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Gâteaux smooth partitions of unity on weakly compactly generated Banach spaces

by

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Abstract. It is shown that any weakly compactly generated Banach space admits Gâteaux smooth partitions of unity.

1. Introduction. It was proved by R. Bonic and J. Frampton ([2]) that if a Banach space X is separable and admits a non-constant function of bounded support of some smoothness class S (formed always by continuous functions —for the definition see [2], [12]), then X admits smooth partitions of unity (subordinated to any open cover and locally finite) of the same class S. For example, any separable Banach space X admits Gâteaux smooth partitions of unity (use [3]) and admits continuously Fréchet smooth partitions of unity iff X^* is separable (use [6], [7], [11]). The word "continuously" in the last sentence can be omitted due to the result of E. Leach and J. Whitfield ([8]).

Furthermore, it was proved by H. Toruńczyk ([12]) that $c_0(\Gamma)$ admits Fréchet C^∞ partitions of unity for an arbitrary set Γ and that smooth partitions of unity on a general reflexive Banach space X can be constructed if there are sufficiently nice norms on X. For example, he proved that any reflexive Banach space admits continuously Fréchet differentiable partitions of unity (by use of Troyanski's renorming result and Asplund averaging technique [13]) and that any Hilbert space has Fréchet C^∞ smooth partitions of unity.

Using [12], it was shown in [5] that weakly compactly generated Banach space admits continuously Fréchet smooth partitions of unity iff it admits a non-constant continuously Fréchet smooth function with bounded support. On the other hand, it was proved by D. Amir and J. Lindenstrauss ([1]) that any weakly compactly generated Banach space admits an equivalent Gâteaux smooth norm.

Here we show that the existence of Gâteaux smooth partitions of unity on weakly compactly generated Banach space can be derived from the results of H. Toruńczyk. In the proof we go through Gâteaux smooth approximations of certain Troyanski's functions G_n on X ([13]) instead of the norms.

We would like to remark that it was proved by V. Meškov ([10]) that on the James space J (with J^{**} separable) ([4]), there is no continuously twice Fréchet differentiable function with bounded nonempty support.

Smooth partitions of unity have many good applications, e.g. the smooth approximations of continuous functions (see e.g. [12]).

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2. Notations and definitions. In the following, $\boldsymbol{R}$ will denote the reals, $\boldsymbol{N}$ the positive integers, $\operatorname{sp} A$, $\overline{\operatorname{sp}}\, A$ the linear, the closed linear hull of A, respectively. Banach space X is *weakly compactly generated* if there is a weakly compact set $K \subset X$ with $\overline{\operatorname{sp}}\, K = X$. A real valued function f on a Banach space X is said to be *Gâteaux smooth* if

$$\lim_{t\to 0} \big(f(x+th)-f(x)\big)t^{-1}$$

exists for any $x, h \in X$ and is for any x a linear continuous functional (in h) on X.

In this paper we consider only norm continuous Gâteaux smooth functions.

3. We start with

LEMMA 1. *Let X be a Banach space and assume there is a weakly compact absolutely convex set $K \subset X\times\boldsymbol{R} = Y$, $\overline{\operatorname{sp}}\, K = Y$, with the property that, if $k \in K$, $f_i \in Y^*$, $i = 1, 2$, are so that $f_i(k) = \sup f_i(K)$, then f_i are linearly dependent.*

Then any convex Lipschitz function f on X is a uniform (on X) limit of convex continuous functions which are Gâteaux smooth.

Proof. Assume the norm on Y to be $|(x, r)| = (|x|^2+r^2)^{1/2}$, $\operatorname{diam} K < 1$ and the Lipschitz constant of f to be 1. Consider the epigraph

$$\operatorname{epi} f = \{(x, r) \in X\times\boldsymbol{R}, f(x) \leqslant r\}$$

and form the sets

$$M_n = \operatorname{epi} f + (1/n)K, \quad n \in \boldsymbol{N}.$$

The sets M_n are convex. Since K is weakly compact, each M_n is closed and it is an epigraph of a convex function m_n. Moreover, the sets $\{(x, r);\ m_n(x) < r\}$ are open, and hence the functions m_n are continuous. If $x \in X$, then $\big(x, m_n(x)\big) \in \big(y, f(y)\big) + (1/n)K$ for some $y \in X$, so $|x-y| < 1/n$ and $|m_n(x)-f(y)| < 1/n$ and thus $|f(x)-m_n(x)| \leqslant |f(x)-f(y)| + |f(y)-m_n(x)| \leqslant 1/n+1/n$ by the Lipschitz property of f. Thus $\sup_{x\in X} |m_n(x)-f(x)| \leqslant 2/n$. Furthermore, any point of the boundary B of $\operatorname{epi} m_n$ is a point of Gâteaux smoothness, for $\operatorname{epi} m_n$ is closed, and whenever $z = \big(x, m_n(x)\big) \in B$, then $z = z_1 + k_1$ for some $z_1 \in \operatorname{epi} f$, $k_1 \in (1/n)K$; and thus whenever $f_i(z) = \sup f_i(\operatorname{epi} m_n)$, $f_i \in Y^*$, $i = 1, 2$, then $f_i(z_1+k_1) = \sup f_i\big(z_1+(1/n)K\big)$ and therefore, by the assumptions on K, f_i are linearly dependent.

LEMMA 2 (in Theorem 1 of [12]). *A Banach space X admits Gâteaux smooth partitions of unity iff there is a set Γ and a homeomorphic embedding u: $X\to c_0(\Gamma)$ with the coordinates $u_\alpha(x) = \big(u(x)\big)_\alpha$ Gâteaux smooth (and continuous) on X.*

LEMMA 3. *Let X be a weakly compactly generated Banach space and let $T = (f_\lambda)_{\lambda\in A}$: $X\to c_0(A)$ be a one-to-one continuous linear operator and $\|\cdot\|$ an equivalent norm on X which is Gâteaux smooth on $X\setminus\{0\}$ (see [1]). Then*

(i) *there exist seminorms G_n, $n \in N$, on X with $G_n(x) \leqslant (n^2+1)\|x\|$ for $x \in X$;*

(ii) *there exist a set Γ and a system of seminorms t_γ, $\gamma \in \Gamma$, on X such that $t_\gamma(x) \leqslant \|x\|$ for $x \in X$, $\gamma \in \Gamma$, $\big(t_\gamma(x)\big) \in c_0(\Gamma)$ for $x \in X$ and each t_γ^2 is Gâteaux smooth on X, and with the property:*

(*) *if $x_k \in X$ $(k = 0, 1, \ldots)$, $\lim_k \|x_k\| = \|x_0\|$, $\lim_k G_n(x_k) = G_n(x_0)$ for all $n \in N$, $\lim_k t_\gamma(x_k) = t_\gamma(x_0)$ for all $\gamma \in \Gamma$, and $\lim_k T(x_k) = T(x_k)$, then*

$$\lim_k \|x_k - x_0\| = 0.$$

Proof. The existence of an operator T and a Gâteaux smooth norm $\|\cdot\|$ on X follows from [1] (Main Theorem and Theorem 3). Furthermore, the proof follows from those of Proposition 1 and Theorem 1 of [13].

THEOREM. *Every weakly compactly generated Banach space X admits Gâteaux smooth partitions of unity.*

Proof. First we show that the assumption of Lemma 1 is satisfied. To this end, consider $Y = X\times\boldsymbol{R}$ and let T: $Y^*\to c_0(A)$ be a linear one-to-one bounded w*-w continuous map constructed in [1] (Proposition 2). Let $|\cdot|$ be a strictly convex equivalent norm for $c_0(A)$ (see [3]) and let K_1 be the closed unit ball in the dual norm $|\cdot|^*$ of the space $c_0^*(A)$. Now, following [9] (the proof of Theorem 3.3), consider the dual operator to T. Let $K = T^*K_1 \subset Y$ (where Y is identified with its canonical image in Y^{**}). The set K has the properties required in the assumption of Lemma 1. In fact, if $y = T^*x \in T^*K_1$, $f_i \in Y^*$ with $f_i(y) = \sup f_i(TK_1)$ for $i = 1, 2$, then $Tf_i(x) = \sup Tf_i(K_1)$ and $Tf_i \in c_0(A)$. By the strict convexity of the norm $|\cdot|$ on $c_0(A)$, the functionals Tf_i are linearly dependent. Since T is one-to-one, f_i are linearly dependent.

Now let $\|\cdot\|$, G_n's, t_γ's and f_λ's be those of Lemma 3. Let $G(n,i)$ be Gâteaux smooth continuous convex functions on X such that $|G(n,i)(x)-G_n(x)|<1/i$ for all $x\in X$, $(n,i)\in N\times N$ (see Lemma 1). Let $I=(N\times N)\cup\{0\}\cup\Gamma\cup\Lambda$ (a disjoint union). Define $u\colon X\to c_0(I)$ by the formula

$$u(x)=\begin{cases}\|x\|^2 & \text{for } \quad a=0,\\ 2^{-n-i}G(n,i)(x) & \text{for } \quad a=(n,i)\in N\times N,\\ t_a^2(x) & \text{for } \quad a\in\Gamma,\\ f_a(x) & \text{for } \quad a\in\Lambda.\end{cases}$$

Then it is easy to see that u is a continuous one-to-one map of X into $c_0(I)$ (use the Lipschitz constants of G_n). Furthermore, all the coordinates $u(x)_a$ are Gâteaux smooth continuous and whenever for some $x_k, x\in X$, $k=1,2,\ldots$, $\lim u(x_k)=u(x)$, then $\lim_k G(n,i)(x_k)=G(n,i)(x)$ and thus also $\lim_k G_n(x_k)=G_n(x)$. Further use Lemma 3 to see that then

$$\lim\|x_k-x\|=0,$$

and therefore u is a homeomorphism into $c_0(I)$. By Lemma 2, X has Gâteaux smooth partitions of unity.

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