\[ |g|_{p,d-1}^{1/p} \lesssim \left( \sum_{m \in \mathbb{N}} \left( \sum_{s=1}^{m} \left( \sum_{s=1}^{m} |\alpha^{m,s}|p \right)^{p/2} \right)^{1/p} \right)^{1/p} \]

with the usual modification for \( q = \infty \). Here \( m' = (m_1, \ldots, m_{n-1}) \) and \( \phi = (\phi_1, \ldots, \phi_{n-1}) \). If \( \phi = (\phi^0, 3) \), then it follows from Theorem 1 (iii) that the operator \( S \),

\[ (Sg)(s) = \sum_{m \in \mathbb{N}} b_m \left[ \sum_{s=1}^{m} \sin \frac{2\pi \tau_s}{2m - m_0} \right] + \sum_{s=1}^{m} b_m \left[ \prod_{j=1}^{n} \frac{\sin 2\pi \tau_j}{2m - m_0} \right], \]

has the desired properties.

**References**


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Banach spaces quasi-reflexive of order one

by

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**Abstract.** It is shown that the nonreflexive Banach space \( J^* \) which is isomorphic to \( J^{*0} \) is not isomorphic to \( J^{*0} \). In fact, \( J^{*0} \) is not isomorphic to any subspace of \( J^* \). Without explicitly describing the norm, it is shown that there is a Banach space which is quasi-reflexive of order one and isomorphic to its first dual. It has a basis with several properties similar to properties of the bases for \( J^* \) and \( J^* \).

It is customary to use \( J \) to indicate any Banach space isomorphic to the space introduced in [5]. Thus \( J \) is isomorphic to a space that is isometric to its second dual [6] and \( J \) is quasi-reflexive of order one (i.e., the quotient of \( J^{*0} \) and the natural image of \( J \) in \( J^{*0} \) has dimension one). If \( J \) is isometric to \( J^{*0} \), then the \( L_1 \)-product of \( J \) and \( J^{*0} \) is quasi-reflexive of order two and isometric to its first dual.

It remains unknown whether \( J \) is isomorphic to some subspace of \( J^{*0} \). However, it seems to be a reasonable but difficult-to-prove conjecture that \( e_n \) is not finitely representable in \( J^{*0} \). In fact, the three-dimensional space \( e_3 \) may not be representable in the predual \( J \) of \( J \). Since \( e_n \) is finitely representable in \( J \) ([4]), the truth of this conjecture would imply \( J \) is not isomorphic to any subspace of \( I \). It also remains unknown whether there is a Banach space that is quasi-reflexive of order one and isomorphic to its dual. The methods of this paper suggest heuristically that no such space exists.

1. The space \( I \) is not isomorphic to any subspace of \( J \). A particular norm will be chosen for \( I \) and the predual \( I \) will be evaluated explicitly. Any such predual is isomorphic to \( J^{*0} \). To prove that \( I \) is not isomorphic to any subspace of \( J \) (Theorem 3), it will be shown that \( I \) contains subspaces nearly isometric to \( e_3 \) (Lemma 2) in such a way that \( T \) being an isomorphism of \( I \) into \( I \) has the impossible consequence (Theorem 3) that, for any \( \theta < 1 \) and any positive integer \( n \), there are members \( (a_1, \ldots, a_n) \) of \( I \) such that \( \sum a_i \geq \theta n \) and \( \sum T a_i \leq \theta T \sqrt{n} / 9 \). This is done by constructing \( f^0 \)-subspaces in \( I \) whose images in \( I \) are similar when

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regarded only as functions on the positive integers, but as normed spaces are more like $l^0$ spaces.

In this paper, the norm to be used for $J$ is

$$\|x\| = \sup \left\{ \sum_{i=1}^n (x(p_{2i-1}) - x(p_{2i}))^2 \right\}^{1/2},$$

where the sup is over all positive integers $n$ and all strictly increasing sequences $(p_k)$ of positive integers. The space $J$ is the Banach space of all sequences $x = (x(i))$ of real numbers such that $\lim x(i) = 0$ and $|x|$ is finite. Let $(e_n)$ be the natural basis for $J$ with the norm (1). If $v_n = \sum e_n$, then

$$\left\| \sum_{n=1}^N x(n) e_n \right\| = \left\| \sum_{n=1}^N x(n) e_n + \sum_{n=1}^N (-x(n)) e_n + \sum_{n=1}^N x(n) e_n + \cdots \right\|$$

$$= \sup \left\{ \sum_{n=1}^N x(n)^2 \right\}^{1/2},$$

where again the sup is over all positive integers $n$ and all strictly increasing sequences $(p_k)$ of positive integers. The natural basis $(e_n)$ for the norm (1) is shrinking ([5]) and the natural basis $(e_n)$ for the norm (2) is boundedly complete ([10], Corollary 6.1, p. 286).

The space $J$ has a unique predual in the sense that, if $X$ is a Banach space for which $X^*$ is isomorphic to $J$, then $X$ is isomorphic to $J^*$ ([19], Theorem 3.6, p. 906).

In order to define $I$ for which $I^*$ is isometric to $J$, the following special conventions will be used. A bump is a sequence of real numbers $a = (a(i))$ for which there is a bounded interval and a number $a$ such that $a(i) = a$ if $i$ is in this interval and $a(n) = 0$ otherwise. The altitude of the bump is $a$ and its sign is the sign of $a$. Two bumps are disjoint if the intersection of their associated intervals is empty; they are strongly disjoint if these intervals are separated by at least one integer, and the first bump contains the second if its interval contains the other interval.

Definition. The space $I$ is the completion of the normed linear space of sequences with finite support for which

$$\|x\| = \inf \left\{ \sum_{n=1}^N \|x_n\| : x = \sum_{n=1}^N x_n \right\},$$

where $f$ is the function defined by $[x] = (\sum_{n=1}^N x_n)^2$ if $x$ is the sum of strongly disjoint bumps whose altitudes are $(a_n)$. It is known that the predual of $J$ is isomorphic to the space obtained by replacing strongly disjoint by disjoint in the preceding definition (let

$X$ on page 279 of [7] be one-dimensional and $x_i = e$ for all $i$). This predual will be used in Lemma 8. However, the use of strongly disjoint bumps makes the following lemma true, which simplifies computations leading to the proof of Theorem 3.

Lemma 1. If $x \in I$ and is the sum of finitely many strongly disjoint bumps, then $[x] = \|x\|$. Prove.

Proof. Let us observe first that if $x$ and $y$ are sums of strongly disjoint bumps whose supports are the same sequence of intervals and the respective altitudes are $(a_n)$ and $(b_n)$, then

$$[x+y] = [x] + [y],$$

since this is equivalent to $\sum (a_n + b_n)^2 \leq \sum a_n^2 + \sum b_n^2$. Now let $x$ be the sum of finitely many strongly disjoint bumps, so all the bumps have supports in some bounded interval $[0, N]$. When estimating $[x]$ by use of (3), we can restrict each $x_0$ to have support in $[0, N]$. There are only a finite number of sets of intervals in $[0, N]$ such that any two intervals are separated by at least one integer. For each such set, it follows from (4) that if $(3)$ there need be at most one $x_0$ whose bumps have the intervals in this set as supports. Therefore, there exist $(x_n^*)$, each of which

$$\|x_n^*\| = \sum_{n=1}^N \|x_n^*\|,$$

Let $I_1$ and $I_2$ be the supports of the first two bumps in $x$, let $a$ be the last point in $I_1$, and let $b$ be the first point in $I_2$. Redefine each $x_0$ so $x_0$ has the value $x_0(a)$ on $I_1$ and is zero to the left of $I_1$. This does not change $\sum_{n=1}^N x_n$. Also, the sum in (5) is not increased, since there is no $k$ for which $\|x_n^*\|$ is increased. Therefore, the sum in (5) is not changed.

Let us now consider bumps of type $B$, for which the support of the bump contains the closed interval $[s, t]$; bumps of type $S$, for which the support of the bump contains $s$ and not $t$; and bumps of type $S^0$, for which the support of the bump contains $s+1$ and not $s$. We need the following facts (i)-(iii).

(i) If $i \neq j$, and $\xi_i$ is a bump in $x_i$ and $\xi_j$ is a bump in $x_j$, where the absolute altitude of $\xi_i$ is $\theta$ times the absolute altitude of $\xi_j$ and $0 < \theta < 1$, then $\mu_i$ can be replaced in (5) by the two vectors $\theta \mu_i$ and $(1-\theta) \mu_i$ so that the new system will have bumps $\theta \xi_i$ and $\xi_j$ that have equal absolute altitudes and have the same signs, respectively, as $\xi_i$ and $\xi_j$.

(ii) All bumps of type $B$ or $S$ have the same sign. Otherwise, it follows from (i) that there is no loss of generality if we assume there are bumps $\xi_i$ and $\xi_j$ of opposite signs and equal absolute altitudes, both of which
contain \( s \). If \( \xi_1 \) is the shorter bump, then \( \xi_1 \) can be discarded and \( \xi_2 \) replaced by \( \xi_1 + \xi_1 \). This is impossible, since it would reduce the sum in (5).

(iii) There is no loss of generality if we assume all bumps of type \( B \) or \( S \) have the same signs. Suppose \( \xi_1 \) and \( \xi_2 \) are bumps of opposite signs, \( \xi_1 \) is a \( B \)-bump belonging to \( a' \), and \( \xi_2 \) is an \( S \)-bump belonging to \( a'' \). Again, we invoke (i) and assume \( \xi_1 \) and \( \xi_2 \) have the same absolute altitudes. Then \( \xi_1 \) is the first bump in \( a' \) and \( \xi_2 \) and \( \xi_1 \) do not have the same right ends, since if they did we could discard \( \xi_2 \) in \( a'' \) and replace \( \xi_1 \) in \( a' \) by \( \xi_1 + \xi_1 \), which would reduce \( [\alpha'] \) and not change \( [\alpha'] \). Therefore \( \xi_2 + \xi_2 \) consists of two bumps, neither of which is of type \( B \) or \( S \). We can replace \( \xi_2 \) in \( a'' \) by the first or the second of these bumps, according as \( \xi_2 \) extends farther to the right than \( \xi_1 \) or \( \xi_1 \) extends farther to the right than \( \xi_2 \), and then replace \( \xi_2 \) in \( a'' \) by the other bump. This changes neither \( [\alpha'] \) nor \( [\alpha''] \), but together with the possible use of (i) it results in a net decrease of at least one in the sum of the number of \( B \)-bumps and the number of \( S \)-bumps. Successive application of this process leads to a representation of \( s \) in (5) for which all remaining \( B \)-bumps and \( S \)-bumps have the same signs.

Since \( s(s+1) = 0 \) and all bumps that are nonzero at \( s+1 \) are of type \( B \), \( B \) or \( S \), it follows from (ii) and (iii) that there are no bumps of type \( B \). Since no bumps have both \( s \) and \( t \) in their support, we shorten all bumps with \( s \) in their support to end at \( s \), shorten all bumps with \( t \) to begin at \( t \), and discard all bumps with support in \( (s, t) \). This does not change \( \sum s^t \) and cannot decrease the sum in (5), so \( s \) is not supported in \( (s, t) \).

The preceding process can be continued inductively to replace \( \sum s^t \) by \( \sum ||s|| \), which satisfies (5) and for which the supports of all bumps are supports of bumps in \( \alpha' \). It follows from (4) that \( ||s|| = ||\alpha'|| \).

**Theorem 1.** The dual of \( I \) is \( J \) and the natural basis of \( I \) is shrinking.

**Proof.** Let \( \langle \alpha \rangle \) be the natural basis for \( I \) and \( \{v_n\} \) be the corresponding coefficient functionals. For a continuous linear functional \( f \) on \( I \), let \( \langle \alpha, f \rangle = f(\alpha) \) for each \( f \) and \( ||f||_J \) be the norm of \( \sum f(\alpha) u_\alpha \) as given by (2). Suppose \( s = \{v(i)\} \) is the sum of \( n \) strongly disjoint bumps and let \( \alpha_n \) and \( \{p_{n-1}, p_n - 1\} \) be the altitude and the support of the \( n \)-th bump. Then Lemma 1 gives \( ||s|| = ||\alpha|| ||s|| \) and we have

\[
||s, f|| = \left| \sum f(\alpha) u(\alpha) \right| \leq \left( \sum \left| f(\alpha) \right| \right) \left( \sum \left| u(\alpha) \right| \right) = \left( \sum \left| f(\alpha) \right| \right)^{\frac{1}{2}} \left( \sum \left| u(\alpha) \right| \right)^{\frac{1}{2}} = \left( ||f||_J ||s|| \right).
\]

Since the unit ball is the closed convex span of such \( s's, \) we have \( ||f|| \leq ||f||_J \). For any \( \epsilon > 0 \), there is a strictly increasing sequence \( \{p_k\} \) such that

\[
||f||_J \leq \sum_{k=1}^{n} \left( \sum_{p_{k-1}}^{p_k} f(\alpha) \right)^{\frac{1}{2}} + \epsilon.
\]

Let \( s \) have the value \( \sum f(\alpha) \) on the interval \( [p_{k-1}, p_k - 1] \). Then

\[
||s, f|| = \sum_{k=1}^{n} \left( \sum_{p_{k-1}}^{p_k} f(\alpha) \right)^{\frac{1}{2}} \geq ||f||_J ||s|| - \epsilon.
\]

Thus \( ||f|| \geq ||f||_J - \epsilon \) for all \( \epsilon > 0 \) and \( ||f|| = ||f||_J \) if \( J^* \) is isometric to \( J \). Also, the coefficient functionals of the natural basis for \( I \) can be identified with the basis \( \{u_n\} \) used in (2) and therefore the natural basis for \( I \) is shrinking ([3], Lemma 1, p. 90).

It is known that a basis \( \{u_n\} \) for a Banach space \( B \) is shrinking if and only if the sequence of coefficient functionals \( \{u_n\} \) is a bounded complete basis for \( B^* \) ([10], Corollary 6.1, p. 286). The space \( B \) has a shrinking basis whose sequence of coefficient functionals \( \{u_n\} \) is a bounded complete basis for \( B^* \) used in (2). Also, \( \{u_1, u_2, u_3, u_4, \ldots\} \) is the natural shrinking basis for \( B^* \) with the norm (1).

**Theorem 2.** Suppose a Banach space \( B \) has a basis \( \{n\} \) with coefficient functionals \( \{u_n\} \). If \( \sum a_n \alpha_n \) is norm-convergent and \( \{a_n, u_1 - u_1, u_2 - u_1, \ldots\} \) is a shrinking basis for \( B^* \) with coefficient functionals \( \{F_n\} \), then

(a) \( \langle \alpha, F_n - F_{n-1} \rangle \) defines an isometry of \( B \) onto the closure of \( \{\sum a_n F_n\} \).

(b) \( B \) is quasi-reflexive of order one.

**Proof.** To show \( \langle \alpha, F_n - F_{n-1} \rangle \) is the natural embedding of \( B \) into \( B^* \), it is sufficient to show that \( \langle F_n - F_{n-1}, u_n \rangle \) is identical to \( \{u_n, e_n\} \), where \( \{u_n, e_n\} \) is equal to \( \langle F_n, u_n - u_{n-1} \rangle \). We have \( \langle F_n - F_{n-1}, u_n \rangle = \langle F_{n+1} - F_n, u_{n+1} \rangle \), which is equal to \( \langle F_n + u_n - u_{n-1}, u_{n-1} \rangle \), so \( \langle F_n - F_{n+1}, u_n \rangle \) equals 1 if \( n = n \), equals 1 if \( n < k \), and equals 0 if \( n > k \). This completes the proof of (a). The isometry in (a) is the natural embedding of \( B \) onto \( B^* \). Since \( \{u_1, u_2 - u_1, \ldots\} \) is a...
\(-u_1, u_2 - u_3, \ldots\) is shrinking, the span of \(F_1\) and \(\mathcal{B}\) is \(B^{**}\), and \(\mathcal{B}\) has the basis \((F_1 - F_{n+1})\). If \(F_1, \mathcal{B}\), then \(F_1 = \sum_i a_i (F_1 - F_{i+1})\) and
\[
(a_1 - 1) F_1 + \sum_i (a_i - a_{i+1}) F_i = a_1 F_{n+1} - \sum_i a_i (F_1 - F_{i+1})
\]
for all \(n\). This implies each \(a_i\) is 1, \(F_1 = \sum_i (F_1 - F_{i+1})\), and the contradiction that \(\sum a_i\) is convergent. Thus \(F_1, \mathcal{B}\) and \(\mathcal{B}\) is quasi-reflexive of order one.

**Lemma 2.** For the space \(I\), any \(0 < 1\), and any positive integer \(n\), there are integers \((r_k; 1 \leq k \leq n)\) with \(r_{k+1} > r_k\) even and the property that
\[
\sum_1^n |a_k| > \sum_1^n |a_i a_k|^r \quad \forall \{a_k\},
\]
if \(|a_k^n| = 1\) for each \(k\), each \(a_k^n\) is the sum of \(s_1, s_2, \ldots, s_n\) strongly disjoint bumps of alternating signs and equal absolute values, each \(s_k\) even and \(s_k > r_k^2 r_{k+1}\), and each bump of \(a_k^n\) contains \(s_k^{\frac{1}{2}}\) of the bumps of \(a_k^n\).

The first step in the proof of this lemma is to show that, for the isometry of \(I\) into \(J^*\) defined in Theorem 2, a norm-one sum \(\sum x_i e_i\) strongly disjoint bumps of alternating signs corresponds to a member of \(J^*\) for which \((y, F) = 1\), where \(y\) is a sum of \(y\) disjoint “humps”. Then it is noted that if each \(y_i\) \(1 \leq k \leq n\) is of the same type as \(y\), with the \(\lambda_{i-1}\) “humps” in \(y_i^{\frac{1}{2}}\) spaced uniformly throughout the intervals on which the slope of \(y_i^{\frac{1}{2}}\) is nonzero, that each \(\lambda_{i-1}\) is sufficiently large, then the linear span of \(y_i^{\frac{1}{2}}\) is nearly isometric to \(\ell_1^2\). The last step is to show that if each \(y_i\) is related to \(x_i^{\frac{1}{2}}\) as \(y\) was related to \(F\), then \(y_i\) is nearly isometric to \(\ell_1^2\), which implies that the linear span of \(y_i^{\frac{1}{2}}\) is nearly isometric to \(\ell_1^2\).

**Proof.** Let \(r\) be a positive integer and \(\mu\) a positive number for which \(\mu^{-2} (2r)^{-1/2}\) is a positive integer. Then let \(p_i; 1 \leq i \leq 4r\) be a strictly increasing sequence of positive integers such that, if \(1 \leq k \leq r,
\]
\[
P_{4k-3} - P_{4k-1} = P_{4k} - P_{4k+1} = \mu^{-2} (2r)^{-1/2}.
\]
Define the function \(y = \langle y(i) \rangle\) on the positive integers by letting \(y\) be linear on all intervals \([P_i, P_{i+1}]\) with \(1 \leq i < 4r\); \(y(i) = 0\) if \(i \leq P_1\), if \(i > P_{4r}\), or if \(i = 4k - 3\) or \(4k + 1\) with \(1 \leq k \leq r\); and \(y(4k - 2) = y(4k - 1) = (2r)^{-1/2}\) if \(1 \leq k \leq r\).

Then with the norm given by (1), we have \(|y| = 1\). Also, \(y\) is the sum of \(r\) disjoint “humps” in the sense that, on the interval \([P_1, P_3]\), \(y\) increases linearly with slope \(\mu\) from 0 to \((2r)^{-1/2}\), \(y\) is constant on the interval \([P_1, P_2]\) on the interval \([P_4, P_5]\), \(y\) decreases linearly with slope \(-\mu\) to 0; \(y\) is zero on the interval \([P_5, P_4]\); etc.

Let \(F(i)\) be the coefficient functionals for the natural shrinking basis for \(J\). As described in the preceding paragraph, let a corresponding linear functional \(F\) be defined by
\[
F = (2r)^{-1/2} \sum_{i=1}^r \left( [-F(P_{4k-1}) + F(P_{4k-2})] + [F(P_{4k-2}) - F(P_{4k-3})] \right).
\]
Then
\[
(y, F) = (2r)^{-1/2} \sum_{i=1}^r \left( [-\ell_1^2(P_{4k-1}) + \ell_1^2(P_{4k-2})] + [\ell_1^2(P_{4k-2}) - \ell_1^2(P_{4k-3})] \right).
\]
Also,
\[
F = (2r)^{-1/2} \sum_{i=1}^r \left( [-F(P_{4k-1}) + F(P_{4k-2})] + [F(P_{4k-2}) - F(P_{4k-3})] \right).
\]
Thus it follows from Theorem 2 that the image \(x\) of \(F\) in the predual \(J\) of \(J\) consists of \(r\) pairs of strongly disjoint bumps, the \(k\)th pair being a bump of altitude \((2r)^{-1/2}\) on the interval \([P_{4k-3}, P_{4k-1}]\) and a bump of altitude \((2r)^{-1/2}\) on the interval \([P_{4k-1}, P_{4k-3}]\). From Lemma 1, we have \(|x| = 1\). Therefore \(|F| = 1\).

It follows by the grueneisen arguments in [4] that if each \(y_i\) \(1 \leq k \leq n\) is of the same type as \(y\) in the preceding paragraphs, but with \(r\) as \(r\) for each \(r_{4k+1}/r_k\) is sufficiently large even integer; and if the “humps” in \(y_i^{\frac{1}{2}}\) are spaced uniformly throughout the intervals on which the slope of \(y_i^{\frac{1}{2}}\) is nonzero, then \(r_{4k+1}/r_k\) is nearly isometric to \(\ell_1^2\). Thus if a set of linear functionals \(F^{(x)}\) is such that \(F^{(x)} \) is nearly isometric to \(\ell_1^2\). If \(F^{(x)}\) corresponds to \(y_{i}^{\frac{1}{2}}\) as \(x\) corresponded to \(y\) in the preceding paragraphs, then \((y', F) = 0\) if \(i > j\). Then \(x\) and \(x'\) is sufficiently large, then \((y', F) = 0\) nearly zero for all \(i < j\). Thus for any \(\theta < 1\) we could have chosen \(y_i^{\frac{1}{2}}\) and \(F^{(x)}\) so that
\[
\sum_{i=1}^r \left( [-\ell_1^2(P_{4k-1}) + \ell_1^2(P_{4k-2})] + [\ell_1^2(P_{4k-2}) - \ell_1^2(P_{4k-3})] \right).
\]
Each \(F^{(x)}\) is the image of an element \(x^{\frac{1}{2}}\) in \(J^{*}\) that has \(r_{4k+1}/r_k\) bumps of alternating signs. Moreover, for each bump \(x^{(x)}\), \(r_{4k+1}/r_k\) bumps of the bumps in \(x^{(x)}\) have support in the support of \(x\). Since the norm of \(I\) is repetition-invariant in the sense that
\[
\|\langle a(\ell) \rangle \| = \langle \|a(\ell)\|, \|a(k - 1)\|, \|a(k)\|, \|a(k + 1)\|, \|a(k + 2)\|, \ldots \rangle \|
\]
for all $x$ and all $k$, Lemma 2 needs no restriction about uniformity of lengths of bumps or uniformity of distances between bumps—only the restriction that, for each bump $b$ in $x$, $\beta_{i+1}/\beta_i$ of the bumps in $\beta_{i+1}$ have support in the support of $b$.

**Theorem 3.** The space $I$ is not isomorphic to any subspace of $J$.

**Proof.** Let $T$ be an isomorphism of $I$ into $J$. We first establish facts (A) and (B):

(A). For any $\epsilon > 0$ and $k > 0$, there exists an integer $n$ such that, for each integer $\lambda > n$, there is an arbitrarily large $k(\lambda)$ such that the bump $\beta_{k(\lambda)}$ with altitude $1$ on the interval $[k, k(\lambda)]$ has the property that $T_b^{\beta_{k(\lambda)}}$ can be made constant on $[x, \lambda]$ without changing its $J$-norm by as much as $\epsilon$.

To see that (A) is true, first choose an increasing sequence of integers $(p_i)$ so that, if $l^{p_i}$ is the bump with altitude $1$ on the interval $[k, p_i]$, then

$$\lim_{i \to \infty} \|T^{l^{p_i}}(i)\alpha(i)\|_{J} = \alpha(i)$$

exists for each $i$. Then there is a $n$ such that, for any $x > n$, the $J$-norm of the sequence $(\alpha(i) - \alpha(i))$ is less than $\frac{\epsilon}{4}$ if $\alpha(i)$ is obtained from $\alpha(i)$ by replacing $\alpha(i)$ by $\alpha(n)$ for all $i$ in the interval $[x, \lambda]$, since otherwise there would be an infinite sequence of disjoint intervals $[x_i, \lambda_i]$ and for each interval a sum of type

$$\sum_{i} |\alpha(p_i) - \alpha(p_i)|^{1/2}$$

with these sums bounded away from zero. This contradicts boundedness of the $J$-norms of $T^{l^{p_i}}$. It follows now that for this $x$ and any $x > n$, there is a $k(\lambda)$ such that $T^{l^{p_i}}(i)\alpha(i)$ approximates $\alpha(i)$ for $i \in [x, \lambda]$, well enough that $T^{l^{p_i}}$ can be made constant on $[x, \lambda]$ without changing its norm by as much as $\epsilon$. Let $k(\lambda) = p_{k(\lambda)}$.

(B). For any positive $a$ and $n$, there exists $k(a, n)$ such that, if $x$ is any bump of altitude $1$ with $\xi(i) = 0$ for $i < k(a, n)$, then $\|T^{l^{p_i}}(i)\alpha(i)|$ is small enough for $i < n$ that deleting the initial segment of $T^{l^{p_i}}(i)$ on $[1, n]$ does not change $\|T^{l^{p_i}}(i)\alpha(i)|$ by more than $\epsilon$.

To establish (B), it is sufficient to show that

$$\lim_{i \to \infty} \sup \{\|T^{l^{p_i}}(i)|: h < s < \xi(i)\} = 0$$

for each $i$.

where $\beta_{i+1}$ is a bump of altitude $1$ on the interval $[s, i]$. If there were an $i$ for which this limit is not zero, there would exist a positive $s$ such that, for any $n$, there is a sum $\alpha$ of $n$ strongly disjoint bumps of absolute altitudes $1$ for which $(T\alpha)(i) > n\delta$, so that $\|T\alpha\|_J > n\delta$. This contradicts $\|x\|_J = \sqrt{n}$, which follows from Lemma 1.

The proof of Theorem 3 will be completed by showing that if there is an isomorphism $T$ of $I$ into $J$, then for any positive integer $n$ and any $\theta < 1$, there is a set $(x_i^j: 1 \leq j \leq n)$ of members of $I$ such that

$$\sum_{i} \|x_i^j\|_J > \theta n$$

with

$$\sum_{i} \|T^{l^{p_i}}(i)\alpha(i)| < \frac{1}{10} \|T\|n^{1/2}$$

which implies $\|T\|^{\beta} > 2\sqrt{n}$ for all $n$. This will be done explicitly only for $n = 3$, since this case clearly is typical.

Choose $r_1, r_2, r_3$ as described in Lemma 2. For an arbitrary positive $n$, let $s_1 = r_1$ and choose $s_2 > r_1/r_2$ and $s_3 > r_2/r_3$ to be even integers such that

$$\sup \{g(i)|: s_1| < i < r_1 s_2$$

if $y$ is the image of an element in $I$ with norm not greater than $(s_1 r_2)^{-1/2} + (s_2 r_3)^{-1/2} + (s_3)^{-1/2}$ and $x$ is the image of an element in $I$ with norm not greater than $(s_1 s_2 s_3)^{-1/2}$. Now $x_1^2$, $x_2^2$, and $x_3^2$ will be constructed to satisfy Lemma 2 for $n = 3$. Choose $\Delta > 0$ so that

$$(x_1^2 + s_1 x_2^2 + s_1 s_2 x_3^2) \Delta < \epsilon$$

With repeated applications of (A) and (B), choose the left end of the first bump in $x_1^2$, then the left end of the first bump in $x_2^2$, then the first $s_1$ bumps in $x_2^2$, then the right end of the first bump in $x_2^2$, then the left end of the second bump in $x_3^2$, then the second set of $s_2$ bumps in $x_3^2$, etc. The right end of the first bump in $x_3^2$ is chosen after the first $s_1$ bumps in $x_3^2$ and the first $s_2$ bumps in $x_3^2$ have been completed. This is continued until all the $s_1$ bumps in $x_1^2, s_1 s_2$ bumps in $x_3^2$, and $s_1 s_2 s_3$ bumps in $x_3^2$ have been completed.

This construction of $x_1^2, x_2^2$ and $x_3^2$ can be done so that:

- For $j = 2$ or $j = 3$, there are $s_{j-1}$ intervals on which each of $x_1^{2-j}$ can be made constant without changing $\|T^{x_1^{2-j}}\|_J$ by more than $\Delta$. Moreover, the image of each bump in $x_3^2$ can be truncated on the left and right without changing the $J$-norm of this image by more than $2\Delta$. This can all be done so the supports of the modified images of bumps from $x_3^2$ lie in disjoint intervals and the set of these intervals can be partitioned into $s_1$ disjoint sets of $\xi$ intervals each, all the intervals in the same set lying in an interval on which $T^{x_1^{2-j}}$ has been made constant. Then

$$\|T^{x_1^{2-j}} + x_2^2 + x_3^2\|_J$$

is not changed by more than $3(s_1 + s_1 s_3 + s_1 s_2 s_3) \Delta < \epsilon$ if all these truncations are made and appropriate sections of images of bumps are replaced by constant functions.
Now consider a sum \( \sum (7) \) of the type used in (1) for estimating the square of the \( J \)-norm for \( w \), where \( w = \sum_{k=1}^{n} T \alpha' \) modified as described above:

\[
(7) \quad \sum_{k=1}^{n} [w(p_{2k-1}) - w(p_{2k})]^2.
\]

There are at most \( 2n \) terms in (7) with the property that one of \( p_{2k-1} \) or \( p_{2k} \) is in one of the \( s \) intervals on which the modified images of bumps in \( \beta' \) are constant, but the other is not in the same such interval. If \( p_i \) is not in an interval on which the modified image of a bump in \( \beta' \) is constant, then \( w(p_i) = T \beta'(p_i) \). Otherwise, \( w(p_i) \) differs from the value at \( p_i \) of the modified \( T \beta' \) by not more than the sum of the sup norm of the image of a bump in \( \beta' \) and the sup norm of the corresponding value of the modified \( T \beta' \) makes a change \( O(\varepsilon) \). Thus the sum of the terms in (7) with at most one of \( p_{2k-1} \) or \( p_{2k} \) in the same “constant section” of the image of a bump in \( \beta' \) differs by \( O(\varepsilon) \) from a sum of type (7) for estimating the squared \( J \)-norm of the modified \( T \beta' \). After deleting these terms, the remaining terms in (7) have the property that both \( p_{2k-1} \) and \( p_{2k} \) are in the same “constant section” of the image of a bump in \( \beta' \), so their sum is a sum of type (1) for estimating the squared \( J \)-norm of the modification of \( T \beta' + T \beta' \).

This and another application of the preceding procedure using the “constant sections” of \( T \beta' \) and the second inequality in (6) enable us to see that the squared \( J \)-norm of the modification of \( T \beta' \) is not larger than \( \sum_{k=1}^{n} [T \beta']^2 \) modified \( \varepsilon \). Since \( \varepsilon \) was arbitrary, it could have been chosen small enough that

\[
\left| \sum_{k=1}^{n} [T \beta']^2 \right| \lesssim \varepsilon / \varepsilon.
\]

2. A Banach space that is quasi-reflexive of order one and isomorphic to its dual. For computational reasons, it will be useful to introduce the concept of double basis and several related concepts, some of which are extensions of familiar properties of bases. It should be noted that repetition-invariance is somewhat dual to the equal-signs-additive property studied by Brunel and Sucheston [1].

A double basis for a Banach space \( X \) is a subset \( \{ a_n : -\infty < n < \infty \} \) such that each \( x \) in \( X \) has a unique representation as \( x = \sum_{n=-\infty}^{\infty} a_n \epsilon_n \), in the sense that

\[
\lim_{n \to \infty, m \to -\infty} \| x - \sum_{k=1}^{n} a(i) \epsilon_i \| = 0.
\]

A bimonotone (double) basis is a (double) basis \( \{ a_n \} \) such that

\[
\left| \sum_{q} a(i) \epsilon_i \right| \geq \left| \sum_{q} a(i) \epsilon_i \right| \text{ if } p < q \leq r < \infty.
\]

A shrinking double basis is a double basis \( \{ a_n \} \) such that each of the basic sequences \( \{ a_n : n < 0 \} \) and \( \{ a_n : n > 0 \} \) is shrinking.

A neighborly (double) basis is a (double) basis \( \{ a_n \} \) such that \( \left| \sum_{n=0}^{\infty} a(i) \epsilon_i \right| \) is not increased if some \( a(k) \) is replaced by either \( a(k-1) \) or \( a(k+1) \). A neighborly basis is of type \( P \) ([9], p. 354), but not conversely. Clearly, neighborliness implies monotonicity. Also, neighborliness implies what we shall call repetition-invariance, namely, for all \( r \),

\[
\left| \sum_{n=-\infty}^{\infty} a(i) \epsilon_i \right| = \left| \sum_{n=-\infty}^{\infty} a(i) \epsilon_i + a(r) \epsilon_{r+1} + \sum_{n=r+1}^{\infty} a(i) \epsilon_i \right|,
\]

and also implies translation-invariance for double bases (but not for ordinary bases), namely,

\[
\left| \sum_{n=-\infty}^{n} a(i) \epsilon_i \right| = \left| \sum_{n=-\infty}^{n} a(i-1) \epsilon_i \right|,
\]

since it is possible to transform \( \sum_{n=-\infty}^{\infty} a(i) \epsilon_i \) into the vector of the second member of (8) or (9) and back again by successive replacements of components, each being replaced by one of the two neighboring components—except that this is not possible for (9) with a basis \( \{ a_n : n > 1 \} \) unless \( a(1) = 0 \).

If \( \{ a_n \} \) is a double basis or a basis, inversion-invariance means that, whenever \( x \) has finite support, there is an \( s \) such that \( \| \sum a(i) \epsilon_i \| = \| \sum a(n - i) \epsilon_i \| \) and, when \( \{ a_n \} \) is a basis (and not a double basis), \( a(0) = 0 \) if \( i > 0 \). With respect to a (double) basis \( \{ a_n \} \), a finite set \( \{ b_k : 1 \leq k \leq s \} \) being disjoint means each \( b_k \) is a linear combination of a finite set of consecutive members of \( \{ a_n \} \) and these sets are pairwise disjoint; if each pair of sets is separated by at least one member of \( \{ a_n \} \), then \( \{ b_k \} \) is strongly disjoint.

After proving a sequence of lemmas, we will obtain a norm for the space of functions with finite support on the set of all integers. This norm is constructed as the limit of a sequence of norms, the first norm being similar to the norm of \( I \) and the second norm similar to the norm of \( J \). Inequality (10) will be preserved in the limit, so the natural basis will be shrinking. Finally, the desired space will be the subspace consisting of functions with support on the positive integers.
LEMMA 3. Let \( \{e_n\} \) be a bimonotone double basis for a Banach space \( X \). Let \( M \) be a number such that,

(i) if \( \{e_n\} \) is strongly disjoint with respect to \( \{e_n\} \), then

\[
\left| \sum_j a^k_j \right| \leq M \left( \sum_j \|a^j_j\|^2 \right)^{1/2}.
\]

(ii) if \( a = \sum_n x(i) e_i \), \( \{p_k\} \) is a strictly increasing sequence of integers, and \( \{p_i\} \) is a sequence such that \( \chi^k(i) = 0 \) if \( i < p_{k-1} \) or \( i > p_k \), and \( \chi^k(i) = \varepsilon(i) \) if \( p_{k-1} \leq i \leq p_k \), then

\[
\frac{1}{M} \left( \sum_i \|\chi^k_i\|^2 \right)^{1/2} \leq \|a\|.
\]

If \( \{u_n\} \) are the coefficient functionals for \( \{e_n\} \) and \( \{u_n - u_{n-1}\} \) is repetition-invariant, then \( \{u_n - u_{n-1}\} \) also satisfies (10) and (11).

Proof. Suppose that \( f = \sum a_n e_n \) with only finitely many nonzero terms, and if \( a = \sum_n \varepsilon(i) e_i \), then

\[
( \varepsilon, f) = \sum_n \varepsilon(i) f(i) = \sum_n \varepsilon(i) [\varepsilon(i) - \varepsilon(i-1)].
\]

Suppose that \( \{f^k: 1 \leq k \leq s\} \) is strongly disjoint with respect to \( \{u_n - u_{n-1}\} \) and let \( f = \sum f^k \). Then there is a strictly increasing sequence \( \{p_k\} \) of integers such that

\[
f^k = \sum_{i=p_{k-1}}^{p_k} f^k(i) \langle u_i - u_{i-1} \rangle \quad \text{if} \quad 1 \leq k \leq s,
\]

and \( f(i) = 0 \) if \( i < p_1 \) or \( i > p_s \). Since \( \{e_n\} \) is bimonotone, it follows from (12) that \( \{a\} \) is such that \( (\varepsilon, f) = \|a\| \|\varepsilon\| \) and \( \varepsilon(i) = 0 \) if \( i < p_1 \) or \( i > p_s \). Now we use the last member of (12) to see that, if \( \chi^k(i) = 0 \) when \( i < p_{k-1} \) or \( i > p_k \), and \( \chi^k(i) = \varepsilon(i) - \varepsilon(i-1) \) when \( p_{k-1} \leq i \leq p_k \), then

\[
(\varepsilon, f) = \sum_{i=p_{k-1}}^{p_k} \chi^k(i) \|a^j_j\|^2 = \sum_{i=p_{k-1}}^{p_k} E^k f^k(i).
\]

Therefore,

\[
\|\sum f^k\| \leq \sum \|f^k\|^2 \leq \sum \|E^k\|^2 \leq \sum \|f^k\|^2 \leq M \left( \sum \|f^k\|^2 \right)^{1/2}.
\]
of all integers and having finite support. Let \( M \) be a positive number and \( \| \cdot \|_b \) be a norm for \( X \) such that, if \( \| \cdot \|_b \) is defined by

\[
\| x \|_b = \left\| \sum_{n=-\infty}^{\infty} \alpha(n) (u_n - u_{n-1}) \right\|_{\ell^2} \quad \text{if} \quad (x(i)) \in X,
\]

where \( \{u_n\} \) is the set of coefficients functions for the double basis \( \{\alpha_n\} \) and \( || \cdot || \) is the dual norm of \( || \cdot ||_b \), then

(a) for \( || \cdot ||_b \), the natural double basis \( \{\alpha_n\} \) for \( X \) is inversion-invariant, neighborly, and satisfies (10) and (11) if \( M = \infty \);

(b) for \( || \cdot ||_b \), \( \{u_n - u_{n-1}\} \) is an inversion-invariant neighborly double basis and satisfies (10) and (11) if \( M = \infty \);

(c) \( ||x||_b \leq ||x||_{\| \cdot \|} \) if \( x \in X \).

If \( ||x||_{\| \cdot \|} = \sqrt{\sum (||x||_{\| \cdot \|})^2 \|} \) for all \( x \in X \), and \( || \cdot ||_b \) is defined relative to \( || \cdot ||_b \), as \( || \cdot ||_b \) was defined relative to \( \| \cdot \| \), then \( || \cdot ||_b \) and \( || \cdot ||_b \) satisfy (a), (b), (c), satisfy (10) and (11) if \( M = \infty \), and also satisfy (d)

\[ ||x||_b \leq ||x||_{\| \cdot \|} \leq ||x||_b \leq ||x|| \text{ if } x \in X. \]

Proof. With \( || \cdot ||_b \) defined as stated, it follows that the natural basis for \( X \) is inversion-invariant and neighborly, since \( || \cdot ||_b \) and \( || \cdot ||_b \) have these properties and neighborliness implies translation-invariance. If the finite set \( \{x^b\} \) is strongly disjoint, then

\[
\| \sum x^b \|_b = \left\| \left( \sum x^b \|_b + \sum \|x^b\| \right) \right\|_{\ell^2} = \| \sum \|x^b\| \|_{\ell^2},
\]

so (10) is satisfied; (11) follows similarly. We have established (a).

Now define \( || \cdot ||_b \) by letting

\[
\|f\|_b = \left\| \sum_{n=-\infty}^{\infty} f(i) (u_n - u_{n-1}) \right\|_{\ell^2} \quad \text{if} \quad f = (f_n) \in X,
\]

where \( || \cdot ||_b \) is the dual norm of \( || \cdot ||_b \). It follows from inversion-invariance and translation-invariance of \( \{\alpha_n\} \) with respect to \( || \cdot ||_b \), that \( \{u_n - u_{n-1}\} \) is inversion-invariant with respect to \( || \cdot ||_b \); it follows from Lemma 4 that \( \{u_n - u_{n-1}\} \) is neighborly with respect to \( || \cdot ||_b \) and it follows from Lemma 3 that \( \{u_n - u_{n-1}\} \) satisfies (10) and (11) with respect to \( || \cdot ||_b \), \( || \cdot ||_b \) and \( \{\alpha_n\} \) is inversion-invariant, neighborly, and satisfies (10) and (11) if \( M = \infty \).

To prove (c), we note that for positive \( a \) and \( b \),

\[
\|a+b\|_{\| \cdot \|}^2 \geq \|a\|_{\| \cdot \|} + \|b\|_{\| \cdot \|},
\]

which is the sum of disjoint (not necessarily strongly disjoint) bumps whose altitudes are \( \{u_n\} \), and

\[
\|x\|_{\| \cdot \|} \leq \|x\|_{\| \cdot \|} \quad \text{if} \quad x \in X,
\]

where the sum is over all positive integers \( a \) and \( b \) with \( a \geq b \) and \( (1/1+a+b)^{-1} \)

With \( f = \sum_{i=0}^{\infty} f(i) e_i \) in \( X \) and \( \varphi = \sum_{i=0}^{\infty} f(i) (u_i - u_{i-1}) \), it follows from the definition of \( || \cdot ||_b \) and (15) that

\[
||\varphi||_{\| \cdot \|} = \sup \{ \| (x, \varphi) \|_{\| \cdot \|} \} = \sup \{ \| (x, \varphi) \|_{\| \cdot \|} \\} \leq ||\varphi||_{\| \cdot \|} \quad \text{if} \quad M = \infty.
\]

It follows from (14) that \( (x, \varphi) \leq \|x\|_b \|\varphi\|_b \) for all \( x \). Since

\[
(x, \varphi) = \sum_{i=0}^{\infty} [f(i) - f(i+1)] e_i \varphi(i) = \sum_{i=0}^{\infty} (x(i) - x(i+1)) f(i),
\]

it follows from translation-invariance of \( || \cdot ||_b \) and replacing \( x \) by \( -x \) that \( (x, \varphi) \leq \|x\|_b \|\varphi\|_b \) for all \( x \).

Thus

\[
||\varphi||_{\| \cdot \|} \leq \|f\|_b + ||f||_b \leq \|f\|_b + ||f||_b \|f\|_b = \|f\|_b,
\]

the second inequality following from (15).

To prove (d), note first that \( || \cdot ||_b \leq || \cdot ||_b \leq || \cdot ||_b \)

Finally, note that \( || \cdot ||_b \leq || \cdot ||_b \) follows from \( || \cdot ||_b \leq || \cdot ||_b \) and the validity of (14) for \( || \cdot ||_b \) and (15) for \( || \cdot ||_b \).

LEMMA 6. Let \( X \) be the linear space of functions defined on the set of all integers and having finite support. Let \( || \cdot ||_b \) and \( || \cdot ||_b \) be defined analogously to (5) and (1), respectively, with

\[
||f||_b = \sum_{n=-\infty}^{\infty} \|f(i) (u_n - u_{n-1}) \|_{\| \cdot \|} \quad \text{if} \quad f = (f_n) \in X,
\]

where \( \|x\|_{\| \cdot \|} \leq || \cdot ||_{\| \cdot \|} \|f \|_{\| \cdot \|} \|f\|_{\| \cdot \|} \)

where the sup is over all positive integers \( a \) and \( b \) with \( a \geq b \) and \( (1/1+a+b)^{-1} \)

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required to be strictly increasing. Just as for the proof of Theorem 1, it can be shown that the dual of $X$ given the norm $\| \cdot \|_b$ is isometric to $X$ with the norm

$$\| \varphi(\cdot, i) \| = 2^{-1/2} \sup \left\{ \sum_{k=1}^{n} \left| \sum_{j \in [k, n]} \varphi(j, i) \right|^{1/2} \right\},$$

where the sup is over all positive integers $n$ and all strictly increasing sequences $(\varphi_n)$ of positive integers. Then the norm of the linear functional

$$\sum_{n=0}^{\infty} \varphi(i)(u_n - u_{i-1}) = \sum_{n=0}^{\infty} \varphi(i) - \varphi(i+1),$$

with $X$ given the norm $\| \cdot \|_b$, is equal to

$$2^{-1/2} \sup \left\{ \sum_{n=1}^{N} [\varphi(p_{k-1}) - \varphi(p_k + 1)]^{1/2} \right\} = \| \varphi(\cdot, i) \|_b.$$

Thus (14) is satisfied. Since $(a - b)^2 \leq 2(a^2 + b^2)$ for numbers $a$ and $b$, it follows from (17) that if $\varphi$ is the sum of disjoint bumps of altitudes $(a_n)$, then

$$\| \varphi \|_b \leq 2^{1/2} \left( \sum_{n=1}^{N} |a_n|^{1/2} \right)^{1/2}.$$

Since the unit ball for $\| \cdot \|_b$ is the closed convex span of such $\varphi$s which also satisfy $2^{1/2} \left( \sum_{n=1}^{N} |a_n|^{1/2} \right)^{1/2} = 1$, we have $\| \varphi \|_b \leq \| \varphi \|_b$ for all $\varphi$ in $X$.

Now we need an $M$ for which (10) and (11) are satisfied for both $\| \cdot \|_b$ and $\| \cdot \|_b$. First, we shall consider $\| \cdot \|_b$. Suppose $(a^2, 1 \leq k < s)$ is strongly disjoint. For each interval $[p_{k-1}, p_k)$ used in (17) that contains a $q$ at which $\varphi = \frac{q}{s} a^2$ is 0, we replace $[\varphi(p_{k-1}) - \varphi(p_k)]^2$ by $2[\varphi(p_{k-1}) - \varphi(q)]^2 + 2[\varphi(q) - \varphi(p_k)]^2$ and obtain

$$\| \varphi \|_b \leq 2^{1/2} \left( \sum_{i=1}^{N} |a_i|^{1/2} \right)^{1/2}.$$

Now suppose $\varphi = \varphi(i)$, $(\varphi_n)$ is a strictly increasing sequence of integers, and for $1 \leq k \leq s$ each $\varphi_k$ is defined by

$$\varphi_k(i) = \begin{cases} 0, & \text{if } i \geq \varphi_{k-1}, \quad i < \varphi_k, \\ \varphi(i) - \varphi(\varphi_{k-1}) & \text{if } \varphi_{k-1} \leq i < \varphi_k. \end{cases}$$

Since $\varphi_k(i) = 0$ if $i \geq \varphi_{k-1}$ or $i \geq \varphi_k$, there are integers $(r^2)$ such that $\varphi_{k-1} \leq r_{k-1} \leq r_k \leq \cdots \leq r_{k+n-1} \leq \varphi_k < r_{k+n}$ and

$$\| \varphi \|_b = \frac{1}{s} \sum_{i=1}^{N} \left[ \varphi_i(r_{j-1}) - \varphi_i(r_j) \right]^2.$$
while

\[ (21) \quad \|x(i)\|_1 = \sup \left\{ \frac{\sum_{n=0}^{\infty} x(i) e_i}{\|g\|} : g \in X^* \right\}. \]

Therefore

\[ \left\| \sum_{i=0}^{\infty} x(i)(F_i - F_{i+1}) \right\| \leqslant \left\| (x(i)) \right\|_1. \]

If \( g = \sum_{i=0}^{\infty} g(i) u_i \), then for any \( n \leq r - 1 \) it is possible to write \( g \) as

\[ g = au_n + f, \quad f = a(u_{n+1} - u_n) + \sum_{i=n+1}^{\infty} a_i (u_i - u_{i-1}), \]

where \( a = \sum_{i=0}^{\infty} g(i) \). Note that \( \|u_{n+1} - u_n\| = \|u_{n+1} - u_n\|_1 \), and it follows from (17) that \( \|u_{n+1} - u_n\| = 1 \). Also, \( \|u_n\| = 2^{-1/2} \), since \( (e_n) \) is bimonotone and it follows from (16) that \( \|e_n\| = 2^{1/2} \). Since \( \|e_n - u_n\| = 1 \), we have

\[ \|g\| \geq \|f\| - \|au_n\| = \|f\| - 2^{-1/2} \|\sum_{i=n+1}^{\infty} a_i (u_i - u_{i-1})\| \geq (1 - 2^{-1/2}) \|f\|. \]

Therefore, it follows from (20) and (21) with \( g = au_n + f \), where \( n \) is chosen so \((a, au_n) = 0\), that we finally have

\[ (1 - 2^{-1/2}) \|x(i)\|_1 \leq \left\| \sum_{i=0}^{\infty} x(i)(F_i - F_{i+1}) \right\| \leqslant \|x(i)\|_1. \]

Since \( (F_i - F_{i+1}) \) satisfies (10) and (11) with \( M = 2^{1/2} \), this implies \( \|x(i)\|_1 \) satisfies (10) and (11) with \( M = 2^{1/2} \).

**Lemma 7.** Let \( X \) be the linear space of functions defined on the set of all integers and having finite support. There is a norm \( \| \cdot \| \) for \( X \) such that \( (e_n) \) is inversion-invariant, translation-invariant, and nearby, and satisfies (10) and (11) with \( M = 2 \). Also, if each of \( \| \cdot \|_1 \) and \( \| \cdot \|_1 \) is taken to be \( \| \cdot \| \), then (14) is satisfied.

**Proof.** From Lemma 6, we have two norms that satisfy (14); satisfy (a), (b), (c) of Lemma 5; and satisfy (10) and (11) with \( M = 2 \). Then successive determinations of new pairs of norms \( (\| \cdot \|_1, \| \cdot \|_1) \) by use of Lemma 5 gives two pointwise-convergent sequences, \( (\| \cdot \|_1) \) and \( (\| \cdot \|_1) \). These sequences converge to the same limit \( \| \cdot \| \), since

\[ \| \cdot \|_1 \leq \| \cdot \|_{1,1} \leq \| \cdot \|_{1,2} \leq \| \cdot \|_1 \leq \| \cdot \|_{1,1}. \]

To show \( \| \cdot \| \) is the desired norm, the only serious question seems to be whether, if each of \( \| \cdot \|_1 \) and \( \| \cdot \|_1 \) is taken to be \( \| \cdot \| \), then (14) is satisfied.

If \( x = \sum_{i=0}^{\infty} x(i) e_i \) belongs to \( X \), let \( \|x\|^* \) be the norm of the linear functional \( \sum_{i=0}^{\infty} x(i) (u_i - u_{i-1}) \), when \( X \) is given the norm \( \| \cdot \| \). Since \( \|x\|^* \) is finite for all \( x \), it follows from (14) being satisfied for each pair \( (\| \cdot \|_1, \| \cdot \|_1) \) that \( \|x\| \leq \|x\|_1 \). Let us show that \( \|x\| \geq \|x\|_1 \). As noted in the proof of Lemma 5, if \( f = \sum_{i=0}^{\infty} f(i) u_i \) belongs to \( X \) and \( \varphi = \sum_{i=0}^{\infty} f(i) (u_i - u_{i-1}) \), then \( \varphi = ||x||^* \). Therefore, \( \|x\|_1 \geq \|x\| \). Since \( \|x\|_1 \geq \|x\|_1 \) for each \( n \), this implies \( \varphi = ||x||^* \) for each \( n \) and each \( x \) in \( X \). Therefore \( \|x\| \leq \|x\|_1 \).

**Theorem 4.** There is a Banach space \( B^* \) that is quasi-reflexive of order one and isomorphic to \( B^* \). The space \( B \) is the completion of a normed linear space of functions on the positive integers with finite support and has the properties:

(i) the natural basis \( (e_n) \) is inversion-invariant, translation-invariant, and nearby, with \( 1 \leq \|e_n\| \leq V/3 \);

(ii) \( \left\| \sum_{n=0}^{\infty} a_n^{1/2} e_n \right\| \leq \left\| \sum_{n=0}^{\infty} a_n^{1/2} e_n \right\| \leq 2 \left\| \sum_{n=0}^{\infty} a_n^{1/2} \right\|^{1/2} \) if \( a \) is a sum of strongly disjoint bumps with altitudes \( (e_n) \);

(iii) if \( (x^n) \) is strongly disjoint, then

\[ \left( 2 \left( 1 + 2 \right) \right)^{1/2} \left\| \sum_{n=0}^{\infty} a_n^{1/2} e_n \right\| \leq \left\| \sum_{n=0}^{\infty} a_n^{1/2} e_n \right\| \leq 2 \left( 1 + 2 \right)^{1/2} \left\| \sum_{n=0}^{\infty} a_n^{1/2} e_n \right\|^{1/2} ; \]

(iv) \( B \) is isomorphic to a Banach space that is isometric to its second dual.

**Proof.** Let \( \| \cdot \| \) be the norm given by Lemma 7 for the space \( X \), with \( (e_n) \) the coefficient functionals for the natural basis \( (e_n) \). As noted in the proof of Lemma 5, \( \|e_n\|_1 = 2^{1/2} \) and \( \|e_n\|_1 = 1 \). Therefore

\[ 1 \leq \|e_n\| \leq \|u_n - u_{n-1}\| = \left\{ \|e_n\|_1 + \|e_n\|_1 \right\} = V/3. \]

Since \( \|e_n\| \geq 1 \), we have \( \|u_n\| \leq 1 \). Since \( (u_n) \) is inversion-invariant and \( (u_n - u_{n-1}) \) is inversion-invariant, then

\[ (22) \quad \text{dist} (u_n, \text{lin} (u_n - u_{n-1} : n \neq 1)) = \text{dist} (u_n, \text{lin} (u_n - u_{n-1} : n \neq 1)) \]

\[ \geq \frac{1}{2} \text{dist} (u_1 - u_e, \text{lin} (u_n - u_{n-1} : n \neq 1)) \]

\[ \geq \frac{1}{2} \|u_1 - u_e\| \geq 1. \]

Let \( B \) be the completion of \( \text{lin} (e_n : n \geq 1) \). Since \( (e_n) \) is inversion-invariant and \( (u_n) \) is inversion-invariant, translation-invariant, and nearby. It follows from (16) that \( (e_n : n \geq 1) \) is a shrinking basis for \( B \). Since \( (e_n) \) is inversion-invariant, translation-invariant, and nearby, the norm of \( \sum_{n=0}^{\infty} a_n u_n \) is independent of whether it is regarded as a member of \( B^* \) or \( X^* \). Since \( (u_n - u_{n-1}) \) is shrinking and basic, and \( u_n \text{lin} (u_n - u_{n-1} : n \geq 1) \), the sequence \( (u_1 - u_e, u_2 - u_1, u_3 - u_2, \ldots) \)
is a shrinking basis for $B^*$. Thus it follows from Theorem 2 that $B$ is quasi-reflexive of order one. Now let

$$ T \left[ \sum_{i=1}^{\infty} \sigma(i)e_i \right] = \sigma(1)u_1 + \sum_{i=1}^{\infty} \sigma(i)(u_i - u_{i-1}). $$

Because (14) is satisfied if each of $\| \|$, and $\| \|$ is taken to be $\|$, each series in (23) converges if and only if the other converges. Thus $T$ is a linear one-to-one map of $B$ onto $B^*$. Since

$$ \| \sigma(1)u_1 + \sum_{i=1}^{\infty} \sigma(i)(u_i - u_{i-1}) \| \leq \| \sigma(1)u_1 \| + \sum_{i=1}^{\infty} \| \sigma(i)e_i \| $n\leq \sigma(1)u_1 \| + \sum_{i=1}^{\infty} \| \sigma(i)u_1 \|,$n\leq \sigma(1)u_1 \| + \sum_{i=1}^{\infty} \| \sigma(i)e_i \| \leq 2 \sum_{i=1}^{\infty} \| \sigma(i)e_i \|,$nwe have $\|T\| \leq 2$. Also, $\|T^{-1}\| \leq 3$, since

$$ \| \sum_{i=1}^{\infty} \sigma(i)e_i \| = \| \sum_{i=1}^{\infty} \sigma(i)(u_i - u_{i-1}) \| $n\leq \| \sigma(1)u_1 \| + \| \sum_{i=1}^{\infty} \sigma(i)(u_i - u_{i-1}) \|,$n\leq \| \sigma(1)u_1 \| + \| \sum_{i=1}^{\infty} \sigma(i)(u_i - u_{i-1}) \|,$nand this and (22) imply

$$ \| \sum_{i=1}^{\infty} \sigma(i)e_i \| \leq 3 \| \sigma(1)u_1 + \sum_{i=1}^{\infty} \sigma(i)(u_i - u_{i-1}) \|. $n$$

Finally, (ii) and (iii) follow from the fact that (ii), (10) and (11) with $M = 2(1 + 2^{10})$, are satisfied by both of the norms used to initiate the convergence to the norm of Lemma 7; (iv) follows from Theorem 4 of [1] and the fact that $\{e_i\}$ being neighborly implies the sequence of coefficient functionals $\{\sigma_i\}$ is equal-signs-additive.

References

