Let $f = g\circ\varphi$ and write $\varphi = u+iv$. Clearly, f is C^1 in $|z|<1$. Because

$$\nabla f = \big(g_x(\varphi)u_x+g_y(\varphi)v_x,\ g_x(\varphi)u_y+g_y(\varphi)v_y\big),$$

from (c) we see that f and ∇f are continuous up to E through triangles $T(z)$'s. From (b), (c), (d) and the Cauchy–Riemann equations, we have $\nabla f \neq (0,0)$ and is not normal to the unit circle. Since the level sets are preserved under conformal mappings, we conclude Theorem 3 from Theorem 2.

EXAMPLES. (1) The following result of Arsove ([1], p. 267) is a simple consequence of Theorem 3 and the Riesz decomposition theorem: if h is a subharmonic function on $\{|z|<1,\ |z-1|<\delta\}$ with positive harmonic majorant then $\lim_{r\to 1} h(re^{i\theta})$ exists for almost all $e^{i\theta}$ in $\{|z|<1,\ |z-1|<\delta\}$.

(2) A and v are defined as in Theorem 3. Then at almost all points of ∂A, where the tangents of ∂A are not horizontal, v has horizontal limit zero.

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Multipliers and unconditional Schauder bases in Besov spaces

by

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Abstract. The paper contains: (a) Explicit representations for distributions belonging to Besov spaces, (b) A system of analytic functions which is an unconditional Schauder basis in Besov spaces, (c) A necessary and sufficient algebraic condition for multipliers in Besov spaces, (d) Remarks on embedding theorems for Besov spaces.

1. Introduction. The paper deals with isotropic Besov spaces $B^s_{p,q}$ ($=$ Lipschitz spaces $\Lambda^s_{p,q}$) defined in $\boldsymbol{R}_n$, where $-\infty<s<\infty$; $1<p<\infty$; and $1\leqslant q\leqslant\infty$. There exists a large variety of different characterizations of distributions belonging to these spaces [3]–[7]. A summary may be found in [9], Chapter 2. In Section 2 of this paper a new representation formula is given. As an immediate consequence there is obtained a common unconditional Schauder basis in all the spaces $B^s_{p,q}$ (provided that $q<\infty$), consisting of entire analytic functions of exponential type, Section 3. A second application of the representation formula yields a necessary and sufficient condition for multipliers in $B^s_{p,q}$, Section 4. A more detailed discussion of this result will be given later on. Section 5 contains remarks on embedding theorems.

2. Representations. $\boldsymbol{R}_n$ denotes the n-dimensional real Euclidean space. The general point in $\boldsymbol{R}_n$ is denoted by $x=(x_1,\dots,x_n)$. S ($=S(\boldsymbol{R}_n)$) is the usual Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions, defined on $\boldsymbol{R}_n$. As usual, S' ($=S'(\boldsymbol{R}_n)$) is the space of tempered distributions, the dual space to S. The Fourier transform in S' is denoted by F, its inverse by F^{-1}. If $f\in S$, then

$$(Ff)(\xi) = (2\pi)^{-\frac{n}{2}}\int_{\boldsymbol{R}_n} e^{-ix\xi}f(x)\,dx,$$

where $x\xi = \sum_{j=1}^{n} x_j\xi_j$. If $-i$ is replaced by i, then one obtains the corresponding formula for F^{-1}.

Use the following decomposition of $\boldsymbol{R}_n$: Let $\sigma=(\sigma_1,\dots,\sigma_n)$ be a vector in $\boldsymbol{R}_n$, where each of the components σ_j is either ± 1 or ± 3, but at least one of them is ± 3. (This means that the vectors σ, where all the components are ± 1, are not considered here.) Let $\sigma^{(l)}$ be the above vectors, numbered in an arbitrary way; $l = 1, \ldots, L_n$; the components are denoted by $\sigma_j^{(l)}$. Here $L_n = 2^n(2^n - 1)$. Define

(1a) $$Q_k^{(l)} = \{x \mid 2^{k-1}(\sigma_j^{(l)} - 1) < x_j \leqslant 2^{k-1}(\sigma_j^{(l)} + 1)\},$$

for $k = 1, 2, 3, \ldots$ and $l = 1, \ldots, L_n$, and

(1b) $$Q_0 = \{x \mid -2 < x_j \leqslant 2\}.$$

The following hold

(2) $$\boldsymbol{R}_n = Q_0 \cup \bigcup_{k=1}^{\infty} \bigcup_{l=1}^{L_n} Q_k^{(l)},$$

(3) $$\bigcup_{l=1}^{L_n} Q_k^{(l)} = \{x \mid -2^{k+1} < x_j \leqslant 2^{k+1}\} - \{x \mid -2^k < x_j \leqslant 2^k\}.$$

The last formula shows the meaning of the decomposition (2): By fixed k the difference of the two cubes on the right-hand side of (3) is divided in the disjoint cubes (1a), where $l = 1, \ldots, L_n$. If $\chi_k^{(l)}$ is the characteristic function of $Q_k^{(l)}$, and if χ_0 is the characteristic function of Q_0, then

(4) $B_{p,q}^s = \{f \mid f \in S',$

$$\|f\|_{B_{p,q}^s} = \|F^{-1}\chi_0 Ff\|_{L_p} + \Big[\sum_{k=1}^{\infty}\sum_{l=1}^{L_n}(2^{sk}\|F^{-1}\chi_k^{(l)}Ff\|_{L_p})^q\Big]^{\frac{1}{q}} < \infty\},$$

where $-\infty < s < \infty$, $1 < p < \infty$; and $1 \leqslant q \leqslant \infty$ (for $q = \infty$ one must replace the l_q-norm in (4) by the corresponding l_∞-norm). Here $\|\cdot\|_{L_p}$ is the usual norm in $L_p(\boldsymbol{R}_n)$. These are the *Besov spaces*: One of the known possibilities to define Besov spaces is a formula of type (4), where $\chi_k^{(l)}$ is replaced by the characteristic functions of the sets on the right-hand side of (3); see [3], p. 374, or [9], Lemma 2.11.2. But the well-known L_p-multiplier properties of characteristic functions of cubes yield (4). Remind that by the Paley–Wiener–Schwartz theorem a distribution $g \in S'$, whose Fourier transform Fg has a compact support, is an entire analytic function of exponential type (see, for instance, [2], 1.7.7). In particular, $F^{-1}\chi_k^{(l)}Ff$ and $F^{-1}\chi_0 Ff$ are entire analytic functions of exponential type. Denote by $\boldsymbol{N}_n$ the set of all lattice points $m = (m_1, \ldots, m_n)$, where m_j are integers. Furthermore, let $xm = \sum_{j=1}^{n} x_j m_j$, where $x \in \boldsymbol{R}_n$ and $m \in \boldsymbol{N}_n$. Similarly, $\sigma^{(l)}x$.

THEOREM 1. *Let $-\infty < s < \infty$; $1 < p < \infty$; and $1 \leqslant q \leqslant \infty$. Then the following three assertions are equivalent:*

(i) $f \in B_{p,q}^s$;

(ii) *$f \in S'$ has the representation*

(5) $$f = \sum_{m \in \boldsymbol{N}_n}\Big[a_m F^{-1}(e^{-i\frac{\pi}{2}xm}\chi_0) + \sum_{k=1}^{\infty}\sum_{l=1}^{L_n} a_{k,m}^{(l)} F^{-1}(e^{-i\pi 2^{-k+1}xm}\chi_k^{(l)})\Big],$$

(6) $$\Big(\sum_{m \in \boldsymbol{N}_n}|a_m|^p\Big)^{\frac{1}{p}} + \Big[\sum_{k=1}^{\infty}\sum_{l=1}^{L_n}\Big(\sum_{m \in \boldsymbol{N}_n}|2^{k(s-\frac{n}{p})+nk}a_{k,m}^{(l)}|^p\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}} < \infty$$

(*with the usual modification for $q = \infty$*);

(iii) *$f \in S'$ has the representation*

(7) $$f = \sum_{m \in \boldsymbol{N}_n}\Big[b_m \prod_{j=1}^{n}\frac{\sin 2x_j}{2x_j - m_j\pi} + \sum_{k=1}^{\infty}\sum_{l=1}^{L_n} b_{k,m}^{(l)} e^{-i2^{k-1}\sigma^{(l)}x}\prod_{j=1}^{n}\frac{\sin 2^{k-1}x_j}{2^{k-1}x_j - m_j\pi}\Big],$$

(8) $$\Big(\sum_{m \in \boldsymbol{N}_n}|b_m|^p\Big)^{\frac{1}{p}} + \Big[\sum_{k=1}^{\infty}\sum_{l=1}^{L_n}\Big(\sum_{m \in \boldsymbol{N}_n}|2^{k(s-\frac{n}{p})}b_{k,m}^{(l)}|^p\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}} < \infty$$

(*with the usual modification for $q = \infty$*).

Remark 1. The proof below shows that the norms described in (6) and (8), and $\|f\|_{B_{p,q}^s}$, are equivalent norms in $B_{p,q}^s$. Furthermore, (iii) is simply a reformulation of (ii), namely:

(9) $$F^{-1}(e^{-i\pi 2^{-k+1}xm}\chi_k^{(l)}) = (2\pi)^{-\frac{n}{2}} 2^{kn} e^{-i2^{k-1}\sigma^{(l)}x}\prod_{j=1}^{n}\frac{\sin 2^{k-1}x_j}{2^{k-1}x_j - m_j\pi}$$

and, consequently,

(10) $$b_{k,m}^{(l)} = (2\pi)^{-\frac{n}{2}} 2^{kn} a_{k,m}^{(l)}$$

and corresponding formulas for the first terms in (5)–(8). The convergence in (5) and (7) is to be understood as convergent series in S'. By the above remarks concerning the Paley–Wiener–Schwartz theorem, or by (9), it follows that

(11) $$F^{-1}(e^{-i\frac{\pi}{2}xm}\chi_0) \quad\text{and}\quad F^{-1}(e^{-i\pi 2^{-k+1}xm}\chi_k^{(l)})$$

are entire analytic functions of exponential type. Furthermore, the proof below yields: If $f \in B_{p,q}^s$ is given, then

(12) $$b_m = (F^{-1}\chi_0 Ff)\Big(\frac{\pi}{2}m\Big), \qquad b_{k,m}^{(l)} = (F^{-1}\chi_k^{(l)}Ff)(\pi 2^{-k+1}m),$$

where $m \in \boldsymbol{N}_n$; $k = 1, 2, \ldots$; and $l = 1, \ldots, L_n$. Finally, we remark that

(ii) and (iii) is also a description of the structure of the Besov spaces: $B^s_{p,q}$ is isomorphic to $l_q(l_p)$. Here $-\infty < s < \infty$; $1 < p < \infty$; $1 \leqslant q \leqslant \infty$. This was proved in [8] (see also [9], 2.11.2), but essentially this result is due to J. Peetre.

Proof of Theorem 1. We give a formal proof and add in the Remark 2 below some rigorous arguments. Develop Ff in $Q^{(l)}_k$ in a Fourier series.

$$Ff = \sum_{m \in N_n} a^{(l)}_{k,m} e^{-i\pi 2^{-k+1} xm}, \qquad x \in Q^{(l)}_k. \tag{13}$$

$$\begin{aligned} a^{(l)}_{k,m} &= \frac{1}{|Q^{(l)}_k|} \int_{Q^{(l)}_k} e^{i\pi 2^{-k+1} ym} Ff\, dy \\ &= 2^{-kn} \int_{R_n} e^{i\pi 2^{-k+1} ym} \chi^{(l)}_k Ff\, dy \\ &= 2^{-kn} (2\pi)^{\frac{n}{2}} (F^{-1} \chi^{(l)}_k Ff)(\pi 2^{-k+1} m). \end{aligned} \tag{14}$$

Consequently,

$$F^{-1} \chi^{(l)}_k Ff = \sum_{m \in N_n} a^{(l)}_{k,m} F^{-1} (\chi^{(l)}_k e^{-i\pi 2^{-k+1} xm}). \tag{15}$$

A corresponding formula holds for Q_0. Summing up all these series one obtains (5), where $a^{(l)}_{k,m}$ are determined by (10) and (12). (Similar formulas hold for the case Q_0.) $F^{-1}\chi^{(l)}_k F$, resp. $F^{-1}\chi_0 F$, is a projection in $L_p(\boldsymbol{R}_n)$; $1 < p < \infty$.

Now we use the following fact: The operator $g \to \{g(\pi 2^{-k+1} m)\}_{m \in N_n}$ is a one-to-one map from the range of the projection $F^{-1}\chi^{(l)}_k F$ in L_p *onto* l_p. This is proved in [8] and [9], 2.11.2.

Applying this assertion to the above situation, it follows that

$$c \|F^{-1} \chi^{(l)}_k Ff\|_{L_p} \leqslant 2^{kn - k\frac{n}{p}} \Big(\sum_{m \in N_n} |a^{(l)}_{k,m}|^p \Big)^{\frac{1}{p}} \leqslant c' \|F^{-1} \chi^{(l)}_k Ff\|_{L_p}, \tag{16}$$

where the positive numbers c and c' are independent of k and l (the last statement is a consequence of an homogeneity-argument).

If $f \in B^s_{p,q}$, then (4) yields (6). If f is given by (5) and (6), then it follows from (5), (16), and (4) that f belongs to $B^s_{p,q}$. This proves the equivalence of (i) and (ii). Denoting temporary the characteristic function of the cube $\{x \mid |x_j| \leqslant 2^{k-1}\}$ by ψ_k. Then the reformulation of (ii), given in (iii), follows from

$$\begin{aligned} [F^{-1}(e^{-i\pi 2^{-k+1} xm} \chi^{(l)}_k(x))](\xi) &= [F^{-1} \chi^{(l)}_k(x)](\xi - \pi 2^{-k+1} m) \\ &= [F^{-1} \psi_k(2^{k-1}\sigma^{(l)} + x)](\xi - \pi 2^{-k+1} m) \\ &= e^{-i 2^{k-1} \sigma^{(l)} \xi} e^{i\pi\sigma^{(l)} m} [F^{-1} \psi_k(x)](\xi - \pi 2^{-k+1} m) \end{aligned}$$

and

$$[F^{-1}\psi_k](\xi) = (2\pi)^{-\frac{n}{2}} 2^n \prod_{j=1}^{n} \frac{\sin 2^{k-1} \xi_j}{\xi_j},$$

and corresponding formulas for the case Q_0. This proves the theorem, including the statements in Remark 1.

Remark 2. In the above proof we did not carefully check the conditions ensuring the convergence of the above series. But it is not very hard to justify the calculations: If $f \in S$, such that Ff has a compact support, then (13), (14), (15), and (5) are meaningful in a classical sense. If $q < \infty$, then functions of such a type are dense in $B^s_{p,q}$. Now (i)→(ii) follows for these spaces by a limit argument. Justify the converse direction: If only a finite number of the coefficients a_m and $a^{(l)}_{k,m}$ is different from zero, then there is no problem. In the case $q < \infty$ one obtains (ii)→(i) again by a limit argument. In particular,

$$f \to \{a_m, a^{(l)}_{m,k}\}$$

is a one-to-one map from $B^s_{p,q}$ onto a weighted space of type $l_q(l_p)$, provided that $q < \infty$. Then the same assertion is true for spaces obtained by real interpolation $(\cdot, \cdot)_{\theta,\infty}$, in particular for

$$B^s_{p,\infty} = (B^{s_0}_{p,1}, B^{s_1}_{p,1})_{\theta,\infty},$$

$-\infty < s_0 < s < s_1 < \infty$; $s = (1-\theta)s_0 + \theta s_1$, and the corresponding weighted spaces of type $l_\infty(l_p)$. (For details of the used interpolation we refer to [9].) This proves the equivalence of (i) and (ii) for $q = \infty$, too.

3. Schauder bases.

THEOREM 2. *Let* $-\infty < s < \infty$; $1 < p < \infty$; *and* $1 \leqslant q < \infty$. *Then*

$$\left\{ \prod_{j=1}^{n} \frac{\sin 2x_j}{2x_j - m_j\pi},\ e^{-i2^{k-1}\sigma^{(l)}x} \prod_{j=1}^{n} \frac{\sin 2^{k-1}x_j}{2^{k-1}x_j - m_j\pi} \right\}_{\substack{k=1,2,3,\ldots \\ l=1,\ldots,L_n \\ m \in N_n}} \tag{17}$$

is an unconditional Schauder basis in $B^s_{p,q}$.

Proof. The proof is an immediate consequence of Theorem 1 and Remark 1.

Remark 3. As remarked above, (17) consists of entire analytic functions of exponential type. Smoothness properties for bases of such a type are necessary: If $\{g_r\}_{r=1}^{\infty}$ is a common Schauder basis in all spaces $B^s_{p,q}$; $q < \infty$; then it follows from Sobolev's embedding theorem, that all the functions g_r are infinitely differentiable functions. If one assumes additionally that for fixed p and q (for instance $p = q = 2$) there exist positive numbers c_r such that

$$\|g_r\|_{B^s_{p,q}} \leqslant c_r^s \quad \text{for } s \geqslant 0, \tag{18}$$

then it follows that $g_r(x)$ is an entire analytic function of exponential type: By Sobolev's embedding theorem, (18) yields for all multi-indices α

$$\sup_x |D^\alpha g_r(x)| \leqslant c\,\|g_r\|_{B^{|\alpha|+1+\frac{n}{p}}_{p,q}} \leqslant C_r^{|\alpha|}$$

by an appropriate choice of C_r. But such a function is an entire analytic function of exponential type. (17) satisfies a condition of type (18).

Remark 4. It will be useful to remind the reformulation given in (9): The system

$$\{F^{-1}(e^{-i\frac{\pi}{2}xm}),\ F^{-1}(e^{-i\pi 2^{-k+1}xm}\chi_k^{(l)})\}_{\substack{k=1,2,3,\dots\\ l=1,\dots,L_n\\ m\in N_n}} \tag{19}$$

is also an unconditional Schauder basis in $B^s_{p,q}$, provided that $-\infty < s < \infty$; $1 < p < \infty$; and $1 \leqslant q < \infty$.

Remark 5. The above argumentation fails for $q = \infty$. The spaces $B^s_{p,\infty}$ are isomorphic to $l_\infty(l_p)$. In particular, $B^s_{p,\infty}$ is not a separable Banach space. If $\overset{0}{B}{}^s_{p,\infty}$ denotes the completion of S in $B^s_{p,\infty}$, then it follows by the above argumentation that $\overset{0}{B}{}^s_{p,\infty}$ is isomorphic to $c_0(l_p)$ (here c_0 is the subspace of l_∞ consisting of all sequences whose components tend to zero.). Then the above procedure yields that (17), resp. (19), is also an unconditional Schauder basis in $\overset{0}{B}{}^s_{p,\infty}$; $-\infty < s < \infty$; $1 < p < \infty$.

4. Multipliers. First we give some general assertions, similarly to corresponding formulas by L. Hörmander [1]. All the used properties for Besov spaces (lifting property, duality, interpolation) may be found in [7] or in [9], Chapter 2. A tempered distribution M is said to be a *multiplier in* $B^s_{p,q}$ if there exists a non-negative number C such that for all $f \in B^s_{p,q}$

$$\|F^{-1}MFf\|_{B^s_{p,q}} \leqslant C\,\|f\|_{B^s_{p,q}} \tag{20}$$

holds. Here $-\infty < s < \infty$; $1 < p < \infty$; and $1 \leqslant q \leqslant \infty$.

Obviously, one may generalize this definition, if one replaces, for instance, $B^s_{p,q}$ on the left-hand side by another Besov space or by a Lebesgue space (= Liouville space = Bessel potential space). But we shall be concerned here only with multipliers in $B^s_{p,q}$ in the above sense.

Explain (20): $F^{-1}MFf$ is meaningful at least for $f \in S$. If there exists a number C with (20) for all $f \in S$, then the bounded and linear operator $F^{-1}MF$, defined on S, can be extended by completion on $B^s_{p,q}$, provided that $q < \infty$ (S is dense in these spaces). (20) for $B^s_{p,\infty}$ will be explained rigorously below. If M is a multiplier in $B^s_{p,q}$ (assume temporary $q < \infty$), then it is also a multiplier in $B^\sigma_{p,q}$, where σ is an arbitrary real number. This is a consequence of the lifting property,

$$\|F^{-1}(1+|\xi|^2)^{\frac{\sigma-s}{2}}Ff\|_{B^s_{p,q}} \sim \|f\|_{B^\sigma_{p,q}}$$

(equivalent norms). This shows that the following definition of a multiplier M in $B^s_{p,\infty}$ is meaningful: For fixed p, where $1 < p < \infty$, there exists a linear operator, mapping $B^\sigma_{p,\infty}$ continuously into $B^\sigma_{p,\infty}$ for all real numbers σ, whose restriction on S coincides with $F^{-1}MF$. In particular, the set of all multipliers in $B^s_{p,q}$ does not depend on s; here $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. But it is also independent of q. This follows from the above statement (independence of s), the interpolation property, and the interpolation formula

$$B^s_{p,r} = (B^{s_0}_{p,q}, B^{s_1}_{p,q})_{\theta,r};$$

where $-\infty < s_0 < s < s_1 < \infty$; $s = (1-\theta)s_0 + \theta s_1$; $1 < p < \infty$; $1 \leqslant q \leqslant \infty$; and $1 \leqslant r \leqslant \infty$. So it is meaningful to denote the set of all multipliers in $B^s_{p,q}$ by $\mathscr{M}_p$. The set of all multipliers in the Lebesgue space L_p is denoted by M_p. Here $1 < p < \infty$.

Lemma. *Let* $1 < p < \infty$ *and* $1/p + 1/p' = 1$. *Then the following hold*

$$M_p \subset \mathscr{M}_p, \tag{21}$$

$$\mathscr{M}_p = \mathscr{M}_{p'}, \tag{22}$$

$$\mathscr{M}_p \subset \mathscr{M}_q \subset \mathscr{M}_2 = M_2 = L_\infty, \tag{23}$$

provided that $|1/q - 1/2| \leqslant |1/p - 1/2|$.

Proof. If $M \in M_p$, then M is also a multiplier in the Lebesgue space (= Bessel potential space = Liouville space) H^s_p, where $-\infty < s < \infty$. The interpolation property and

$$B^{s\theta}_{p,q} = (L_p, H^s_p)_{\theta,q}; \quad s \neq 0;\ 0 < \theta < 1;\ 1 \leqslant q \leqslant \infty;$$

yield $M \in \mathscr{M}_p$. This proves (21). The operator $F^{-1}MF$ is formally self-adjoint. Hence, (22) is a consequence of the duality property $(B^0_{p,2})' = B^0_{p',2}$. Let $1 < p < q \leqslant 2$. Then the first two assertions in (23) follow from (22) and the interpolation formula

$$B^0_{q,q} = (B^0_{p,p}, B^0_{p',p'})_{\theta,q}; \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{p'}.$$

Finally, $B^0_{2,2} = L_2$ and the well-known fact $M_2 = L_\infty$ yield the equalities in (23).

Remark 6. In addition to (21) the following holds:

An essentially bounded function $M(x)$ with compact support belongs to $\mathcal{M}_p$ if and only if it belongs to M_p.

Assume, without loss of generality, supp $M(x) \subset Q_0$. If $M \in \mathcal{M}_p$ and if $f \in S$, then (4) yields

$$\begin{aligned}\|F^{-1}MFf\|_{L_p} &= \|F^{-1}\chi_0 F(F^{-1}MFF^{-1}\chi_0 Ff)\|_{L_p} \\ &= \|F^{-1}MF(F^{-1}\chi_0 Ff)\|_{B^s_{p,q}} \leqslant C\|F^{-1}\chi_0 Ff\|_{B^s_{p,q}} \\ &= C\|F^{-1}\chi_0 Ff\|_{L_p} \leqslant C'\|f\|_{L_p}.\end{aligned}$$

Hence $M \in M_p$. The converse assertion follows from (21).

The main aim of this section is the characterization of $\mathcal{M}_p$. Let $M \in L_\infty$. Then the Fourier coefficients of M in the cubes $Q^{(l)}_k$ and Q_0 are denoted by $M^{(l)}_{k,m}$ and M_m:

$$M^{(l)}_{k,m} = 2^{-kn} \int_{Q^{(l)}_k} M(x) e^{i\pi 2^{-k+1}xm} dx, \tag{24a}$$

$$M_m = 4^{-n} \int_{Q_0} M(x) e^{i\frac{\pi}{2}xm} dx, \tag{24b}$$

$m \in N_n$; $k = 1, 2, 3, \ldots$; $l = 1, \ldots, L_n$.

THEOREM 3. *If $1 < p < \infty$; then $\mathcal{M}_p$ is the set of all essentially bounded functions in R_n (with respect to the Lebesgue measure) with the additional property that there exists a positive number B such that*

$$\Big(\sum_{t \in N_n}\Big|\sum_{m \in N_n} c_m M_{t-m}\Big|^p\Big)^{\frac{1}{p}} \leqslant B\Big(\sum_{m \in N_n}|c_m|^p\Big)^{\frac{1}{p}}, \tag{25a}$$

$$\Big(\sum_{t \in N_n}\Big|\sum_{m \in N_n} c_m M^{(l)}_{k,t-m}\Big|^p\Big)^{\frac{1}{p}} \leqslant B\Big(\sum_{m \in N_n}|c_m|^p\Big)^{\frac{1}{p}} \tag{25b}$$

hold for all sequences $\{c_m\}_{m \in N_n}$, for which the right-hand side of (25) is finite, for all numbers k, where $k = 1, 2, 3, \ldots$; and for all numbers l, where $l = 1, \ldots, L_n$.

Proof. $\mathcal{M}_p \subset L_\infty$ was proved in the above lemma. Let $M(x) \in L_\infty$ and $f \in S$. Let f be given by (5). Then

$$F^{-1}MFf = \sum_{m \in N_n}\Big[a_m F^{-1}\, M(x) e^{-i\frac{\pi}{2}xm}\chi_0) + \sum_{k=1}^{\infty}\sum_{l=1}^{L_n} a^{(l)}_{k,m} F^{-1}\big(M(x)e^{-i\pi 2^{-k+1}xm}\chi^{(l)}_k\big)\Big] \tag{26}$$

holds. Since $M \in L_\infty$, the last series converges in L_2 and so also in S'.

Let

$$F^{-1}MFf = \sum_{m \in N_n}\Big[b_m F^{-1}(e^{-i\frac{\pi}{2}xm}\chi_0) + \sum_{k=1}^{\infty}\sum_{l=1}^{L_n} b^{(l)}_{k,m}F^{-1}(e^{-i\pi 2^{-k+1}xm}\chi^{(l)}_k)\Big] \tag{27}$$

be the representation of $F^{-1}MFf$ in the sense of Theorem 1 (ii). Develop $M(x)e^{-i\frac{\pi}{2}xm}$ in Q_0 in a Fourier series,

$$M(x)e^{-i\frac{\pi}{2}xm} = \sum_{t \in N_n} M_{t-m} e^{-i\frac{\pi}{2}xt}, \quad x \in Q_0.$$

Corresponding formulas hold for the Fourier series of $M(x)e^{-i\pi 2^{-k+1}xm}$ in $Q^{(l)}_k$. Comparison of (26) and (27) yields

$$b_t = \sum_{m \in N_n} a_m M_{t-m}; \quad b^{(l)}_{k,t} = \sum_{m \in N_n} a^{(l)}_{k,m} M^{(l)}_{k,t-m}. \tag{28}$$

If (25) is satisfied, then it follows from (28) and from Theorem 1 (ii) that M belongs to $\mathcal{M}_p$.

Prove the converse assertion. Let $M \in \mathcal{M}_p$. If f is given by (5), where $a^{(l)}_{k,m} = 0$, then (25a) is a consequence of (6), (20), and (28). In the same way one obtains (25b).

Remark 7. The proof shows that C in (20) and B in (25) are correlated by $C = cB$, where c depends on n, s, p, q, but not on M.

Remark 8. Using Remark 6 one obtains the following statement: *An essentially bounded function $M(x)$ with* supp $M(x) \subset Q_0$ *belongs to M_p if and only if (25a) is satisfied.*

(Obviously, there is no difficulty to replace "supp $M(x) \subset Q_0$" by "$M(x)$ has a compact support".)

EXAMPLE. *Let $M \in L_\infty$ and let*

$$\sum_{m \in N_n}|M_m| \leqslant B \quad \text{and} \quad \sum_{m \in N_n}|M^{(l)}_{k,m}| \leqslant B \tag{29}$$

for all $k = 1, 2, 3, \ldots$ and all $l = 1, \ldots, L_n$. Then (25) is satisfied, and consequently, $M \in \mathcal{M}_p$ for all p, where $1 < p < \infty$.

This statement contains multipliers of Hörmander type: Let

$$M_0(x) = \chi_0 M(x), \qquad M_k^{(l)}(x) = \chi_k^{(l)} M(x), \tag{30}$$

$$\|D^\alpha M_0\|_{L_2(Q_0)} \leqslant C, \qquad \|D^\alpha M_k^{(l)}\|_{L_2(Q_k^{(l)})} \leqslant C 2^{k\frac{n}{2}-k|\alpha|}, \tag{31}$$

for $|\alpha| \leqslant 1 + \left[\frac{n}{2}\right]$. (Here $\left[\frac{n}{2}\right]$ is the largest integer smaller than or equal to $\frac{n}{2}$.) First assume additionally that $M_0(x)$, resp. $M_k^{(l)}(x)$, vanishes near the boundary of Q_0, resp. $Q_k^{(l)}$. Using (24) and the fact that the sequence of the Fourier coefficients belongs to l_2, provided that the corresponding function belongs to L_2, it follows that

$$|M_{k,m}^{(l)}| \leqslant (1+|m|)^{-\left[\frac{n}{2}\right]-1} |\tilde{M}_{k,m}^{(l)}|, \qquad \sum_{m \in N_n} |\tilde{M}_{k,m}^{(l)}|^2 \leqslant C',$$

where C' depends only on C, but not on k and l. For M_m holds a corresponding formula. But

$$\sum_{m \in N} |M_{k,m}^{(l)}| \leqslant \Big(\sum_{m \in N_n} (1+|m|)^{-2\left[\frac{n}{2}\right]-2}\Big)^{\frac{1}{2}} \Big(\sum_{m \in N_n} |\tilde{M}_{k,m}^{(l)}|^2\Big)^{\frac{1}{2}} \leqslant C'',$$

and a corresponding formula for $\sum_{m \in N_n} |M_m|$. Hence, (29) is satisfied.

If $M(x) \in \mathscr{M}_p$, then also $M(x+h) \in \mathscr{M}_p$. Let M be a function satisfying (31). Using an appropriate partition of unity, M can be represented as a finite sum of functions satisfying the counterpart to (31) and the above additional property, and of functions of such a type shifted by constant vectors. This proves $M \in \mathscr{M}_p$.

5. Embeddings. In Section 3 and in Section 4 two applications of Theorem 1 are considered. The question arises whether other results can be obtained from Theorem 1. We add here some remarks concerning embedding theorems.

(a) *Embeddings for different metrics.* In the embedding theory for Besov spaces the following two well-known assertions play an important role:

$$B_{p,r}^s \subset B_{q,r}^t; \quad t - \frac{n}{q} = s - \frac{n}{p}; \quad 1 < p \leqslant q < \infty; \quad 1 \leqslant r \leqslant \infty; \tag{32}$$

and

$$B_{p,1}^{n/p} \subset C; \quad 1 < p < \infty. \tag{33}$$

Here $C = C(\boldsymbol{R}_n)$ is the space of complex-valued continuous functions obtained by completion of S in the supremum-norm. Most of the other

embedding theorems for different metrics can be obtained from (32) and (33) by interpolation, [9], § 2.8. Since $\|\cdot\|_{l_q} \leqslant \|\cdot\|_{l_p}$ for $p \leqslant q$, (32) is an immediate consequence of Theorem 1 (ii).

Prove (33). Let 0 be the element of N_n whose components are zero. Then, by (7),

$$f(0) = b_0 + \sum_{k=1}^{\infty} \sum_{l=1}^{L_n} b_{k,0}^{(l)}.$$

Using the fact that (8) with $q = 1$ and $s = n/p$ is an equivalent norm in $B_{p,1}^{n/p}$, it follows that

$$|f(0)| \leqslant c \|f\|_{B_{n,1}^{n/p}}.$$

Replacing $f(x)$ by $f(x+h)$, one obtains

$$|f(h)| \leqslant c \|f\|_{B_{p,1}^{n/p}}, \qquad h \in \boldsymbol{R}_n.$$

This proves (33).

(b) *Traces.* Beside the embedding theorems for different metrics, the traces on hyperplanes of lower dimension for functions belonging to Besov spaces are of interest. To indicate the dimension, the above Besov spaces are denoted now by $B_{p,q}^s(\boldsymbol{R}_n)$. Let $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1})$. Then the well-known assertions hold:

(i) (direct embedding) $f(x) \to f(x', 0)$ *gives a linear and bounded operator from* $B_{p,q}^s(\boldsymbol{R}_n)$ *onto* $B_{p,q}^{s-1/p}(\boldsymbol{R}_{n-1})$, *provided that* $\infty > s > 1/p$. *Here* $1 < p < \infty$; $1 \leqslant q \leqslant \infty$.

(ii) (inverse embedding) *There exists a linear and bounded operator* S *from* $B_{p,q}^{s-1/p}(\boldsymbol{R}_{n-1})$ *into* $B_{p,q}^s(\boldsymbol{R}_n)$ *such that* $(Sg)(x', 0) = g(x')$ *for all* $g(x') \in B_{p,q}^{s-1/p}(\boldsymbol{R}_{n-1})$. *Here* $-\infty < s < \infty$; $1 < p < \infty$; $1 \leqslant q \leqslant \infty$.

That $f(x) \to f(x', 0)$ is a bounded map from $B_{p,q}^s(\boldsymbol{R}_n)$ into $B_{p,q}^{s-1/p}(\boldsymbol{R}_{n-1})$ can be obtained from (33) by interpolation, [9], 2.9.3. We do not go into details here. We prove here the part (ii), which includes that the map $f(x) \to f(x', 0)$ of part (i) is a map "onto" (after it is proved that it is a map "into"). Let $g(x') \in B_{p,q}^{s-1/p}(\boldsymbol{R}_{n-1})$. Use the representation (7), (8), that means

$$g(x') = \sum_{m' \in N_{n-1}} \Big[b_{m'} \prod_{j=1}^{n-1} \frac{\sin 2x_j}{2x_j - m_j\pi} + \sum_{k=1}^{\infty} \sum_{l=1}^{L_{n-1}} b_{k,m'}^{(l)} e^{-i2^{k-1}\sigma'(l)x} \prod_{j=1}^{n-1} \frac{\sin 2^{k-1} x_j}{2^{k-1} x_j - m_j \pi} \Big],$$

$$\|g\|_{B^{s-1/p}_{p,q}(R_{n-1})} \sim \Big(\sum_{m'\in N_{n-1}} |b_{m'}|^p\Big)^{1/p} + \Big[\sum_{k=1}^{\infty}\sum_{l=1}^{L_{n-1}}\Big(\sum_{m'\in N_{n-1}} |2^{k(s-n/p)} b^{(l)}_{k,m'}|^p\Big)^{q/p}\Big]^{1/q},$$

with the usual modification for $q = \infty$. Here $m' = (m_1, \ldots, m_{n-1})$ and $\sigma'^{(l)} = (\sigma_1^{(l)}, \ldots, \sigma_{n-1}^{(l)})$. If $\sigma^{(l)} = (\sigma'^{(l)}, 3)$, then it follows from Theorem 1 (iii) that the operator S,

$$(Sg)(x) = \sum_{\substack{m\in N_n \\ m=(m',0)}} \Big[b_{m'} \prod_{j=1}^{n} \frac{\sin 2x_j}{2x_j - m_j\pi} + \sum_{k=1}^{\infty}\sum_{l=1}^{L_{n-1}} b^{(l)}_{k,m'} e^{-i2^{k-1}\sigma^{(l)}x} \prod_{j=1}^{n} \frac{\sin 2^{k-1}x_j}{2^{k-1}x_j - m_j\pi}\Big],$$

has the desired properties.

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Banach spaces quasi-reflexive of order one

by

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Abstract. It is shown that the nonreflexive Banach space J which is isomorphic to J^{**} is not isomorphic to J^*. In fact, J^* is not isomorphic to any subspace of J. Without explicitly describing the norm, it is shown that there is a Banach space which is quasi-reflexive of order one and isomorphic to its first dual. It has a basis with several properties similar to properties of the bases for J and J^*.

It is customary to use J to indicate any Banach space isomorphic to the space introduced in [5]. Thus J is isomorphic to a space that is isometric to its second dual [6] and J is quasi-reflexive of order one (i.e., the quotient of J^{**} and the natural image of J in J^{**} has dimension one). If J is isometric to J^{**}, then the l_2-product of J and J^* is quasi-reflexive of order two and isometric to its first dual.

It remains unknown whether J is isomorphic to some subspace of J^*. However, it seems to be a reasonable but difficult-to-prove conjecture that c_0 is not finitely representable in J^*. In fact, the three-dimensional space $l_\infty^{(3)}$ may not be representable in the predual I of J. Since c_0 is finitely representable in J ([4]), the truth of this conjecture would imply J is not isomorphic to any subspace of I. It also remains unknown whether there is a Banach space that is quasi-reflexive of order one and isometric to its dual. The methods of this paper suggest heuristically that no such space exists.

1. The space I is not isomorphic to any subspace of J. A particular norm will be chosen for J and the predual I will be evaluated explicitly. Any such predual is isomorphic to J^*. To prove that I is not isomorphic to any subspace of J (Theorem 3), it will be shown that I contains subspaces nearly isometric to $l_1^{(n)}$ (Lemma 2) in such a way that T being an isomorphism of I into J has the impossible consequence (Theorem 3) that, for any $\theta < 1$ and any positive integer n, there are members $\{x^1, \ldots, x^n\}$ of I such that $\|\sum_1^n x^j\| > \theta n$ and $\|\sum_1^n Tx^j\| < \|T\|\sqrt{n}/\theta$. This is done by constructing $l_1^{(n)}$-subspaces in I whose images in J are similar when

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