$$\|g\|_{B^{s-1/p}_{p,q}(R_{n-1})} \sim \Big(\sum_{m'\in N_{n-1}} |b_{m'}|^p\Big)^{1/p} + \Big[\sum_{k=1}^{\infty}\sum_{l=1}^{L_{n-1}}\Big(\sum_{m'\in N_{n-1}} |2^{k(s-n/p)} b^{(l)}_{k,m'}|^p\Big)^{q/p}\Big]^{1/q},$$

with the usual modification for $q = \infty$. Here $m' = (m_1, \ldots, m_{n-1})$ and $\sigma'^{(l)} = (\sigma_1^{(l)}, \ldots, \sigma_{n-1}^{(l)})$. If $\sigma^{(l)} = (\sigma'^{(l)}, 3)$, then it follows from Theorem 1 (iii) that the operator S,

$$(Sg)(x) = \sum_{\substack{m\in N_n\\ m=(m',0)}} \Big[b_{m'} \prod_{j=1}^{n} \frac{\sin 2x_j}{2x_j - m_j\pi} + \sum_{k=1}^{\infty}\sum_{l=1}^{L_{n-1}} b^{(l)}_{k,m'} e^{-i2^{k-1}\sigma^{(l)}x} \prod_{j=1}^{n} \frac{\sin 2^{k-1}x_j}{2^{k-1}x_j - m_j\pi}\Big],$$

has the desired properties.

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Banach spaces quasi-reflexive of order one

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Abstract. It is shown that the nonreflexive Banach space J which is isomorphic to J^{**} is not isomorphic to J^*. In fact, J^* is not isomorphic to any subspace of J. Without explicitly describing the norm, it is shown that there is a Banach space which is quasi-reflexive of order one and isomorphic to its first dual. It has a basis with several properties similar to properties of the bases for J and J^*.

It is customary to use J to indicate any Banach space isomorphic to the space introduced in [5]. Thus J is isomorphic to a space that is isometric to its second dual [6] and J is quasi-reflexive of order one (i.e., the quotient of J^{**} and the natural image of J in J^{**} has dimension one). If J is isometric to J^{**}, then the l_2-product of J and J^* is quasi-reflexive of order two and isometric to its first dual.

It remains unknown whether J is isomorphic to some subspace of J^*. However, it seems to be a reasonable but difficult-to-prove conjecture that c_0 is not finitely representable in J^*. In fact, the three-dimensional space $l_\infty^{(3)}$ may not be representable in the predual I of J. Since c_0 is finitely representable in J ([4]), the truth of this conjecture would imply J is not isomorphic to any subspace of I. It also remains unknown whether there is a Banach space that is quasi-reflexive of order one and isometric to its dual. The methods of this paper suggest heuristically that no such space exists.

1. The space I is not isomorphic to any subspace of J. A particular norm will be chosen for J and the predual I will be evaluated explicitly. Any such predual is isomorphic to J^*. To prove that I is not isomorphic to any subspace of J (Theorem 3), it will be shown that I contains subspaces nearly isometric to $l_1^{(n)}$ (Lemma 2) in such a way that T being an isomorphism of I into J has the impossible consequence (Theorem 3) that, for any $\theta < 1$ and any positive integer n, there are members $\{x^1, \ldots, x^n\}$ of I such that $\|\sum_1^n x^j\| > \theta n$ and $\|\sum_1^n Tx^j\| < \|T\|\sqrt{n}/\theta$. This is done by constructing $l_1^{(n)}$-subspaces in I whose images in J are similar when

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regarded only as functions on the positive integers, but as normed spaces are more like $l_2^{(n)}$-spaces.

In this paper, the norm to be used for J is

$$\|x\| = \sup\Big\{\sum_{k=1}^{n}[x(p_{2k-1})-x(p_{2k})]^2\Big\}^{1/2}, \tag{1}$$

where the sup is over all positive integers n and all *strictly increasing* sequences $\{p_k\}$ of positive integers. The space J is the Banach space of all sequences $x=\{x(i)\}$ of real numbers such that $\lim_{i\to\infty} x(i)=0$ and $\|x\|$ is finite. Let $\{e_n\}$ be the natural basis for J with the norm (1). If $u_n=\sum_{i=1}^{n} e_i$, then

$$\Big\|\sum_1^\infty x(i)u_i\Big\| = \Big\|\Big[\sum_1^\infty x(i)\Big]e_1+\Big[\sum_2^\infty x(i)\Big]e_2+\Big[\sum_3^\infty x(i)\Big]e_3+\dots\Big\| = \sup\Big\{\sum_{k=1}^{n}\Big[\sum_{p_{2k-1}}^{p_{2k}-1} x(i)\Big]^2\Big\}^{1/2}, \tag{2}$$

where again the sup is over all positive integers n and all *strictly increasing* sequences $\{p_k\}$ of positive integers. The natural basis $\{e_n\}$ for the norm (1) is shrinking ([5]) and the natural basis $\{u_n\}$ for the norm (2) is boundedly complete ([10], Corollary 6.1, p. 286).

The space J has a unique predual in the sense that, if X is a Banach space for which X^* is isomorphic to J, then X is isomorphic to J^* ([2], Theorem 3.6, p. 908).

In order to define I for which I^* is isometric to J, the following special conventions will be used. A *bump* is a sequence of real numbers $x=\{x(i)\}$ for which there is a bounded interval and a number a such that $x(i)=a$ if i is in this interval and $x(i)=0$ otherwise. The *altitude* of the bump is a and its *sign* is the sign of a. Two bumps are *disjoint* if the intersection of their associated intervals is empty; they are *strongly disjoint* if these intervals are separated by at least one integer, and the first bump *contains* the second if its interval contains the other interval.

DEFINITION. The space I is the completion of the normed linear space of sequences with finite support for which

$$\|x\| = \inf\Big\{\sum_{k=1}^{n}[\![x^k]\!]:\ x=\sum_{k=1}^{n}x^k\Big\}, \tag{3}$$

where $[\![\ ]\!]$ is the function defined by $[\![x]\!]=\big(\sum a_i^2\big)^{1/2}$ if x is the sum of *strongly disjoint* bumps whose altitudes are $\{a_i\}$.

It is known that the predual of J is isomorphic to the space obtained by replacing *strongly disjoint* by *disjoint* in the preceding definition (let X on page 279 of [7] be one-dimensional and $x_i=e$ for all i). This predual will be used in Lemma 6. However, the use of strongly disjoint bumps makes the following lemma true, which simplifies computations leading to the proof of Theorem 3.

LEMMA 1. *If x belongs to I and is the sum of finitely many strongly disjoint bumps, then $[\![x]\!]=\|x\|$.*

Proof. Let us observe first that if x and y are sums of strongly disjoint bumps whose supports are the same sequence of intervals and the respective altitudes are $\{a_i\}$ and $\{b_i\}$, then

$$[\![x+y]\!] \leqslant [\![x]\!]+[\![y]\!], \tag{4}$$

since this is equivalent to $\big[\sum(a_i+b_i)^2\big]^{1/2}\leqslant\big[\sum a_i^2\big]^{1/2}+\big[\sum b_i^2\big]^{1/2}$. Now let x be the sum of finitely many strongly disjoint bumps, so all the bumps have supports in some bounded interval $[0,N]$. When estimating $\|x\|$ by use of (3), we can restrict each x^k to have support in $[0,N]$. There are only a finite number of sets of intervals in $[0,N]$ such that any two intervals are separated by at least one integer. For each such set, it follows from (4) that in (3) there need be at most one x^k whose bumps have the intervals in this set as supports. Therefore, there exist $\{x^k\}$ with $x=\sum_{k=1}^{n}x^k$ and

$$\|x\| = \sum_{k=1}^{n}[\![x^k]\!]. \tag{5}$$

Let I_1 and I_2 be the supports of the first two bumps in x, let s be the last point in I_1, and let t be the first point in I_2. Redefine each x^k so x^k has the value $x^k(s)$ on I_1 and is zero to the left of I_1. This does not change $\sum_{k=1}^{n}x^k$. Also, the sum in (5) is not increased, since there is no k for which $[\![x^k]\!]$ is increased. Therefore, the sum in (5) is not changed.

Let us now consider bumps of *type B*, for which the support of the bump contains the closed interval $[s,t]$; bumps of *type S*, for which the support of the bump contains s and not t; and bumps of *type S*^{1}, for which the support of the bump contains $s+1$ and not s. We need the following facts (i)–(iii).

(i) If $i\neq j$ and ξ_1 is a bump in x^i and ξ_2 is a bump in x^j, where the absolute altitude of ξ_2 is θ times the absolute altitude of ξ_1 and $0<\theta\leqslant 1$, then x^i can be replaced in (5) by the two vectors θx^i and $(1-\theta)x^i$ so that the new system will have bumps $\theta\xi_1$ and ξ_2 that have equal absolute altitudes and have the same signs, respectively, as ξ_1 and ξ_2.

(ii) *All bumps of type B or S have the same signs.* Otherwise, it follows from (i) that there is no loss of generality if we assume there are bumps ξ_1 and ξ_2 of opposite signs and equal absolute altitudes, both of which contain s. If ξ_1 is the shorter bump, then ξ_1 can be discarded and ξ_2 replaced by $\xi_1+\xi_2$. This is impossible, since it would reduce the sum in (5).

(iii) *There is no loss of generality if we assume all bumps of type B or S^1 have the same signs.* Suppose ξ_1 and ξ_2 are bumps of opposite signs, ξ_1 is a B-bump belonging to x^i, and ξ_2 is an S^1-bump belonging to x^j. Again, we invoke (i) and assume ξ_1 and ξ_2 have the same absolute altitudes. Then ξ_2 is the first bump in x^j and ξ_1 and ξ_2 do not have the same right ends, since if they did we could discard ξ_2 in x^j and replace ξ_1 in x^i by $\xi_1+\xi_2$, which would reduce $[\![x^i]\!]$ and not change $[\![x^j]\!]$. Therefore $\xi_1+\xi_2$ consists of two bumps, neither of which is of type B or S^1. We can replace ξ_1 in x^i by the first or the second of these bumps, according as ξ_2 extends farther to the right than ξ_1 or ξ_1 extends farther to the right than ξ_2, and then replace ξ_2 in x^j by the other bump. This changes neither $[\![x^i]\!]$ nor $[\![x^j]\!]$, but together with the possible use of (i) it results in a net decrease of at least one in the sum of the number of B-bumps and the number of S^1-bumps. Successive application of this process leads to a representation of x in (5) for which all remaining B-bumps and S^1-bumps have the same signs.

Since $x(s+1)=0$ and all bumps that are nonzero at $s+1$ are of type B, S or S^1, it follows from (ii) and (iii) that there are no bumps of type B. Since no bumps have both s and t in their support, we shorten all bumps with s in their support to end at s, shorten all bumps with t to begin at t, and discard all bumps with support in (s,t). This does not change $\sum_{k=1}^{n}x^k$ and cannot decrease the sum in (5), so no bumps had support in (s,t).

The preceding process can be continued inductively to replace $\sum_{k=1}^{n}x^k$ by $\sum_{k=1}^{m}\bar{x}^k$ that satisfies (5) and for which the supports of all bumps are so supports of bumps in x. It follows from (4) that $[\![x]\!]=\|x\|$.

THEOREM 1. *The dual of I is J and the natural basis of I is shrinking.*

Proof. Let $\{e_j\}$ be the natural basis for I and $\{u_j\}$ be the corresponding coefficient functionals. For a continuous linear functional f on I, let $(e_j,f)=f(j)$ for each j and $\|f\|_J$ be the norm of $\sum_1^\infty f(i)u_i$ as given by (2). Suppose $x=\{x(i)\}$ is the sum of n strongly disjoint bumps and let a_k and $[p_{2k-1},p_{2k}-1]$ be the altitude and the support of the kth bump. Then Lemma 1 gives $\|x\|=\left(\sum a_k^2\right)^{1/2}$ and we have

$$|(x,f)|=\Big|\sum f(i)x(i)\Big|\leqslant\sum_{k=1}^{n}\Big|\sum_{p_{2k-1}}^{p_{2k}-1}f(i)\Big|\,|a_k|$$
$$\leqslant\Big\{\sum_{k=1}^{n}\Big[\sum_{p_{2k-1}}^{p_{2k}-1}f(i)\Big]^2\Big\}^{1/2}\Big[\sum_{k=1}^{n}a_k^2\Big]^{1/2}\leqslant\|f\|_J\,\|x\|.$$

Since the unit ball is the closed convex span of such x's, we have $\|f\|\leqslant\|f\|_J$. For any $\varepsilon>0$, there is a strictly increasing sequence $\{p_k\}$ such that

$$\|f\|_J<\Big\{\sum_{k=1}^{n}\Big[\sum_{p_{2k-1}}^{p_{2k}-1}f(i)\Big]^2\Big\}^{1/2}+\varepsilon.$$

Let x have the value $\sum_{p_{2k-1}}^{p_{2k}-1}f(i)$ on the interval $[p_{2k-1},p_{2k}-1]$. Then

$$|(x,f)|=\sum_{k=1}^{n}\Big[\sum_{p_{2k-1}}^{p_{2k}-1}f(i)\Big]^2\geqslant(\|f\|_J-\varepsilon)\|x\|.$$

Thus $\|f\|\geqslant(\|f\|_J-\varepsilon)$ for all $\varepsilon>0$ and $\|f\|=\|f\|_J$, so I^* is isometric to J. Also, the coefficient functionals of the natural basis for I can be identified with the basis $\{u_n\}$ for J used in (2) and therefore the natural basis for I is shrinking ([3], Lemma 1, p. 90).

It is known that a basis $\{e_n\}$ for a Banach space B is *shrinking* if and only if the sequence of coefficient functionals $\{u_n\}$ is a boundedly complete basis for B^* ([10], Corollary 6.1, p. 286). The space I has a shrinking basis whose sequence of coefficient functionals $\{u_n\}$ is a boundedly complete basis for J, the basis used in (2). Also, $\{u_1,u_2-u_1,u_3-u_2,\ldots\}$ is the natural shrinking basis for J with the norm (1). The next theorem shows this behavior implies quasi-reflexivity of order one.

THEOREM 2. *Suppose a Banach space B has a basis $\{e_n\}$ with coefficient functionals $\{u_n\}$. If $\sum_1^\infty e_i$ is not norm-convergent and $\{u_1,u_2-u_1,u_3-u_2,\ldots\}$ is a shrinking basis for B^* with coefficient functionals $\{F_i\}$, then*

(a) *$e_n\leftrightarrow F_n-F_{n+1}$ defines an isometry of B onto the closure of $\{\sum_1^n a_iF_i$: $n\geqslant1$ and $\sum_1^n a_i=0\}$,*

(b) *B is quasi-reflexive of order one.*

Proof. To show $e_n\leftrightarrow F_n-F_{n+1}$ is the natural embedding of B into B^{**}, it is sufficient to show that (F_n-F_{n+1},u_k) is identical to $(u_k,e_n)=\delta_n^k$. We have $(F_n-F_{n+1},u_k)=(F_n,u_k)-(F_{n+1},u_k)$, which is equal to

$$[(F_n,u_k-u_{k-1})+(F_n,u_{k-1}-u_{k-2})+\ldots+(F_n,u_2-u_1)+(F_n,u_1)]-$$
$$-[(F_{n+1},u_k-u_{k-1})+(F_{n+1},u_{k-1}-u_{k-2})+\ldots+(F_{n+1},u_2-u_1)+(F_{n+1},u_1)],$$

so (F_n-F_{n+1},u_k) equals $1-0=1$ if $k=n$, equals $1-1=0$ if $n<k$, and equals $0-0=0$ if $n>k$. This completes the proof of (a). The isometry in (a) is the natural embedding of B onto $\hat{B}\subset B^{**}$. Since $\{u_1,u_2-$

$-u_1, u_3-u_2, \ldots\}$ is shrinking, the span of F_1 and $\hat{B}$ is B^{**}, and $\hat{B}$ has the basis $\{F_n - F_{n+1}\}$. If $F_1 \in \hat{B}$, then $F_1 = \sum_1^\infty a_i(F_i - F_{i+1})$ and

$$(a_1-1)F_1 + \sum_2^n (a_i - a_{i-1})F_i = a_n F_{n+1} - \sum_{n+1}^\infty a_i(F_i - F_{i+1})$$

for all n. This implies each a_i is 1, $F_1 = \sum_1^\infty (F_i - F_{i+1})$, and the contradiction that $\sum_1^\infty e_i$ is convergent. Thus $F_1 \notin \hat{B}$ and B is quasi-reflexive of order one.

LEMMA 2. *For the space I, any $\theta < 1$, and any positive integer n, there are integers $\{r_k\colon 1 \leqslant k \leqslant n\}$ with r_{k+1}/r_k even and the property that*

$$\sum_1^n |a_k| \geqslant \Big\| \sum_1^n a_k x^k \Big\| \geqslant \theta \sum_1^n |a_k| \quad \textit{for all } \{a_k\}$$

if $\|x^k\| = 1$ for each k, each x^k is the sum of $s_1 s_2 \ldots s_k$ strongly disjoint bumps of alternating signs and equal absolute altitudes, each s_k is even and $s_k \geqslant r_k/r_{k-1}$, and each bump of x^k contains s_{k+1} of the bumps of x^{k+1}.

[The first step in the proof of this lemma is to show that, for the isometry of I into J^* given by Theorem 2, a norm-one sum x of $2r$ strongly disjoint bumps of alternating signs corresponds to a member F of J^* for which $(y, F) = 1$, where y in J is a sum of r disjoint "humps". It then is noted that if each y^k for $1 \leqslant k \leqslant n$ is of the same type as y, with the r_{k+1} "humps" in y^{k+1} spaced uniformly throughout the intervals on which the slope of y^k is nonzero, and if each r_{k+1}/r_k is sufficiently large, then the linear span of $\{y^k\}$ is nearly isometric to l_∞^n. The last step is to show that if each y^k is related to F^k as y was related to F, then lin $\{F^k\}$ is nearly isometric to $l_1^{(n)}$, which implies that the linear span of the x^k's that correspond to the F^k's is nearly isometric to $l_1^{(n)}$.]

Proof. Let r be a positive integer and μ a positive number for which $\mu^{-1}(2r)^{-1/2}$ is a positive integer. Then let $\{p_i\colon 1 \leqslant i \leqslant 4r\}$ be a strictly increasing sequence of positive integers such that, if $1 \leqslant k \leqslant r$,

$$p_{4k-2} - p_{4k-3} = p_{4k} - p_{4k-1} = \mu^{-1}(2r)^{-1/2}.$$

Define the function $y = \{y(i)\}$ on the positive integers by letting y be linear on all intervals $[p_i, p_{i+1}]$ with $1 \leqslant i < 4r$; $y(i) = 0$ if $i \leqslant p_1$, if $i \geqslant p_{4r}$, or if i is $4k-3$ or $4k$ with $1 \leqslant k \leqslant r$; and

$$y(4k-2) = y(4k-1) = (2r)^{-1/2} \quad \text{if } 1 \leqslant k \leqslant r.$$

Then with the norm given by (1), we have $\|y\| = 1$. Also, y is the sum of r disjoint "humps" in the sense that, on the interval $[p_1, p_2]$, y increases

linearly with slope μ from 0 to $(2r)^{-1/2}$; y is constant on the interval $[p_2, p_3]$; on the interval $[p_3, p_4]$, y decreases linearly with slope $-\mu$ to 0; y is zero on the interval $[p_4, p_5]$; etc.

Let $\{F(i)\}$ be the coefficient functionals for the natural shrinking basis for J. For y as described in the preceding paragraph, let a corresponding linear functional F be defined by

$$F = (2r)^{-1/2} \sum_{k=1}^r \{[-F(p_{4k-3}) + F(p_{4k-2})] + [F(p_{4k-1}) - F(p_{4k})]\}.$$

Then

$$(y, F) = (2r)^{-1/2} \sum_{k=1}^r \{[-0 + (2r)^{-1/2}] + [(2r)^{-1/2} - 0]\} = 1.$$

Also,

$$F = (2r)^{-1/2} \sum_{k=1}^r \Big\{ \sum_{i=1}^{p_{4k-2}-p_{4k-3}} [-F(p_{4k-3}+i-1) + F(p_{4k-3}+i)] \Big\} + $$
$$+ (2r)^{-1/2} \sum_{k=1}^r \Big\{ \sum_{i=1}^{p_{4k}-p_{4k-1}} [F(p_{4k-1}+i-1) - F(p_{4k-1}+i)] \Big\}.$$

Thus it follows from Theorem 2 that the image x of F in the predual I of J consists of r pairs of strongly disjoint bumps, the kth pair being a bump of altitude $-(2r)^{-1/2}$ on the interval $[p_{4k-3}, p_{4k-2}-1]$ and a bump of altitude $(2r)^{-1/2}$ on the interval $[p_{4k-1}, p_{4k}-1]$. From Lemma 1, we have $\|x\| = 1$. Therefore $\|F\| = 1$.

It follows by the gruesome arguments in [4] that if each y^k for $1 \leqslant k \leqslant n$ is of the same type as y in the preceding paragraphs, but with $r = r_k$; if each r_{k+1}/r_k is a sufficiently large even integer; and if the "humps" in y^{k+1} are spaced uniformly throughout the intervals on which the slope of y^k is nonzero, then lin $\{y^k\colon 1 \leqslant k \leqslant n\}$ is nearly isometric to $l_\infty^{(n)}$. Thus if a set of linear functionals $\{F^k\}$ is such that $\|F^k\| = 1$ and (y^i, F^k) is nearly δ_i^k for all i and k, then $\{F^k\}$ has the property that lin $\{F^k\}$ is nearly isometric to $l_1^{(n)}$. If F^k corresponds to y^k as F corresponded to y in the preceding paragraphs, then $(y^i, F^j) = 0$ if $i > j$. If each r_{k+1}/r_k is sufficiently large, then (y^i, F^j) is nearly zero for all $i < j$. Thus for any $\theta < 1$ we could have chosen $\{y^k\}$ and $\{F^k\}$ so that

$$\Big\| \sum_1^n a_k F^k \Big\| \geqslant \theta \sum_1^n |a_k| \quad \text{for all } \{a_k\}.$$

Each F^k is the image of an element x^k in I that has $2r_k$ bumps of alternating signs. Moreover, for each bump ξ in x^k, r_{k+1}/r_k of the bumps in x^{k+1} have support in the support of ξ. Since the norm of I is *repetition-invariant* in the sense that

$$\|\{x(i)\}\| = \|\{x(i), \ldots, x(k-1), x(k), x(k), x(k+1), x(k+2), \ldots\}\|$$

for all x and all k, Lemma 2 needs no restriction about uniformity of lengths of bumps or uniformity of distances between bumps—only the restriction that, for each bump ξ in x^k, r_{k+1}/r_k of the bumps in x^{k+1} have support in the support of ξ.

THEOREM 3. *The space I is not isomorphic to any subspace of J.*

Proof. Let T be an isomorphism of I into J. We first establish facts (A) and (B):

(A). For any $\varepsilon > 0$ and $k > 0$, there exists an integer $\varkappa$ such that, for each integer $\lambda > \varkappa$, there is an arbitrarily large $k(\lambda)$ such that the bump $\xi^{k(\lambda)}$ with altitude 1 on the interval $[k, k(\lambda)]$ has the property that $T\xi^{k(\lambda)}$ can be made constant on $[\varkappa, \lambda]$ without changing its J-norm by as much as ε.

To see that (A) is true, first choose an increasing sequence of integers $\{p_j\}$ so that, if ξ^{p_j} is the bump with altitude 1 on the interval $[k, p_j]$, then

$$\lim_{j\to\infty} (T\xi^{p_j})(i) = a(i)$$

exists for each i. Then there is a $\varkappa$ such that, for any $\lambda > \varkappa$, the J-norm of the sequence $\{a(i) - a^\lambda(i)\}$ is less than $\frac{1}{2}\varepsilon$ if $\{a^\lambda(i)\}$ is obtained from $\{a(i)\}$ by replacing $a(i)$ by $a(\varkappa)$ for all i in the interval $[\varkappa, \lambda]$, since otherwise there would be an infinite sequence of disjoint intervals $[\varkappa_i, \lambda_i]$ and for each interval a sum of type

$$\Big\{\sum_{j=1}^{m_i} [a(p_{2j-1}^i) - a(p_{2j}^i)]^2\Big\}^{1/2}, \qquad \varkappa_i \leqslant p_1^i < \ldots < p_{2m_i}^i \leqslant \lambda_i,$$

with these sums bounded away from zero. This contradicts boundedness of the J-norms of $T\xi^{p_j}$. It follows now that for this $\varkappa$ and any $\lambda > \varkappa$, there is a $j(\lambda)$ such that $(T\xi^{p_{j(\lambda)}})(i)$ approximates $a(i)$ for $i \in [\varkappa, \lambda]$ well enough that $T\xi^{p_{j(\lambda)}}$ can be made constant on $[\varkappa, \lambda]$ without changing its norm by as much as ε. Let $k(\lambda) = p_{j(\lambda)}$.

(B). For any positive ε and $\varkappa$, there exists $k(\varepsilon, \varkappa)$ such that, if ξ is any bump of altitude 1 with $\xi(i) = 0$ for $i \leqslant k(\varepsilon, \varkappa)$, then $|(T\xi)(i)|$ is small enough for $i \leqslant \varkappa$ that deleting the initial segment of $T\xi$ on $[1, \varkappa]$ does not change $\|T\xi\|_J$ by more than ε.

To establish (B), it is sufficient to show that

$$\limsup_{k\to\infty} \{|(T\xi^{s,t})(i)|:\ k \leqslant s \leqslant t\} = 0 \qquad \text{for each } i,$$

where $\xi^{s,t}$ is a bump of altitude 1 on the interval $[s, t]$. If there were an i for which this limit is not zero, there would exist a positive δ such that, for any n, there is a sum x of n strongly disjoint bumps of absolute altitudes 1 for which $(Tx)(i) > n\delta$, so that $\|Tx\|_J > n\delta$. This contradicts $\|x\|_I = \sqrt{n}$, which follows from Lemma 1.

The proof of Theorem 3 will be completed by showing that if there is an isomorphism T of I into J, then for any positive integer n and any $\theta < 1$, there is a set $\{x^j:\ 1 \leqslant j \leqslant n\}$ of members of I such that

$$\Big\|\sum_1^n x^j\Big\|_I > \theta n \quad \text{and} \quad \Big\|\sum_1^n Tx^j\Big\|_J < \frac{1}{\theta}\, \|T\| \sqrt{n},$$

which implies $\|T\|\,\|T^{-1}\| > \theta^2\sqrt{n}$ for all n. This will be done explicitly only for $n = 3$, since this case clearly is typical.

Choose r_1, r_2, r_3 as described in Lemma 2. For an arbitrary positive ε, let $s_1 = r_1$ and choose $s_2 \geqslant r_2/r_1$ and $s_3 \geqslant r_3/r_2$ to be even integers such that

$$\text{(6)} \qquad \sup\{|y(i)|\} < \varepsilon/s_1, \qquad \sup\{|z(i)|\} < \varepsilon/s_1 s_2,$$

if y is the image of an element in I with norm not greater than $(s_1 s_2)^{-1/2} + (s_1 s_2 s_3)^{-1/2}$ and z is the image of an element in I with norm not greater than $(s_1 s_2 s_3)^{-1/2}$. Now x^1, x^2, and x^3 will be constructed to satisfy Lemma 2 for $n = 3$. Choose $\Delta > 0$ so that

$$3(s_1 + s_1 s_2 + s_1 s_2 s_3)\Delta < \varepsilon.$$

With repeated applications of (A) and (B), choose the left end of the first bump in x^1, then the left end of the first bump in x^2, then the first s_3 bumps in x^3, then the right end of the first bump in x^2, then the left end of the second bump in x^2, then the second set of s_3 bumps in x^3, etc. The right end of the first bump in x^1 is chosen after the first s_2 bumps in x^2 and the first $s_2 s_3$ bumps in x^3 have been completed. This is continued until all the s_1 bumps in x^1, $s_1 s_2$ bumps in x^2, and $s_1 s_2 s_3$ bumps in x^3 have been completed.

This construction of x^1, x^2 and x^3 can be done so that:

For $j = 2$ or $j = 3$, there are s_{j-1} intervals on each of which Tx^{j-1} can be made constant without changing $\|Tx^{j-1}\|_J$ by more than Δ. Moreover, the image of each bump in x^j can be truncated on the left and right without changing the J-norm of this image by more than 2Δ. This can all be done so the supports of the modified images of bumps from x^j lie in disjoint intervals and the set of these intervals can be partitioned into s_{j-1} disjoint sets of s_j intervals each, all the intervals in the same set lying in an interval on which Tx^{j-1} has been made constant. Then

$$\|Tx^1 + Tx^2 + Tx^3\|_J$$

is not changed by more than $3(s_1 + s_1 s_2 + s_1 s_2 s_3)\Delta < \varepsilon$ if all these truncations are made and appropriate sections of images of bumps are replaced by constant functions.

Now consider a sum (7) of the type used in (1) for estimating the square of the J-norm for w, where w is $\sum_{j=1}^{3} Tx^j$ modified as described above:

$$\sum_{k=1}^{n} [w(p_{2k-1}) - w(p_{2k})]^2. \tag{7}$$

There are at most $2s_1$ terms in (7) with the property that one of p_{2k-1} or p_{2k} is in one of the s_1 intervals on which the modified images of bumps in x^1 are constant, but the other is not in the same such interval. If p_i is not in an interval on which the modified image of a bump in x^1 is constant, then $w(p_i) = Tx^1(p_i)$. Otherwise, $w(p_i)$ differs from the value at p_i of the modified Tx^1 by not more than the sum of the sup norm of the image of a bump in x^2 and the sup norm of a bump in x^3. Because of (6), this difference is $O(\varepsilon)/s_1$, so replacing this value of w in (7) by the corresponding value of the modified Tx^1 makes a change $O(\varepsilon)/s_1$ in (7). Thus the sum of the terms in (7) with at most one of p_{2k-1} or p_{2k} in the same "constant section" of the image of a bump in x^1 differs by $O(\varepsilon)$ from a sum of type (7) for estimating the squared J-norm of the modified Tx^1. After deleting these terms, the remaining terms in (7) have the property that both p_{2k-1} and p_{2k} are in the same "constant section" of the image of a bump in x^1, so their sum is a sum of type (1) for estimating the squared J-norm of the modification of Tx^2+Tx^3.

This and another application of the preceding procedure using the "constant sections" of Tx^2 and the second inequality in (6) enable us to see that the squared J-norm of the modification of $\sum_{1}^{3} Tx^j$ is not larger than $\sum_{1}^{3} \|(Tx^j)_{\text{modified}}\|^2 + O(\varepsilon)$. Since ε was arbitrary, it could have been chosen small enough that

$$\Big\| \sum_{1}^{3} Tx^j \Big\| < \frac{1}{\theta}\Big[\sum_{1}^{3} \|Tx^j\|^2\Big]^{1/2} \leqslant \frac{1}{\theta}\|T\|\sqrt{3}.$$

2. A Banach space that is quasi-reflexive of order one and isomorphic to its dual. For computational reasons, it will be useful to introduce the concept of double basis and several related concepts, some of which are extensions of familiar properties of bases. It should be noted that repetition-invariance is somewhat dual to the equal-signs-additive property studied by Brunel and Sucheston [1].

A *double basis* for a Banach space X is a subset $\{e_n\colon -\infty < n < \infty\}$ such that each x in X has a unique representation as $x = \sum_{-\infty}^{\infty} x(i)e_i$ in the sense that

$$\lim_{m,n\to\infty} \Big\| x - \sum_{-m}^{n} x(i)e_i \Big\| = 0.$$

A *bimonotone* (*double*) *basis* is a (double) basis $\{e_n\}$ such that

$$\Big\|\sum_{p}^{s} x(i)e_i\Big\| \geqslant \Big\|\sum_{q}^{r} x(i)e_i\Big\| \quad \text{if} \quad p \leqslant q \leqslant r \leqslant s.$$

A *shrinking double basis* is a double basis $\{e_n\}$ such that each of the basic sequences $\{e_n\colon n<0\}$ and $\{e_n\colon n>0\}$ is shrinking.

A *neighborly* (*double*) *basis* is a (double) basis $\{e_n\}$ such that $\|\sum_{-\infty}^{\infty} x(i)e_i\|$ is not increased if some $x(k)$ is replaced by either $x(k-1)$ or $x(k+1)$. A neighborly basis is of type P ([9], p. 354), but not conversely. Clearly, neighborliness implies monotonicity. Also, neighborliness implies what we shall call *repetition-invariance*, namely, for all r,

$$\Big\|\sum_{-\infty}^{\infty} x(i)e_i\Big\| = \Big\|\sum_{-\infty}^{r} x(i)e_i + x(r)e_{r+1} + \sum_{r+1}^{\infty} x(i)e_{i+1}\Big\|, \tag{8}$$

and also implies *translation-invariance* for double bases (but not for ordinary bases), namely,

$$\Big\|\sum_{-\infty}^{\infty} x(i)e_i\Big\| = \Big\|\sum_{-\infty}^{\infty} x(i-1)e_i\Big\|, \tag{9}$$

since it is possible to transform $\sum_{-\infty}^{\infty} x(i)e_i$ into the vector of the second member of (8) or (9) and back again by successive replacements of components, each being replaced by one of the two neighboring components—except that this is not possible for (9) with a basis $\{e_n\colon n\geqslant 1\}$ unless $x(1)=0$.

If $\{e_n\}$ is a double basis or a basis, *inversion-invariance* means that, whenever x has finite support, there is an n such that $\|\sum x(i)e_i\| = \|\sum x(n-i)e_i\|$ and, when $\{e_n\}$ is a basis (and *not* a double basis), $x(i)=0$ if $i \geqslant n$.

With respect to a (double) basis $\{e_n\}$, a finite set $\{z^k\colon 1\leqslant k\leqslant s\}$ being *disjoint* means each z^k is a linear combination of a finite set of consecutive members of $\{e_n\}$ and these sets are pairwise disjoint; if each pair of sets is separated by at least one member of $\{e_n\}$, then $\{z^k\}$ is *strongly disjoint*.

After proving a sequence of lemmas, we will obtain a norm for the space of functions with finite support on the set of all integers. This norm is constructed as the limit of a sequence of norms, the first norm being similar to the norm of I and the second norm similar to the norm of J. Inequality (10) will be preserved in the limit, so the natural basis will be shrinking. Finally, the desired space will be the subspace consisting of functions with support on the positive integers.

LEMMA 3. *Let $\{e_n\}$ be a bimonotone double basis for a Banach space X. Let M be a number such that,*

(i) *if $\{z^k\}$ is strongly disjoint with respect to $\{e_n\}$, then*

$$\Big\|\sum_1^s z^k\Big\| \leqslant M\Big[\sum_1^s \|z^k\|^2\Big]^{1/2}; \tag{10}$$

(ii) *if $z = \sum_{-\infty}^{\infty} z(i)e_i$, $\{p_k\}$ is a strictly increasing sequence of integers, and $\{\zeta^k\}$ is a sequence such that $\zeta^k(i) = 0$ if $i \leqslant p_{2k-1}$ or $i > p_{2k}$, and $\zeta^k(i) = z(i) - z(p_{2k-1})$ if $p_{2k-1} \leqslant i \leqslant p_{2k}$, then*

$$\frac{1}{M}\Big[\sum_1^s \|\zeta^k\|^2\Big]^{1/2} \leqslant \|z\|. \tag{11}$$

If $\{u_n\}$ are the coefficient functionals for $\{e_n\}$ and $\{u_n - u_{n-1}\}$ is repetition-invariant, then $\{u_n - u_{n-1}\}$ also satisfies (10) *and* (11).

Proof. Note that if $f = \sum_{-\infty}^{\infty} f(i)(u_i - u_{i-1})$ with only finitely many nonzero terms, and if $x = \sum_{-\infty}^{\infty} x(i)e_i$, then

$$(x, f) = \sum_{-\infty}^{\infty}[f(i) - f(i+1)]x(i) = \sum_{-\infty}^{\infty} f(i)[x(i) - x(i-1)]. \tag{12}$$

Suppose $\{f^k\colon 1 \leqslant k \leqslant s\}$ is strongly disjoint with respect to $\{u_n - u_{n-1}\}$ and let $f = \sum_1^s f^k$. Then there is a strictly increasing sequence $\{p_k\}$ of integers such that

$$f^k = \sum_{p_{2k-1}}^{p_{2k}} f^k(i)(u_i - u_{i-1}) \quad \text{if} \quad 1 \leqslant k \leqslant s,$$

and $f(i) = 0$ if $i \leqslant p_1$, $i > p_{2s}$, or $p_{2k} < i \leqslant p_{2k+1}$ and $1 \leqslant k < s$.

Since $\{e_n\}$ is bimonotone, it follows from (12) that there is an x such that $(x, f) = \|f\|\,\|x\|$ and $x(i) = 0$ if $i < p_1$ or $i > p_{2s}$. Now we use the last member of (12) to see that, if $\xi^k(i) = 0$ when $i \leqslant p_{2k-1}$ or $i > p_{2k}$, and $\xi^k(i) = x(i) - x(p_{2k-1})$ when $p_{2k-1} \leqslant i \leqslant p_{2k}$, then

$$(x, f) = \sum_{k=1}^{s}\sum_{p_{2k-1}+1}^{p_{2k}} f^k(i)[x(i) - x(i-1)] = \sum_{k=1}^{s}(\xi^k, f^k).$$

Therefore,

$$\Big\|\sum_1^s f^k\Big\| \leqslant \frac{\sum\|\xi^k\|\,\|f^k\|}{\|x\|} \leqslant \frac{\sum\|\xi^k\|\,\|f^k\|}{[\sum\|\xi^k\|^2]^{1/2}/M} \leqslant M\Big[\sum_1^s \|f^k\|^2\Big]^{1/2}.$$

Now suppose $f = \sum_{-\infty}^{\infty} f(i)(u_i - u_{i-1})$ and $\{p_k\}$ is a strictly increasing sequence of integers. Since $\{u_n - u_{n-1}\}$ is repetition-invariant, we can assume without loss of generality that $f(p_{2k-1}) = f(p_{2k-1}+1)$ if $1 \leqslant k \leqslant s$. For $1 \leqslant k \leqslant s$, let φ^k satisfy $\varphi^k(i) = 0$ if $i \leqslant p_{2k-1}$ or $i > p_{2k}$, and

$$\varphi^k(i) = f(i) - f(p_{2k-1}) \quad \text{if} \quad p_{2k-1} \leqslant i \leqslant p_{2k}.$$

Since $\{e_n\}$ is bimonotone and $\varphi^k(p_{2k-1}) = \varphi^k(p_{2k-1}+1) = 0$, it follows from (12) that there is an x^k for each k such that

$$(x^k, \varphi^k) = \sum_{p_{2k-1}+2}^{p_{2k}} \varphi^k(i)[x^k(i) - x^k(i-1)] = \|\varphi^k\|\,\|x^k\|$$

and $\|x^k\| = \|\varphi^k\|$, where $x^k(i) = 0$ if $i \leqslant p_{2k-1}$ or $i > p_{2k}$. Then $\{x^k\}$ is strongly disjoint and, since $f(x^k) = (x^k, \varphi^k)$ for each k,

$$\|f\| \geqslant \frac{\sum_1^s (x^k, \varphi^k)}{\big\|\sum_1^s x^k\big\|} = \frac{\sum_1^s \|\varphi^k\|^2}{\big\|\sum_1^s x^k\big\|} \geqslant \frac{\sum_1^s \|\varphi^k\|^2}{M[\sum\|x^k\|^2]^{1/2}} = \frac{1}{M}\Big[\sum_1^s \|\varphi^k\|^2\Big]^{1/2}.$$

LEMMA 4. *Let $\{e_n\}$ be a neighborly double basis for a Banach space B. If $\{u_n\}$ are the coefficient functionals, then $\{u_n - u_{n-1}\}$ is a neighborly double basis for its closed linear span.*

Proof. Suppose $f = \sum_{-\infty}^{\infty} f(i)(u_i - u_{i-1})$, with only finitely many nonzero terms. Define g by letting $g(i) = f(i)$ if $i \neq r+1$ and $g(r+1) = f(r)$. Because of (12) and $\{e_n\}$ being bimonotone, there is an x such that

$$\|g\| = \frac{\sum_{-\infty}^{\infty}[g(i) - g(i+1)]x(i)}{\|x\|}. \tag{13}$$

Since $g(r) - g(r+1) = 0$ and $\{e_n\}$ is neighborly, we can assume $x(r) = x(r+1)$. Then

$$\begin{aligned}&[f(r) - f(r+1)]x(r) + [f(r+1) - f(r+2)]x(r+1)\\ &\qquad = [f(r) - f(r+2)]x(r+1) = [g(r+1) - g(r+2)]x(r+1),\end{aligned}$$

so the numerator in (13) is not changed if g is replaced by f. Therefore, $\|f\| \geqslant \|g\|$. A similar argument can be used if $g(r-1) = f(r)$. Now that $\{u_n - u_{n-1}\}$ is known to be neighborly, it follows that it also is bimonotone and therefore basic.

LEMMA 5. *Let X be the linear space of functions defined on the set* *of all integers and having finite support. Let $\bar{M}$ be a positive number and $\| \|_1$ be a norm for X such that, if $\| \|_2$ is defined by*

$$\|x\|_2 = \Big\| \sum_{-\infty}^{\infty} x(i)(u_i - u_{i-1}) \Big\| \quad \text{if} \quad \{x(i)\} \in X, \tag{14}$$

where $\{u_n\}$ is the set of coefficient functionals for the double basis $\{e_n\}$ and $\| \|$ is the dual norm of $\| \|_1$, then

(a) *for $\| \|_1$, the natural double basis $\{e_n\}$ for X is inversion-invariant, neighborly, and satisfies* (10) *and* (11) *if $M = \bar{M}$;*

(b) *for $\| \|_2$, $\{u_n - u_{n-1}\}$ is an inversion-invariant neighborly double basis and satisfies* (10) *and* (11) *if $M = \bar{M}$;*

(c) $\|x\|_2 \leqslant \|x\|_1$ *if $x \in X$.*

If $|||x|||_1 = [\frac{1}{2}(\|x\|_1^2 + \|x\|_2^2)]^{1/2}$ for all x in X, and $||| \, |||_2$ is defined relative to $||| \, |||_1$ as $\| \|_2$ was defined relative to $\| \|_1$, then $||| \, |||_1$ and $||| \, |||_2$ satisfy (a), (b), (c), *satisfy* (10) *and* (11) *if $M = \bar{M}$, and also satisfy*

(d) $\|x\|_2 \leqslant |||x|||_2 \leqslant |||x|||_1 \leqslant \|x\|_1$ *if $x \in X$.*

Proof. With $||| \, |||_1$ defined as stated, it follows that the natural basis for X is inversion-invariant and neighborly, since $\| \|_1$ and $\| \|_2$ have these properties and neighborliness implies translation-invariance. If the finite set $\{z^k\}$ is strongly disjoint, then

$$\Big|\Big|\Big| \sum z^k \Big|\Big|\Big|_1 = \Big[\tfrac{1}{2}\Big(\Big\| \sum z^k \Big\|_1^2 + \Big\| \sum z^k \Big\|_2^2\Big)\Big]^{1/2}$$
$$\leqslant \Big[\tfrac{1}{2}\Big(\bar{M}^2 \sum \|z^k\|_1^2 + \bar{M}^2 \sum \|z^k\|_2^2\Big)\Big]^{1/2} = \bar{M}\Big[\sum |||z^k|||_1^2\Big]^{1/2},$$

so (10) is satisfied; (11) follows similarly. We have established (a).

Now define $||| \, |||_2$ by letting

$$|||f|||_2 = \Big|\Big|\Big| \sum_{-\infty}^{\infty} f(i)(u_i - u_{i-1}) \Big|\Big|\Big| \quad \text{if} \quad f = \{f_n\} \in X, \tag{14'}$$

where $||| \, |||$ is the dual norm of $||| \, |||_1$. It follows from inversion-invariance and translation-invariance of $\{e_n\}$ with respect to $||| \, |||_1$ that $\{u_n - u_{n-1}\}$ is inversion-invariant with respect to $||| \, |||$; it follows from Lemma 4 that $\{u_n - u_{n-1}\}$ is neighborly with respect to $||| \, |||$; and it follows from Lemma 3 that $\{u_n - u_{n-1}\}$ satisfies (10) and (11) with respect to $||| \, |||$ if $M = \bar{M}$. Therefore, with respect to $||| \, |||_2$, $\{e_n\}$ is inversion-invariant, neighborly, and satisfies (10) and (11) if $M = \bar{M}$.

To prove (c), we first note that, for positive a and b,

$$[\tfrac{1}{2}(a^2 + b^2)]^{1/2} \geqslant \tfrac{1}{2}(a+b) \geqslant [\tfrac{1}{2}(1/a + 1/b)]^{-1}. \tag{15}$$

With $f = \sum_{-\infty}^{\infty} f(i)e_i$ in X and $\varphi = \sum_{-\infty}^{\infty} f(i)(u_i - u_{i-1})$, it follows from the definition of $||| \, |||_2$ and (15) that

$$|||f|||_2 = \sup\{(x, \varphi)/|||x|||_1\} = \sup\{(x, \varphi)/[\tfrac{1}{2}(\|x\|_1^2 + \|x\|_2^2)]^{1/2}\}$$
$$\leqslant \sup\big\{\tfrac{1}{2}\big((x, \varphi)/\|x\|_1 + (x, \varphi)/\|x\|_2\big)\big\}.$$

It follows from (14) that $(x, \varphi) \leqslant \|f\|_2 \|x\|_1$ for all x. Since

$$(x, \varphi) = \sum_{-\infty}^{\infty} [f(i) - f(i+1)]x(i) = \sum_{-\infty}^{\infty} [x(i) - x(i-1)]f(i),$$

it follows from translation-invariance of $\| \|_2$ and replacing x by $-x$ that $(x, \varphi) \leqslant \|x\|_2 \|f\|_1$ for all x. Thus

$$|||f|||_2 \leqslant \tfrac{1}{2}(\|f\|_2 + \|f\|_1) \leqslant [\tfrac{1}{2}(\|f\|_1^2 + \|f\|_2^2)]^{1/2} = |||f|||_1,$$

the second inequality following from (15).

To prove (d), note first that $\| \|_2 \leqslant ||| \, |||_1 \leqslant \| \|_1$ follows from $\| \|_2 \leqslant \| \|_1$ and the definition of $||| \, |||_1$, and that we have just proved $||| \, |||_2 \leqslant ||| \, |||_1$. Finally, note that $\| \|_2 \leqslant ||| \, |||_2$ follows from $||| \, |||_1 \leqslant \| \|_1$ and the validity of (14) for $\| \|_2$ and (14)′ for $||| \, |||_2$.

Lemma 6. *Let X be the linear space of functions defined on the set of all integers and having finite support. Let $\| \|_1$ and $\| \|_2$ be defined analogously to* (3) *and* (1), *respectively, with*

$$\| \|_1 = 2^{1/2} \inf\Big\{\sum_{k=1}^{n} [\![x^k]\!] : x = \sum_{k=1}^{n} x^k\Big\}, \tag{16}$$

where $[\![x]\!] = \big(\sum a_n^2\big)^{1/2}$ if x is the sum of disjoint (not necessarily strongly disjoint) bumps whose altitudes are $\{a_n\}$, and

$$\|x\|_2 = 2^{-1/2} \sup\Big\{\sum_{k=1}^{n} [x(p_{2k-1}) - x(p_{2k})]^2\Big\}^{1/2}, \tag{17}$$

where the sup is over all positive integers n and all increasing (but not necessarily strictly increasing) sequences $\{p_k\}$ of positive integers. Then for both $\| \|_1$ and $\| \|_2$, the natural basis $\{e_n\}$ for X is inversion-invariant and neighborly. Also, these norms satisfy (14), *they satisfy $\|x\|_2 \leqslant \|x\|_1$ if $x \in X$, $\| \|_2$ satisfies* (10) *and* (11) *with $M = 2^{1/2}$, and $\| \|_1$ satisfies* (10) *and* (11) *with $M = 2(1 + 2^{1/2})$.*

Proof. It is easy to see that the natural basis for X, given $\| \|_2$, is inversion-invariant and neighborly; this also is true for $\| \|_1$, since it is true for $[\![\ ]\!]$. But this would not have been true if the bumps in (16) were required to be strongly disjoint, or if the sequence $\{p_k\}$ in (17) were required to be strictly increasing. Just as for the proof of Theorem 1, it can be shown that the dual of X given the norm $\| \|_1$ is isometric to X with the norm

$$\|\{x(i)\}\| = 2^{-1/2}\sup\Big\{\sum_{k=1}^{n}\Big[\sum_{p_{2k-1}}^{p_{2k}} x(i)\Big]^2\Big\}^{1/2},$$

where the sup is over all positive integers n and all *strictly increasing* sequences $\{p_k\}$ of positive integers. Then the norm of the linear functional

$$\sum_{-\infty}^{\infty} x(i)(u_i-u_{i-1}) = \sum_{-\infty}^{\infty}[x(i)-x(i+1)]u_i,$$

with X given the norm $\| \|_1$, is equal to

$$2^{-1/2}\sup\Big\{\sum_{k=1}^{n}[x(p_{2k-1})-x(p_{2k}+1)]^2\Big\}^{1/2} = \|\{x(i)\}\|_2.$$

Therefore (14) is satisfied. Since $(a-b)^2 \leqslant 2(a^2+b^2)$ for numbers a and b, it follows from (17) that if x is the sum of disjoint bumps of altitudes $\{a_n\}$, then

$$\|x\|_2 \leqslant 2^{-1/2}\Big(\sum 4a_n^2\Big)^{1/2} = 2^{1/2}\Big(\sum a_n^2\Big)^{1/2}. \tag{18}$$

Since the unit ball for $\| \|_1$ is the closed convex span of such x's which also satisfy $2^{1/2}(\sum a_n^2)^{1/2} = 1$, we have $\|x\|_2 \leqslant \|x\|_1$ for all x in X.

Now we only need an M for which (10) and (11) are satisfied for both $\| \|_1$ and $\| \|_2$. First, we shall consider $\| \|_2$. Suppose $\{z^k\colon 1 \leqslant k \leqslant s\}$ is strongly disjoint. For each interval $[p_{2k-1}, p_{2k}]$ used in (17) that contains a q at which $z = \sum_1^s z^k$ is 0, we replace $[z(p_{2k-1})-z(p_{2k})]^2$ by $2[z(p_{2k-1})-z(q)]^2+2[z(q)-z(p_{2k})]^2$ and obtain

$$\|z\|_2 \leqslant 2^{-1/2}\Big(\sum_1^s 4\|z^k\|_2^2\Big)^{1/2} = 2^{1/2}\Big(\sum_1^s \|z^k\|_2^2\Big)^{1/2}.$$

Now suppose $z = \{z(i)\}$, $\{q_k\}$ is a strictly increasing sequence of integers, and for $1 \leqslant k \leqslant s$ each ζ^k is defined by

$$\zeta^k(i) = \begin{cases} 0 & \text{if} \quad i \leqslant q_{2k-1} \text{ or } i > q_{2k}, \\ z(i)-z(q_{2k-1}) & \text{if} \quad q_{2k-1} \leqslant i \leqslant q_{2k}. \end{cases}$$

Since $\zeta^k(i) = 0$ if $i \leqslant q_{2k-1}$ or $i > q_{2k}$, there are integers $\{r_j^k\}$ such that $q_{2k-1} \leqslant r_1^k \leqslant r_2^k \leqslant \ldots \leqslant r_{2m_k-1}^k \leqslant q_{2k} < r_{2m_k}^k$ and

$$\|\zeta^k\|_2^2 = \tfrac{1}{2}\sum_{j=1}^{m_k}[\zeta^k(r_{2j-1}^k)-\zeta^k(r_{2j}^k)]^2. \tag{19}$$

Then the sum of the first $m-1$ terms in the right member of (19) is not less than the last term, since otherwise the first m_k-1 terms could be replaced by $[\zeta^k(q_{2k-1})-\zeta^k(r_{2m_k-1}^k)]^2$ to increase the right member of (19). Therefore

$$\|\zeta^k\|_2^2 \leqslant \sum_{j=1}^{m_k-1}[\zeta^k(r_{2j-1}^k)-\zeta^k(r_{2j}^k)]^2 = \sum_{j=1}^{m_k-1}[z(r_{2j-1}^k)-z(r_{2j}^k)]^2.$$

Since $\|z\|_2^2$ is at least as large as $\sum_{k=1}^{s}\sum_{j=1}^{m_k-1}\frac{1}{2}[z(r_{2j-1}^k)-z(r_{2j}^k)]^2$, we have

$$\|z\|_2 \geqslant 2^{-1/2}\Big[\sum_1^s\|\zeta^k\|_2^2\Big]^{1/2}.$$

Thus (10) and (11) are satisfied for $\| \|_2$ with $M = 2^{1/2}$.

Now let x have the norm $\| \|_1$. We know that $\{e_n\}$ is a neighborly double basis for X; if $\{u_n\}$ are the coefficient functionals, then $\{u_n-u_{n-1}\}$ is a neighborly double basis for its closed linear span and

$$\Big\|\sum_{-\infty}^{\infty}f(i)(u_i-u_{i-1})\Big\| = \|\{f(i)\}\|_2.$$

Let $\{F_n\}$ be the coefficient functionals for $\{u_n-u_{n-1}\}$, considered as a basis for its closed linear span. Then it follows from Lemma 4 that $\{F_n-F_{n+1}\}$ is neighborly. Since $\| \|_2$ satisfies (10) and (11) for $M = 2^{1/2}$, it follows from Lemma 3 that this also is true for $\| \|$ defined by letting $\|\{x(i)\}\|$ be the norm of the linear functional $\sum_{-\infty}^{\infty}x(i)(F_i-F_{i+1})$ as a linear functiona on lin $\{u_n-u_{n-1}\}$. Also,

$$\Big\|\sum_{-\infty}^{\infty}x(i)(F_i-F_{i+1})\Big\| \tag{20}$$

$$= \sup\left\{\frac{\big(\sum_{-\infty}^{\infty}f(i)(u_i-u_{i-1}),\ \sum_{-\infty}^{\infty}x(i)(F_i-F_{i+1})\big)}{\big\|\sum_{-\infty}^{\infty}f(i)(u_i-u_{i-1})\big\|} : \{f(i)\}\in X\right\}$$

$$= \sup\left\{\frac{\sum_{-\infty}^{\infty}x(i)[f(i)-f(i+1)]}{\|f\|} : f = \sum_{-\infty}^{\infty}f(i)(u_i-u_{i-1})\right\}$$

$$= \sup\left\{\frac{\big(\sum_{-\infty}^{\infty}x(i)e_i, f\big)}{\|f\|} : f\in \text{lin}\,\{u_n-u_{n-1}\}\right\},$$

while

$$\|\{x(i)\}\|_1 = \sup\left\{\frac{\left(\sum_{-\infty}^{\infty} x(i)e_i,\, g\right)}{\|g\|} : g \in X^*\right\}. \tag{21}$$

Therefore $\left\|\sum_{-\infty}^{\infty} x(i)(F_i - F_{i+1})\right\| \leqslant \|\{x(i)\}\|_1$. If $g = \sum_r^s g(i)u_i$, then for any $n \leqslant r-1$ it is possible to write g as

$$g = au_n + f, \qquad f = a(u_{n+1}-u_n) + \sum_{n+2}^{s} a_i(u_i - u_{i-1}),$$

where $a = \sum_r^s g(i)$. Note that $\|u_{n+1}-u_n\| = \|e_{n+1}\|_2$, and it follows from (17) that $\|e_{n+1}\|_2 = 1$. Also, $\|u_n\| = 2^{-1/2}$, since $\{e_n\}$ is bimonotone and it follows from (16) that $\|e_n\|_1 = 2^{1/2}$. Since $\{u_n - u_{n-1}\}$ is bimonotone, we have

$$\|g\| \geqslant \|f\| - \|au_n\| = \|f\| - 2^{-1/2}\|a(u_{n+1}-u_n)\| \geqslant (1-2^{-1/2})\|f\|.$$

Therefore, it follows from (20) and (21) with $g = au_n + f$, where n is chosen so $(x, au_n) = 0$, that we finally have

$$(1-2^{-1/2})\|\{x(i)\}\|_1 \leqslant \left\|\sum_{-\infty}^{\infty} x(i)(F_i - F_{i+1})\right\| \leqslant \|\{x(i)\}\|_1.$$

Since $\{F_i - F_{i+1}\}$ satisfies (10) and (11) with $M = 2^{1/2}$, this implies $\|\ \|_1$ satisfies (10) and (11) with $M = 2(1+2^{1/2})$.

LEMMA 7. *Let X be the linear space of functions defined on the set of all integers and having finite support. There is a norm $\|\ \|$ for X such that $\{e_n\}$ is inversion-invariant, neighborly, and satisfies* (10) *and* (11) *with $M = 2(1+2^{1/2})$. Also, if each of $\|\ \|_1$ and $\|\ \|_2$ is taken to be $\|\ \|$, then* (14) *is satisfied.*

Proof. From Lemma 6, we have two norms that satisfy (14); satisfy (a), (b), (c) of Lemma 5; and satisfy (10) and (11) with $M = 2(1+2^{1/2})$. Then successive determinations of new pairs of norms $(\|\ \|_1^{(n)}, \|\ \|_2^{(n)})$ by use of Lemma 5 gives two pointwise-convergent sequences, $\{\|\ \|_1^{(n)}\}$ and $\{\|\ \|_2^{(n)}\}$. These sequences converge to the same limit $\|\ \|$, since

$$\|\ \|_1^{(n+1)} = [\tfrac{1}{2}(\|\ \|_1^{(n)})^2 + \tfrac{1}{2}(\|\ \|_2^{(n)})^2]^{1/2}, \qquad \|\ \|_2^{(n)} \leqslant \|\ \|_2^{(n+1)} \leqslant \|\ \|_1^{(n+1)} \leqslant \|\ \|_1^{(n)}.$$

To show $\|\ \|$ is the desired norm, the only serious question seems to be whether, if each of $\|\ \|_1$ and $\|\ \|_2$ is taken to be $\|\ \|$, then (14) is satisfied.

If $x = \sum_{-\infty}^{\infty} x(i)e_i$ belongs to X, let $\|x\|^*$ be the norm of the linear functional $\sum_{-\infty}^{\infty} x(i)(u_i - u_{i-1})$, when X is given the norm $\|\ \|$. Since $\|\ \| \leqslant \|\ \|_1^{(n)}$ for all n, it follows from (14) being satisfied for each pair $(\|\ \|_1^{(n)}, \|\ \|_2^{(n)})$ that $\|\ \|^* \geqslant \|\ \|_2^{(n)}$ for each n. Therefore, $\|\ \|^* \geqslant \|\ \|$. Let us show that $\|\ \|^* \leqslant \|\ \|$. As noted in the proof of Lemma 5, if $f = \sum_{-\infty}^{\infty} f(i)e_i$ belongs to X and $\varphi = \sum_{-\infty}^{\infty} f(i)(u_i - u_{i-1})$, then $(x, \varphi) \leqslant \|x\|_2^{(n)}\|f\|_1^{(n)}$. Since $\|x\| \geqslant \|x\|_2^{(n)}$ for each n, this implies $(x, \varphi) \leqslant \|x\|\,\|f\|_1^{(n)}$ for each n and each x in X. Therefore $\|f\|^* \leqslant \|f\|_1^{(n)}$ for each n.

THEOREM 4. *There is a Banach space B that is quasi-reflexive of order one and isomorphic to B^*. The space B is the completion of a normed linear space of functions on the positive integers with finite support and has the properties:*

(i) *the natural basis $\{e_n\}$ is inversion-invariant, translation-invariant, and neighborly, with $1 \leqslant \|e_n\| \leqslant \sqrt{3/2}$;*

(ii) $[\sum a_n^2]^{1/2} \leqslant \|x\| \leqslant 2^{1/2}[\sum a_n^2]^{1/2}$ *if x is a sum of strongly disjoint bumps with altitudes $\{a_n\}$;*

(iii) *if $\{x^k\}$ is strongly disjoint, then*

$$[2(1+2^{1/2})]^{-1}\left[\sum \|x^k\|^2\right]^{1/2} \leqslant \left\|\sum x^k\right\| \leqslant 2(1+2^{1/2})\left[\sum \|x^k\|^2\right]^{1/2};$$

(iv) *B is isomorphic to a Banach space that is isometric to its second dual.*

Proof. Let $\|\ \|$ be the norm given by Lemma 7 for the space X, with $\{u_n\}$ the coefficient functionals for the natural basis $\{e_n\}$. As noted in the proof of Lemma 6, $\|e_n\|_1 = 2^{1/2}$ and $\|e_n\|_2 = 1$. Therefore,

$$1 \leqslant \|e_n\| = \|u_n - u_{n-1}\| \leqslant [\tfrac{1}{2}(\|e_n\|_1)^2 + \tfrac{1}{2}(\|e_n\|_2)^2] = \sqrt{3/2}.$$

Since $\|e_n\| \geqslant 1$, we have $\|u_n\| \leqslant 1$. Since $\{u_n: -\infty < n < \infty\}$ is inversion-invariant and $\{u_n - u_{n-1}: -\infty < n < \infty\}$ is bimonotone,

$$\begin{aligned} \text{dist}\,(u_1, \text{lin}\{u_n - u_{n-1}: n \neq 1\}) &= \text{dist}\,(u_0, \text{lin}\,\{u_n - u_{n-1}: n \neq 1\}) \\ &\geqslant \tfrac{1}{2}\text{dist}\,(u_1 - u_0, \text{lin}\,\{u_n - u_{n-1}: n \neq 1\}) \\ &\geqslant \tfrac{1}{2}\|u_1 - u_0\| \geqslant \tfrac{1}{2}. \end{aligned} \tag{22}$$

Let B be the completion of $\text{lin}\,\{e_n: n \geqslant 1\}$. Since $\{e_n: -\infty < n < \infty\}$ is inversion-invariant and neighborly, $\{e_n: n \geqslant 1\}$ is inversion-invariant, translation-invariant, and neighborly. It follows from (10) that $\{e_n: n \geqslant 1\}$ is a shrinking basis for B. Since $\{e_n: -\infty < n < \infty\}$ is bimonotone, the norm of $\sum_1^{\infty} a_i u_i$ is independent of whether it is regarded as a member of B^* or X^*. Since $\{u_n - u_{n-1}: -\infty < n < \infty\}$ is shrinking and basic, and $u_1 \notin \text{cl}\,[\text{lin}\,\{u_n - u_{n-1}: n \geqslant 1\}]$, the sequence $\{u_1, u_2 - u_1, u_3 - u_2, \ldots\}$

is a shrinking basis for B^*. Thus it follows from Theorem 2 that B is quasi-reflexive of order one. Now let

$$T\Big[\sum_1^\infty x(i)e_i\Big] = x(1)u_1 + \sum_2^\infty x(i)(u_i - u_{i-1}). \tag{23}$$

Because (14) is satisfied if each of $\|\ \|_1$ and $\|\ \|_2$ is taken to be $\|\ \|$, each series in (23) converges if and only if the other converges. Thus T is a linear one-to-one map of B onto B^*. Since

$$\Big\|x(1)u_1 + \sum_2^\infty x(i)(u_i - u_{i-1})\Big\| \leqslant \|x(1)u_1\| + \Big\|\sum_2^\infty x(i)(u_i - u_{i-1})\Big\|$$
$$\leqslant |x(1)| + \Big\|\sum_1^\infty x(i)e_i\Big\|$$
$$\leqslant \|x(1)e_1\| + \Big\|\sum_1^\infty x(i)e_i\Big\| \leqslant 2\Big\|\sum_1^\infty x(i)e_i\Big\|,$$

we have $\|T\| \leqslant 2$. Also, $\|T^{-1}\| \leqslant 3$, since

$$\Big\|\sum_1^\infty x(i)e_i\Big\| = \Big\|\sum_1^\infty x(i)(u_i - u_{i-1})\Big\|$$
$$\leqslant \|x(1)u_0\| + \Big\|x(1)u_1 + \sum_2^\infty x(i)(u_i - u_{i-1})\Big\|,$$
$$\leqslant |x(1)| + \Big\|x(1)u_1 + \sum_2^\infty x(i)(u_i - u_{i-1})\Big\|,$$

and this and (22) imply

$$\Big\|\sum_1^\infty x(i)e_i\Big\| \leqslant 3\Big\|x(1)u_1 + \sum_2^\infty x(i)(u_i - u_{i-1})\Big\|.$$

Finally, (ii) and (iii) follow from the fact that (ii), and (10) and (11) with $M = 2(1+2^{1/2})$, are satisfied by both of the norms used to initiate the convergence to the norm of Lemma 7; (iv) follows from Theorem 4 of [1] and the fact that $\{e_n\}$ being neighborly implies the sequence of coefficient functionals $\{u_n\}$ is equal-signs-additive.

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