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A conjecture of Ulam on the invariance of measure in Hilbert's cube*

by

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Abstract. We prove in this paper that the standard product measure in Hilbert's cube I^ω is invariant relative to various metrizations of I^ω.

1. Two sets X, Y in a metric space S are called *isometric* if there exists an isometry of X onto Y, i.e. a distance preserving map of X onto Y. A measure over a σ-algebra of subsets of S is called *invariant* if isometric measurable sets have the same measure.

Let I be the closed unit interval $[0, 1]$, I^ω the Hilbert cube and μ the standard probability measure in I^ω.

Given a sequence $a = (a_0, a_1, \ldots)$ of positive real numbers with $\sum a_i^2 < \infty$, we can introduce in I^ω the distance function

$$d_a(x, y) = \left(\sum a_i^2 (x_i - y_i)^2\right)^{1/2}.$$

S.M. Ulam asked whether μ is invariant relative to d_a. It is the purpose of this paper to prove some theorems related to this conjecture. The conjecture, however, remains open and we give only a reduction to a problem of finite-dimensional geometry (Theorem 5). First we apply a theorem proved in [2] to show that, if two *open* sets in I^ω are isometric, then their measures are equal (Theorem 2). But I was unable to extend this to closed sets, which would be enough to settle Ulam's conjecture. We prove, however, that μ is invariant with respect to some other metrisations of I^ω (Theorem 3). We prove also a theorem on the extension of invariant Borel measures from Borel subspaces of metric spaces to invariant Borel measures over the whole space (Theorem 6). Some of these results were announced in [3].

I am indebted to A. Iwanik and A. Ehrenfeucht for their criticism of a first draft of this paper.

2. First I shall prove a theorem which implies the conjecture under the additional assumption that X and Y are open in I^ω.

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We need a slight refinement of Theorem 1 of [2].

Let S be a compact metric space. For any set $X \subseteq S$ and any $t > 0$ we put

$$E(X, t) = \min\{\operatorname{card} K\colon K \text{ is a covering of } X \text{ with sets of diameters} < t\}.$$

A set $A \subseteq S$ is called *thin* if for every $\varepsilon > 0$ there exists a $\delta > 0$ and an open set $V \subseteq S$ such that $A \subseteq V$ and for every compact set $C \subseteq V$ and every positive $t < \delta$ we have

$$E(C, t) \leqslant \varepsilon E(S, t).$$

THEOREM 1. *If S is a compact metric space, then there exists a complete Borel measure ν on S such that $\nu(S) = 1$, if $A \subseteq S$ and A is thin then $\nu(A) = 0$, and for any sets $U, V \subseteq S$ both open in S, if U is isometric to V then $\nu(U) = \nu(V)$.*

This theorem, except for the part about thin sets is proved in [2], and it is obvious that the construction of the measure given there on p. 109 satisfies also this additional part. We could also refine the above theorem in the style of [2], removing the supposition that S is compact and substituting the conclusion that $\nu(S) = 1$ by $\nu(C_0) = 1$ for some C_0 compact and thick in S and considering thinness relative to C_0.

PROPOSITION. *If $A \subseteq S$ and for every positive integer n there exists a set V, including A, open in S and such that there are disjoint sets $V_1, \ldots, V_n$ in S, all isometric to V (the V_i need not be open in S), then A is thin.*

Proof. Given $\varepsilon > 0$ choose n such that $1/n < \varepsilon$. Let V be as in the assumption. Then for any compact set $C \subseteq V$ we have n disjoint compact sets $C_i \subseteq V_i$, all isometric to C. Put $\delta = \frac{1}{2}\min\{d(x, y)\colon x \in C_i,\ y \in C_j$ for $i \neq j\}$. Then, of course,

$$E(C, t) \leqslant \frac{1}{n} E(S, t)$$

for every $t < \delta$. ∎

LEMMA 1. *If $S = I^\omega$ and d is a translation invariant metrisation of S, i.e., $d(x, y) = d(x+z, y+z)$ whenever $x, y, x+z, y+z \in S$, where $+$ denotes vector addition, then the measure ν of Theorem 1 is unique and $\nu = \mu$.*

Proof. From the translation invariance of d we get that of ν over open sets. Consider a cover of I^n with m^n isometric n-cubes with non overlapping interiors. Let C_{mn} be the collection of cylinders in I^ω over the interiors of those cubes. We want to show that

$$\nu(K) = \mu(K) \quad \text{for all } K \in C_{mn}. \tag{0}$$

By translation invariance over open sets, we have $\nu(K_1) = \nu(K_2)$ for all $K_1, K_2 \in C_{mn}$. Hence, it is sufficient to establish that $\nu\big(\operatorname{Boundary}(\bigcup C_{mn})\big) = 0$. This follows from Theorem 1; indeed by Proposition, it is easy to check that $\operatorname{Boundary}(\bigcup C_{mn})$ is a finite union of thin sets.

Now, since the cylinders of $\bigcup_{m,n=1}^{\infty} C_{mn}$ generate the Borel σ-algebra of I^ω, hence, by well known facts, (0) implies $\nu = \mu$. ∎

THEOREM 2. *If $A, B \subseteq I^\omega$ are open sets for which there exists a translation invariant metrisation d of I^ω consistent with the product topology and an isometry f of A onto B with respect to d, then $\mu(X) = \mu\big(f(X)\big)$ for every μ-measurable set $X \subseteq A$.*

Proof. Theorem 2 follows from Theorem 1, Lemma 1 and the regularity of μ. ∎

Remarks and problems. 1. Theorem 1 yields a refinement of the translation-invariance of the Haar measure in compact metric groups with a left (right) invariant metric. See also [2] for the locally compact case.

2. The proof of Theorem 1 given in [2] uses the axiom of choice for uncountable families of sets. Can one prove Theorem 2 without using this axiom? (Perhaps the ideas of Cartan and Loomis [1], [4] could be helpful in this problem.)

3. In connection with Theorem 2 let us recall that *an open set in a compact metric space cannot be isometric to a proper subset or superset of itself* (see [0]). This fails in general for F_σ or G_δ sets (the set $\{e^{ni}\colon n = 0, 1, 2, \ldots\}$ in the unit circle and its complement are counter-examples).

4. Let $A \subseteq I^\omega$ be a closed set with $\mu(A) > 0$. Let f be an isometry of A into I^ω with respect to d_a. Must then f be extendible to an isometry of B onto B, where B is the space of all bounded sequences of reals with the metric d_a?

3. Now we consider the following metrisation of I^ω compatible with the product topology. Let $a = (a_0, a_1, \ldots)$ be a sequence of positive real numbers with $a_i \to 0$ and let

$$\varrho_a(x, y) = \max\{a_i|x_i - y_i|\colon i = 0, 1, 2, \ldots\}.$$

THEOREM 3. *If there exist constants $a < 1$ and N such that*

$$a_{n+1} \leqslant a a_n \quad \textit{for all } n \geqslant N \tag{1}$$

then μ is invariant relative to ϱ_a.

First we prove a theorem which synthesises the construction of Hausdorff measures for certain metric spaces. Let S be a complete metric space and C a non-empty compact subset of S. For every $t > 0$ we denote by $E(t)$ the minimum number of sets of diameters less than t necessary to cover C.

THEOREM 4. *Suppose that there exists a constant $\gamma > 0$ such that for every finite sequence $t_1, \ldots, t_n$ of positive numbers for which there exists a covering of C by n sets of diameters less than $t_1, \ldots, t_n$, respectively, we have*

$$\sum_{k=1}^{n} \frac{1}{E(t_k)} \geqslant \gamma. \tag{2}$$

Then there exists an invariant Borel measure ν over S such that $\nu(C) = 1$.

Proof in outline. We put $h(t) = 1/E(t)$. We define the Hausdorff outer h-measure χ^* for every $X \subseteq S$ by the standard formula

$$\chi^*(X) = \lim_{d\downarrow 0} \inf_{t} \sum_i h(t_i),$$

where $t = (t_1, t_2, \ldots)$ runs over all finite or infinite sequences of positive reals with $t_i \leqslant d$ for all i and for which there exist sets of diameters less than $t_1, t_2, \ldots$, respectively, covering X. It is easy to check that χ^* is a Carathédory outer measure invariant under isometries of sets (see [6]). Hence, (see [7]) all Borel sets are χ^*-measurable and the restriction χ of χ^* to Borel sets is an invariant Borel measure. By the definition of h it follows that $\chi(C) \leqslant 1$. Since C is compact the assumption (2) implies that $\chi(C) > 0$. We set $\nu(X) = \chi(X)/\chi(C)$ for every Borel set $X \subseteq S$ and Theorem 4 follows. ■

LEMMA 2. *If $C = I^\omega$ and the metric in C is ϱ_a with a satisfying* (1), *then the hypothesis of Theorem 4 is true.*

Proof. For any $t > 0$ we put

$$M = \max\{i\colon\ a_i \geqslant t\}.$$

Let a and N be as in (1) and

$$\gamma_0 = \exp\left(-a_N \sum_{i=0}^{N} \frac{1}{a_i} - \frac{1}{1-a}\right).$$

We assume that t is so small that $M > N$. Using a covering of I^ω with parallelepipeds we see that

$$E(t) \leqslant \prod_{i=0}^{M} \left(\frac{a_i}{t} + 1\right). \tag{3}$$

The following inequalities follow from (1).

$$\prod_{i=0}^{M} \left(\frac{a_i}{t} + 1\right) \Big/ \prod_{i=0}^{M} \frac{a_i}{t} = \prod_{i=0}^{M} \left(1 + \frac{t}{a_i}\right)$$

$$\leqslant \exp\left(\sum_{i=0}^{M} \frac{t}{a_i}\right) \leqslant \exp\left(\sum_{i=0}^{M} \frac{a_M}{a_i}\right)$$

$$= \exp\left(\sum_{i=0}^{N} \frac{a_M}{a_i} + \sum_{i=N+1}^{M-1} \prod_{j=i}^{M-1} \frac{a_{j+1}}{a_j} + 1\right)$$

$$\leqslant \exp\left(a_N \sum_{i=0}^{N} \frac{1}{a_i} + \sum_{i=N+1}^{M} a^{M-i}\right) \leqslant \gamma_0^{-1}.$$

Therefore, by (3),

$$E(t) \leqslant \gamma_0^{-1} \prod_{i=0}^{M} \frac{a_i}{t}. \tag{4}$$

If $X \subseteq I^\omega$ is a set of diameter $\leqslant t$ with respect to ϱ_a, then the projection of X into the ith axis of I^ω, where $i \leqslant M$, is included in a segment of length t/a_i. Hence, the outer measure

$$\mu^*(X) \leqslant \prod_{i=0}^{M} \frac{t}{a_i}.$$

Now we put

$$M(k) = \max\{i\colon\ a_i \geqslant t_k\}, \quad \text{for } k = 1, \ldots, n.$$

It follows that, if there exists a sequence of n sets covering I^ω of diameters less than $t_1, \ldots, t_n$, respectively, then

$$\sum_{k=1}^{n} \prod_{i=0}^{M(k)} \frac{t_k}{a_i} \geqslant 1.$$

Hence, by (4), if $M(k) > N$ for $k = 1, \ldots, n$, then

$$\sum_{k=1}^{n} \frac{1}{E(t_k)} \geqslant \gamma_0 \sum_{k=1}^{n} \prod_{i=0}^{M(k)} \frac{t_k}{a_i} \geqslant \gamma_0.$$

To get rid of the assumptions $M(k) > N$, we put

$$\gamma = \min\big(\gamma_0, 1/E(t)\big),$$

where t is the largest number such that $M > N$, and Lemma 2 follows. ■

Proof of Theorem 3. Since ϱ_a is translation-invariant Theorem 3 follows from Lemma 1, Theorem 4 and Lemma 2. ■

Remark 5. It is doubtful whether condition (1) is essential for the conclusion of Theorem 3 because, in a sense, there seem to be very few isometric pairs of sets in I^ω with respect to ϱ_a. But I was not able to establish the hypothesis of Theorem 4 except in this case. See [2] for related open problems.

4. Now we return to the metric d_a of Section 1, where a is a sequence of positive numbers with $\sum a_i^2 < \infty$. We put

$$r_m = \Big(\sum_{i=m}^{\infty} a_i^2\Big)^{1/2}$$

and define the parallelepiped

$$C_a^m = [0, a_0]\times[0, a_1]\times \ldots \times[0, a_{m-1}].$$

Let d_m be the ordinary Euclidean distance in C_a^m and λ^m the ordinary Lebesgue measure in C_a^m. For every set $P \subseteq C_a^m$ and every $t > 0$ we put

$$P^{(t)} = \{x \in C_a^m\colon\ d_m(x, P) < t\}, \tag{5}$$

i.e., $P^{(t)}$ is the open t-neighborhood of P. Let now $t(a, m)$ be the smallest real number such that for every closed set $P \subseteq C_a^m$ and every function $F\colon P \to C_a^m$ such that

$$|d_m(x, y) - d_m(F(x), F(y))| \leqslant r_m, \quad \text{for all } x, y \in P,$$

we have

$$\lambda^m\big((F(P))^{(t(a,m))}\big) \geqslant \lambda^m(P). \tag{6}$$

It is easy to check that such a smallest number exists.

The theorem which follows reduces Ulam's conjecture to the following one: If $\sum a_i^2 < \infty$, then

$$\lim_{m\to\infty} t(a, m) = 0. \tag{7}$$

THEOREM 5. *If a satisfies* (7), *then μ is invariant with respect to d_a.*

Proof. By the regularity of μ and the compactness of I^ω, it is enough to prove that closed isometric sets have equal measures. Thus let $X, Y \subseteq I^\omega$ be closed sets and $f\colon X\to Y$ be an isometry of X onto Y with respect to d_a. Let $p_m\colon I^\omega \to C_a^m$ be defined by

$$p_m(x) = (a_0x_0, a_1x_1, \ldots, a_{m-1}x_{m-1}) \quad \text{for } m = 1, 2, \ldots$$

For every Borel set $Z \subseteq C_a^m$ we put

$$\mu_m(Z) = \frac{\lambda^m(Z)}{a_0a_1\ldots a_{m-1}}.$$

We need several auxiliary facts. Since μ is a product measure,

$$\mu_m(p_m(Z))\downarrow \mu(Z) \quad \text{for every closed set } Z \subseteq I^\omega. \tag{8}$$

For any $Z \subseteq I^\omega$ and $t > 0$ we put

$$Z^{(t)} = \{x \in I^\omega\colon\ d_a(x, Z) < t\}.$$

Then, by (5), we get

$$p_m(Z^{(t)}) = (p_m(Z))^{(t)}. \tag{9}$$

Let us prove that

(10) *If $Z \subseteq I^\omega$ is a closed set and $t_m \downarrow 0$, then*

$$\mu_m(p_m(Z^{(t_m)})) \to \mu(Z).$$

By (8), for any positive integer n we can easily prove

$$\mu(Z) \leqslant \mu_m(p_m(Z)) \leqslant \mu_m(p_m(Z^{(t_n)}))\downarrow \mu(Z^{(t_n)}).$$

Since $\mu(Z^{(t_n)})\downarrow\mu(Z)$, we get (10).

We define a function $F\colon p_m(X)\to C_a^m$ by putting

$$F = p_m \circ f \circ q,$$

where $q\colon p_m(X)\to X$ is any function such that

$$p_m(q(u)) = u \quad \text{for all } u \in p_m(X).$$

Since $f(X) = Y$, we have

$$F(p_m(X)) \subseteq p_m(Y). \tag{11}$$

Let us prove that

$$|d_m(u, v) - d_m(F(u), F(v))| \leqslant r_m \quad \text{for all } u, v \in p_m(X). \tag{12}$$

It is clear that

$$d_m(u, v) \leqslant d_a(q(u), q(v)) \leqslant d_m(u, v) + r_m,$$

$$d_a(f(q(u)), f(q(v))) = d_a(q(u), q(v)),$$

since f is an isometry, and that

$$d_a(f(q(u)), f(q(v))) - r_m \leqslant d_m(F(u), F(v)) \leqslant d_a(f(q(u)), f(q(v))).$$

Hence (12) follows.

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Now we conclude the proof of Theorem 5. For any $\eta > 0$ there exists an M such that for all $m > M$ we have

$$\begin{aligned} \mu(Y)+\eta &\geqslant \mu_m\big(p_m(Y^{(t(a,m))})\big) && \text{[by (7) and (10)]}\\ &= \mu_m\big((p_m(Y))^{(t(a,m))}\big) && \text{[by (9)]}\\ &\geqslant \mu_m\big((F(p_m(X)))^{(t(a,m))}\big) && \text{[by (11)]}\\ &\geqslant \mu_m\big(p_m(X)\big) && \text{[by (6) and (12)]}\\ &\geqslant \mu(X). && \text{[by (8)]} \end{aligned}$$

We conclude that $\mu(Y) \geqslant \mu(X)$. And, by symmetry, $\mu(X) = \mu(Y)$. ■

5. The metrics d_a and ϱ_a extend in a natural way to the space B of all bounded sequences of reals. Can one extend μ to the σ-algebra of Borel subsets of B preserving its invariance? Theorem 4 and Lemma 2 enable this to be done for the metrics ϱ_a satisfying (1), but we can prove a more general theorem.

Let M be a metric space and M_0 a Borel subset of M. Let μ_0 be an invariant Borel (not necessarily finite) measure over M_0.

THEOREM 6. *μ_0 can be extended to an invariant Borel measure over M.*

Proof. For every Borel set $X \subseteq M$ we put

$$\mu(X) = \sup_{(A,f)} \sum_{i=1}^{\infty} \mu_0\big(f_i(A_i)\big),$$

where $(A,f) = \big((A_1, A_2, \ldots), (f_1, f_2, \ldots)\big)$ runs over all pairs such that A_i are disjoint Borel subsets of X, f_i is an isometry of A_i into M_0 and the $f_i(A_i)$ are Borel sets. We have to prove the following three things.

(i) μ is countably additive;

(ii) μ is invariant;

(iii) μ is an extension of μ_0.

To show (i) notice first from the definition of μ that, if $X_1, X_2, \ldots$ are disjoint Borel sets in M, then

$$\mu\Big(\bigcup_{n=1}^{\infty} X_n\Big) \geqslant \sum_{n=1}^{\infty} \mu(X_n). \tag{13}$$

To prove the reverse inequality choose any $\varepsilon > 0$ and (A,f) satisfying the above conditions with $X = \bigcup_{n=1}^{\infty} X_n$ such that

$$\mu\Big(\bigcup_{n=1}^{\infty} X_n\Big) \leqslant \sum_{i=1}^{\infty} \mu_0\big(f_i(A_i)\big) + \varepsilon. \tag{14}$$

Since the X_n are Borel, the sets $A_{in} = A_i \cap X_n$ are Borel relative to A_i. Since f_i is an isometry, $f_i(A_{in})$ are Borel relative to $f_i(A_i)$. Since the $f_i(A_i)$ are Borel, the $f_i(A_{in})$ are Borel in M_0. Hence

$$\begin{aligned} \sum_{i=1}^{\infty} \mu_0\big(f_i(A_i)\big) &= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu_0\big(f_i(A_{in})\big) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \mu_0\big(f_i(A_{in})\big) \leqslant \sum_{n=1}^{\infty} \mu(X_n). \end{aligned} \tag{15}$$

By (13), (14) and (15), we get (i).

(ii) follows since isometry of Borel sets preserves Borel subsets.

(iii) follows from the invariance of μ_0. ■

THEOREM 7. *If M is a metric space which has an uncountable compact subset, then there exists an invariant Borel measure over M which is finite and positive on some compact sets and vanishes on points.*

Proof. If M is compact, then this theorem can be proved by an obvious modification of an argument of Oxtoby [5], p. 220 ff. In view of Theorem 6 this implies the general case. ■

6. We take this opportunity to recall the following results which permit to solve a question raised in [2].

Let C be the axiom of choice for countable families of sets. R. M. Solovay [8] proved among other things that C *is insufficient to prove the existence of sets of real numbers without the property of Baire.* He also announced in [8], p. 3, a theorem (proved independently by David Pincus, *The strength of the Hahn–Banach theorem*, Victoria Symposium on Nonstandard Analysis, Springer 1947, pp. 203–248) which can be stated as follows.

THEOREM 8. C *implies that if there exists a countably complete Boolean algebra of sets B and a finite and finitely additive measure on B which is not countably additive, then there exists a set of reals without the property of Baire.*

By those results Theorem 3 of [2] cannot be proved without using the axiom of choice for uncountable families of sets.

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On a class of Banach spaces

by

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Abstract. A Banach space E with E^{**}/E separable is the direct sum of a reflexive subspace and a separable one.

The Banach spaces we use here are defined over the field K of the real or complex numbers. If $\langle E, F\rangle$ is a dual pair of vector spaces, with the bilinear form $\langle x, y\rangle$, $x\in E$, $y\in F$, we represent by $\sigma(E, F)$ the locally convex topology on E such that the origin of E has as neighbourhood sub-basis $\{U_y: y\in F\}$, being $U_y = \{x\in E: |\langle x, y\rangle| \leqslant 1\}$. If E is a Banach space, we consider it as a subspace of its second conjugate E^{**} by means of the canonical injection. If F is a subspace of E, we denote by $F^{\perp}$ the subspace of E^* orthogonal to F and by $F^{\perp\perp}$ the subspace of E^{**} orthogonal to $F^{\perp}$. We say that E is *weakly compactly generated space*, or WCG space, if there is in E a weakly compact fundamental set.

THEOREM. *Let E be a Banach space such that E^{**}/E is separable. Then E is a direct sum of a reflexive subspace and a separable subspace (clearly, every separable subspace of E has its second dual separable).*

We shall need the following lemmas:

LEMMA 1. *Let F be a closed subspace of a Banach space X. Assume that every $x^{**}\in X^{**}$ that belongs to the $\sigma(X^{**}, X^*)$-closure of a countable bounded subset of X is of the form $x^{**} = x+f^{\perp\perp}$ with $x\in X$ and $f^{\perp\perp}\in F^{\perp\perp}$. Then the space X/F is reflexive.*

Proof. Let $(\bar{x}_n)$ be a bounded sequence in X/F. If φ is the canonical mapping of X onto X/F, let (x_n) be a bounded sequence in X such that $\varphi(x_n) = \bar{x}_n$, $n = 1, 2, \ldots$ If x_0^{**} is an accumulation point of (x_n) in $X^{**}[\sigma(X^{**}, X^*)]$, we set

$$x_0^{**} = x_0+f_0^{\perp\perp}, \quad x_0\in X, \quad f_0^{\perp\perp}\in F^{\perp\perp}.$$

If u is an element of $F^{\perp}$, the sequence of elements of K, $(u(x_n))$, has an accumulation point $x_0^{**}(u)$. On the other hand,

$$u(x_n) = u(\bar{x}_n), \quad x_0^{**}(u) = (x_0+f^{\perp\perp})(u) = u(x_0),$$