si T est une injection continue de E dans F, et si E et F ont des carrés liés par T, nous dirons qu'ils *ont des carrés homothétiques*: les points (u, v) forment un carré dans E, et les points

$$\frac{u}{\sup(\|u\|_E, \|v\|_F)}, \quad \frac{v}{\sup(\|u\|_E, \|u\|_F)},$$

considérés comme des points de F (c'est-à-dire E étant plongé dans F par T) forment un carré dans F.

Ces définitions s'étendent au cas où sont donnés n espaces de Banach $X_1, \ldots, X_n$, et des opérateurs linéaires continus T_i: $X_i \to X_{i+1}$, $i = 1, \ldots, n-1$. Là encore, si les T_i sont des injections, nous parlerons de carrés homothétiques dans $(X_1, \ldots, X_n)$.

Nous renvoyons à [1] pour la définition et les premières propriétés des opérateurs uniformément convexifiants. Nous emploierons seulement la caractérisation ci-dessous, qui est équivalente à la définition.

Rappel. T, opérateur linéaire continu de E dans F, n'est pas uni-