Relations between certain problems of Banach

by

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In his treatise on linear transformations, S. BANACH[2]) lists among others the following unsolved problems (Cf. (B), pp. 144—145):

(a) To every closed linear manifold $\mathfrak{M}$ of L_p, $1 < p \neq 2$, does there exist, a closed linear manifold $\mathfrak{N}$ such that every $f \in L_p$ may be represented in a unique way as $g + h$, $g \in \mathfrak{M}$, $h \in \mathfrak{N}$?

(b) To every closed linear manifold $\mathfrak{M}$ of l_p, $1 < p \neq 2$, does there exist, a closed linear manifold $\mathfrak{N}$, such that every $f \in l_p$ may be represented in a unique way as $g + h$, $g \in \mathfrak{M}$, $h \in \mathfrak{N}$?

(c) Is every infinite–dimensional closed linear manifold $\mathfrak{M}$ of l_p isomorphic with l_p?

We will show in this note, that the answer to (a) is the same as the answer to (b) and indeed depends on the limit of properties of $l_{p,n}$ as n approaches infinity. Also, that if the answer to (b) is yes, the answer to (c) is also yes.

1. Let $\varDelta$ denote a separable space with a p–norm, i. e. $\varDelta$ is either L_p or l_p, or the set $l_{p,n}$ of ordered n–tuples of real numbers $\{a_1, \ldots, a_n\}$ with the norm $\|\{a_1, \ldots, a_n\}\| = (|a_1|^p + \ldots + |a_n|^p)^{1/p}$. We also let $l_p = l_{p,\infty}$.

Let $\mathfrak{M}$ be a closed linear manifold in $\varDelta$, i. e. $\mathfrak{M}$ is a closed set such that $f \in \mathfrak{M}$ and $g \in \mathfrak{M}$ imply $af + bg \in \mathfrak{M}$, where a and b are any two real numbers.

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[1]) National Research Fellow. Some of these results were obtained while the author was stationed at Brown University, Providence, R. I.

[2]) S. Banach, Théorie des opérations linéaires, Warsaw (1932). We shall refer to this book as (B).

Let R denote the set of real numbers $0 \leqslant a \leqslant \infty$, and let $r(a, b) = a/(1+a) - b/(1+b)$ $(\infty/(1+\infty) = 1)$. It is easy to see that R with the metric $|r(a, b)|$ is a metric HAUSDORFF–space and is in a one-to-one correspondence with the closed interval $\langle 0, 1 \rangle$.

If $\mathfrak{M}$ is a closed linear manifold in $\varDelta$, a limited transformation E such that $E\varDelta = \mathfrak{M}$, $E^2 = E$, is said to project $\varDelta$ on $\mathfrak{M}$.

If E is a limited transformation, we denote by $|E|$ the bound of E.

Lemma 1.1. *Let $\mathfrak{M}$ be a closed linear manifold in $\varDelta$. The existence of a closed linear manifold $\mathfrak{N}$ such that every $f \in \varDelta$ may be represented uniquely as $h+g$, $h \in \mathfrak{M}$, $g \in \mathfrak{N}$, is equivalent to the existence of a projection E of $\varDelta$ on $\mathfrak{M}$.*

Proof. Suppose $\mathfrak{N}$ exists. Let E be the transformation, which is such that $Ef = h$. Owing to the properties of $\mathfrak{N}$, this is single-valued, linear and defined everywhere. Now let f_i be a sequence, which approaches f and such that if $f_i = h_i + g_i$, $h_i \in \mathfrak{M}$, $g_i \in \mathfrak{N}$, the h_i form a convergent sequence with the limit h'. Then h' is $\in \mathfrak{M}$, and the sequence $g_i = f_i - h_i$ also converges to a $g' \in \mathfrak{N}$. By continuity we have $f = h' + g'$. The uniqueness of the resolution of f now implies that $Ef = h'$, or that E is closed. Theorem 7 of (B) Chap. III, p. 41, now implies that E is bounded. Since the range of E is included in $\mathfrak{M}$ and for every $f \in \mathfrak{M}$, $Ef = f$, we see that the range of E is $\mathfrak{M}$, and $E^2 = E$ or E is a projection of $\varDelta$ on $\mathfrak{M}$.

Now suppose E exists. Let $\mathfrak{N}$ be the set of g's in $\varDelta$ for which $Eg = 0$. Since E is bounded and linear, $\mathfrak{N}$ is a closed linear manifold. Now if f is $\in \varDelta$, $f = Ef + (1-E)f$, where Ef is $\in \mathfrak{M}$, and $(1-E)f$ is $\in \mathfrak{N}$, since $E(1-E)f = (E-E^2)f = 0$. Now if h is $\in \mathfrak{M}$, $h = Ef$ for some $f \in \varDelta$, and hence $Eh = E^2 f = Ef = h$. Now if h is $\in \mathfrak{M}.\mathfrak{N}$, $0 = Eh = h$, or $\mathfrak{M}.\mathfrak{N} \subset \{0\}$ or $\mathfrak{M}.\mathfrak{N} = \{0\}$. Now let f again be $\in \varDelta$, $f = h + g = h' + g'$, $h, h' \in \mathfrak{M}$, $g, g' \in \mathfrak{N}$. Then $h - h' = g' - g$. Now $h - h'$ is $\in \mathfrak{M}$, $g' - g$ is $\in \mathfrak{N}$, hence $h - h' = g' - g$ is $\in \mathfrak{M}.\mathfrak{N} = \{0\}$, or $h - h' = g' - g = 0$. This shows that $f \in \varDelta$ can only be expressed in one way as $h+g$, $h \in \mathfrak{M}$, $g \in \mathfrak{N}$.

We prefer to consider problems (a) and (b) in the following equivalent form:

(a) To every closed linear manifold $\mathfrak{M}$ of L_p, $1 < p \neq 2$, is there a projection of L_p on $\mathfrak{M}$?

(b) To every closed linear manifold $\mathfrak{M}$ of l_p, $1 < p \neq 2$, is there a projection of l_p on $\mathfrak{M}$?

Let $\varDelta_1, \ldots, \varDelta_n$, $n = 1, 2, \ldots, \infty$ (if $n = \infty$, omit $\varDelta_n$), be a set of spaces. Let $(\varDelta_1 \times \varDelta_2 \times \ldots \varDelta_n)_p$ (if $n = \infty$, omit $\varDelta_n$) denote the space of ordered sets of elements $\{f_1, f_2, \ldots, f_n\}$ (if $n = \infty$, omit f_n), $f_\alpha \in \varDelta_\alpha$, such that $\sum_{\alpha=1}^{n} \|f_\alpha\|^p < \infty$, with a norm defined by the equation

$$\|\{f_1, f_2, \ldots, f_n\}\| = \Big(\sum_{\alpha=1}^{\infty} \|f_\alpha\|^p\Big)^{1/p}.$$

$\varDelta_\alpha \simeq \varDelta_\beta$ is to mean that there exists a one-to-one isometric mapping of $\varDelta_\alpha$ on $\varDelta_\beta$.

Lemma 1.2. (a) $(l_{p,m_1} \times l_{p,m_2} \times \ldots \times l_{p,m_n})_p \simeq l_{p,m}$ *for* $n = 1, 2, \ldots, \infty$, $m_\alpha = 1, 2, \ldots, \infty$, if $\sum_{\alpha=1}^{n}{}' m_\alpha = m$.

(b) $(\varDelta_1 \times \varDelta_2 \times \ldots \varDelta_n)_p \simeq L_p$ *for* $n = 1, 2, \ldots, \infty$ *if* $\varDelta_\alpha = L_p$, *for each* α.

Proof. (a) is obvious. To show (b), it is only necessary to divide the interval $(0, 1)$ into n intervals (if $n = \infty$, $\aleph_0$ intervals) and in each interval set up a one-to-one isometric mapping in the obvious manner between the functions, defined on this interval, measurable and with the p'th power summable and L_p. When this has been done, a function on the interval $(0, 1)$ corresponds to an element of $(\varDelta_1 \times \varDelta_2 \times \ldots \varDelta_n)_p$, and it is easily seen that this sets up a one-to-one isometric correspondence of L_p and $(\varDelta_1 \times \varDelta_2 \times \ldots \varDelta_n)_p$.

2. Let $\mathfrak{M}$ be a closed linear manifold in $\varDelta$. We define a function $C(\mathfrak{M})$, which takes on values in R as follows. If there exists no projection of $\varDelta$ on $\mathfrak{M}$, then $C(\mathfrak{M}) = \infty$. Otherwise $C(\mathfrak{M}) = $ gr. l. b. $(|E|, E\varDelta = \mathfrak{M}, E^2 = E)$. Similarly we define the function $\bar{C}(\varDelta)$ as l. u. b. $(C(\mathfrak{M}), \mathfrak{M} \subset \varDelta)$.

Lemma 2.1. *Let $\varDelta_1$ and $\varDelta_2$ be such that $\varDelta_1$ is isometrically isomorphic with a closed linear manifold $\mathfrak{M}$ of $\varDelta_2$. Let $\mathfrak{M}$ be such that there exists a projection E of $\varDelta$ on $\mathfrak{M}$ with $|E| = 1$, $\mathfrak{N}$ the set of f's $\in \varDelta_2$, for which $Ef = 0$. Let $\mathfrak{P}$ be any closed linear manifold of $\varDelta_2$ such that if $f \in \mathfrak{P}$, $f = h + g$, $h \in \mathfrak{P}.\mathfrak{M}$,*

$g \in \mathfrak{P}.\mathfrak{N}$. *Let* $\mathfrak{P}_1$ *in* Δ_1 *be the manifold which corresponds to* $\mathfrak{P}.\mathfrak{M}$. *Then* $C(\mathfrak{P}_1) \leqslant C(\mathfrak{P})$.

Proof. If $C(\mathfrak{P}) = \infty$, our statement is true. Suppose $C(\mathfrak{P}) < \infty$. Let F be any projection of Δ_2 on $\mathfrak{P}$. Then EF is a projection on $\mathfrak{P}.\mathfrak{M}$. For if f_1 is $\in \Delta_2$, $f = Ff_1 = h + g$, $h \in \mathfrak{P}.\mathfrak{M}$, $g \in \mathfrak{P}.\mathfrak{N}$, and $EFf = h$ or the range of EF is included in $\mathfrak{P}.\mathfrak{M}$. Also for every $h \in \mathfrak{P}.\mathfrak{M}$, we have $EFh = Eh = h$. This with our previous statement shows that $(EF)^2 = EF$ and that the range of EF is exactly $\mathfrak{P}.\mathfrak{M}$.

Let $(EF)'$ be EF considered only on $\mathfrak{M}$. Obviously $(EF)'$ projects $\mathfrak{M}$ on $\mathfrak{P}.\mathfrak{M}$. Let G be the corresponding transformation on Δ_1. Then $C(\mathfrak{P}_1) \leqslant |G|' = |(EF)'| \leqslant |EF| \leqslant |E| \cdot |F| = |F|$ or $C(\mathfrak{P}_1) \leqslant |F|$. Since F was any projection on $\mathfrak{P}$, we have $C(\mathfrak{P}_1) \leqslant C(\mathfrak{P})$.

Lemma 2.2. *If* Δ_1 *and* Δ_2 *are as in Lemma* 2.1, $\overline{C}(\Delta_1) \leqslant \overline{C}(\Delta_2)$. *In particular if* $\Delta_2 = (\Delta_0 \times \Delta_1)_p$, $\overline{C}(\Delta_1) \leqslant \overline{C}(\Delta_2)$.

Proof. Let $\mathfrak{P}_1$ be any closed linear manifold of Δ_1, $\mathfrak{P}$ the corresponding set of elements in $\mathfrak{M}$. $\mathfrak{P}$ is a closed linear manifold satisfying the conditions given in Lemma 2.1, since $\mathfrak{P}.\mathfrak{M} = \mathfrak{P}$, $\mathfrak{P}.\mathfrak{N} = \{0\}$. Lemma 2.1 now implies that $C(\mathfrak{P}_1) \leqslant C(\mathfrak{P}) \leqslant \overline{C}(\Delta_2)$. But $\mathfrak{P}_1$ was any closed linear manifold in Δ_1, hence $\overline{C}(\Delta_1) \leqslant \overline{C}(\Delta_2)$.

To show the second statement, we take $\mathfrak{M} \subset (\Delta_0 \times \Delta_1)_p$, as the set of elements $\{0, f\}$ of $(\Delta_0 \times \Delta_1)_p$, E as the transformation of $(\Delta_0 \times \Delta_1)_p$, such that $E\{f, g\} = \{0, g\}$. One readily sees that $\mathfrak{M}$ is isometrically isomorphic with Δ_1, and that E projects Δ_2 on $\mathfrak{M}$ and $|E'| = 1$. We may now apply the first part of this lemma to obtain the desired result.

Lemma 2.3. *If* $\Delta \simeq (\Delta_1 \times \Delta_2 \times \ldots \Delta_n)_p$ *and* k *is the least upper bound of those numbers* k_1 *for which there are an infinite number of the* α*'s with* $\overline{C}(\Delta_\alpha) > k_1$, *then there exists a manifold* $\mathfrak{P} \subset \Delta$, *such that* $C(\mathfrak{P}) \geqslant k$. (Note k may be ∞).

Proof. It follows from the definition of k, that if ε is > 0, then there exists an infinite number of the α's for which $r(\overline{C}(\Delta_\alpha), k) \geqslant -\varepsilon$. Hence if $\{\varepsilon\}$ is a sequence of numbers > 0, and such that $\varepsilon_i \to 0$, then we can find a sequence of integers $\{\alpha_i\}$ such that $\alpha_i < \alpha_{i+1}$, for which $r(\overline{C}(\Delta_{\alpha_i}), k) \geqslant -\varepsilon_i/2$.

Now since $r(\overline{C}(\Delta_{\alpha_i}), k) \geqslant -\varepsilon_i/2$, we can find a $\mathfrak{P}$, in Δ_{α_i}, such that $r(C(\mathfrak{P}_{\alpha_i}), k) > -\varepsilon_i$. Let $\mathfrak{P}$ be the closed linear manifold consisting of those elements $\{f_1, f_2, f_3, \ldots\} \in \Delta$, such that $f_\beta = 0$, if β is not $\in \{\alpha_i\}$ and $f_{\alpha_i} \in \mathfrak{P}_{\alpha_i}$. As we saw in the proof of Lemma 2.2, Δ_{α_i} and Δ are as Δ_1 and Δ_2 in Lemma 2.1 and it is easily seen that $\mathfrak{P}$ satisfies the conditions given im Lemma 2.1 also. Thus Lemma 2.1 now implies that $C(\mathfrak{P})$ is $\geqslant C(\mathfrak{P}_{\alpha_i})$ hence $r(C(\mathfrak{P}), k) \geqslant -\varepsilon_i$ for every i. This implies that $r(C(\mathfrak{P}), k) \geqslant 0$, $C(\mathfrak{P}) \geqslant k$.

3. Lemma 3.1. $\overline{C}(L_p) \geqslant \overline{C}(l_{p,\infty})$.

Proof. In (*B*) Theorem 9, Chap. XII, p. 206, it is shown that the manifold $\mathfrak{M} \subset L_p$, determined by the functions y_i is isometrically isomorphic with l_p, when

$$y_i(t) = 2^{i/p} \text{ for } 1/2^i \leqslant t \leqslant 1/2^{i-1}, \quad y_i(t) = 0 \text{ otherwise.}$$

Now if $z(t)$ is $\in L_p$, let

$$Ez(t) = \sum_{i=1}^{\infty} \int_0^1 z(s) y_i^{p-1}(s)\, ds\, y_i(t).$$

Then by a direct calculation one can verify that $|E| = 1$ and that if z is $\in \mathfrak{M}$ (i. e. $z = \sum_{i=1}^{\infty} \alpha_i y_i$, $\sum_{i=1}^{\infty} |\alpha_i|^p < \infty$) $Ez = z$. Hence E projects L_p on $\mathfrak{M}$ and we may apply Lemma 2.2 so that it yields $\overline{C}(L_p) \geqslant \overline{C}(l_p)$.

Lemma 3.2. *There exists a linear manifold* $\mathfrak{M} \subset L_p$, *such that* $C(\mathfrak{M}) = \overline{C}(L_p)$.

Proof. This follows from Lemma 1.2 (b) (with $n = \infty$) and Lemma 2.3, for k is in this case $\overline{C}(L_p)$.

Lemma 3.3. *There exists a linear manifold* $\mathfrak{M} \subset l_{p,\infty}$, *such that* $C(\mathfrak{M}) = \overline{C}(l_{p,\infty})$.

Proof. In Lemma 1.2 (a) let $n_\alpha = \infty$ for every α. Then apply Lemma 2.3.

Lemma 3.4. $\overline{C}(l_{p,n}) \geqslant \overline{C}(l_{p,m})$ *if* $n \geqslant m$.

Proof. This follows from Lemma 1.2 and Lemma 2.2.

Theorem I. *$C(\mathfrak{M})$ and $\bar{C}(\Delta)$ are to be as in § 2. There exists an $\mathfrak{M}$ in L_p, and an $\mathfrak{N}$ in $l_{p,\infty}$, such that $C(\mathfrak{M})=\bar{C}(L_p)$, and $C(\mathfrak{N})=\bar{C}(l_{p,\infty})$. Furthermore*

$$1=\bar{C}(l_{p,1})\leqslant\bar{C}(l_{p,2})\leqslant\ldots\leqslant C(l_{p,\infty})\leqslant\bar{C}(L_p).$$

The Lemmas of this section imply this Theorem.

4. Lemma 4.1. *Let $f_1,f_2,\ldots,f_n$, be n linearly independent elements of L_p, $n<\infty$. There exists a constant C depending only on the set $f_1,f_2,\ldots,f_n$, such that if*

$$\|\sum_{i=1}^{\infty}\alpha_i f_i\|=1, \qquad (\alpha)$$

then $|\alpha_i|\leqslant C$.

Proof. Since the f_i's are linearly independent, to each i there exists a linear functional F_i with a norm $k_i>0$, such that $F_i(f_i)=1$, $F_i(f_j)=0$ for $i\neq j$ (Cf. (*B*) Chap. IV, lemma p. 57). Then for any set of numbers $\beta_1,\ldots,\beta_n$ such that $\beta_i=0$, we have

$$1/k_i=F_i(f_i+\sum_{j=1}^{n}\beta_j f_j)/k_i\leqslant\|f_i+\sum_{j=1}^{n}\beta_j f_j\|.$$

Hence $\|yf_i+\sum_{j=1}^{n}y\beta_j f_j\|\geqslant|y|/k_i$.

Thus in (α), $|\alpha_i|/k_i\leqslant 1$, or $|\alpha_i|\leqslant k_i$. To complete the proof let $C=\max_i k_i$.

Lemma 4.2. *Let $f_1,\ldots,f_n$ be n linearly independent elements of L_p, $n<\infty$. Then to every $\varepsilon>0$, there exists an integer m, and n functions $h_1,\ldots,h_n$, which are constant in each interval $(p/m,\ (p+1)/m)$, $p=0,\ldots,m-1$ and such that if (α) holds* (Lemma 4.1), *then*

$$\|\sum_{i=1}^{n}\alpha_i(f_i-h_i)\|\leqslant\varepsilon \qquad (\beta).$$

Proof. The hypothesis of Lemma 4.1 holds and we may apply it to obtain the C. Now to each f_i we can find a continuous function g_i, such that $\|f_i-g_i\|\leqslant\varepsilon/2Cn$. The g_i's are continuous and hence they are uniformly continuous. Thus there exists a δ, such that if $|x_1-x_2|<\delta$, then

$$|g_i(x_1)-g_i(x_2)|\leqslant\varepsilon/2Cn,\quad i=1,2,\ldots,n \qquad (\gamma).$$

Now let m be any integer such that $1/m<\delta$. Let h_i be the step-function such that if $p/m\leqslant x<(p+1)/m$, $p=0,1,\ldots,m-1$, $h_i(x)=g_i(p/m)$, $h_i(1)=g_i(1)$. It follows from (γ) that $\|h_i-g_i\|\leqslant\varepsilon/2Cn$ and since $\|f_i-g_i\|\leqslant\varepsilon/2Cn$, we obtain $\|f_i-h_i\|\leqslant\varepsilon/Cn$.

Now if (α) holds,

$$\|\sum_{i=1}^{n}\alpha_i(f_i-h_i)\|\leqslant\sum_{i=1}^{n}|\alpha_i|\cdot\|f_i-h_i\|\leqslant\sum_{i=1}^{n}|\alpha_i|\varepsilon/Cn\leqslant\sum_{i=1}^{n}\varepsilon/n=\varepsilon.$$

Lemma 4.3. *Let $f_1,\ldots,f_n$, $h_1,\ldots,h_n$, ε, be as in* Lemma 4.2. *If $\varepsilon<1$, the h_i are linearly independent.*

Proof. Let $\beta_1,\ldots,\beta_n$ be a set of numbers such that $\beta_i=0$ for a fixed i, then (Cf. proof of Lemma 4.1)

$$\|f_i+\sum_{j=1}^{n}\beta_j f_j\|\geqslant k_i$$

for a $k_i>0$ and independent of the β_j's, or

$$\|(1/k_i)f_i+\sum_{j=1}^{n}(\beta_j/k_i)f_j\|=t_i\geqslant 1,$$

where t_i now depends on the β_j's, so we obtain

$$\|(1/k_it_i)f_i+\sum_{j=1}^{n}(\beta_j/k_it_i)f_j\|=1.$$

Hence by (β) (Lemma 4.2)

$$\|(1/k_it_i)h_i+\sum_{i=1}^{n}(\beta_j/k_it_i)h_j\|\geqslant 1-\varepsilon$$

or

$$\|h_i+\sum_{j=1}^{n}\beta_j h_j\|\geqslant k_it_i(1-\varepsilon)\geqslant k_i(1-\varepsilon).$$

Hence if $\varepsilon<1$, the h_i are linearly independent.

5. Lemma 5.1. *Let $\mathfrak{M}$ and $\mathfrak{M}_0$ be two n-dimensional manifolds in Δ, $1\leqslant n<\infty$. Let us suppose that there exists n linearly independent elements $f_1,\ldots,f_n$, $f_i\in\mathfrak{M}$, and n linearly independent elements $g_1,\ldots,g_n$, $g_i\in\mathfrak{M}_0$, such that if*

$$\|\sum_{i=1}^{n}\alpha_i f_i\|=1,$$

then $\|\sum_{i=1}^{n}\alpha_i(f_i-g_i)\|\leqslant\varepsilon$, with $(\varepsilon/(1-\varepsilon))C(\mathfrak{M}_0)<1$, $\varepsilon<1$.

Then

$$C(\mathfrak{M}) \leqslant C(\mathfrak{M}_0)\,[(1+(\varepsilon/(1-\varepsilon))\,C(\mathfrak{M}_0))/(1-(\varepsilon/(1-\varepsilon))\,C(\mathfrak{M}_0))].$$

Proof. Under our hypotheses, if $\|\sum_{i=1}^{n}\alpha_i g_i\| \leqslant 1-\varepsilon$, then $\|\sum_{i=1}^{n}\alpha_i(f_i-g_i)\| \leqslant \varepsilon$. For let us suppose that $\|\sum_{i=1}^{n}\alpha_i g_i\| \leqslant 1-\varepsilon$, and $\|\sum_{i=1}^{n}\alpha_i(f_i-g_i)\| = k$. Then

$$\|\sum_{i=1}^{n}\alpha_i f_i\| \leqslant \|\sum_{i=1}^{n}\alpha_i g_i\| + \|\sum_{i=1}^{n}\alpha_i(f_i-g_i)\| \leqslant 1-\varepsilon+k$$

or $\|\sum_{i=1}^{n}(\alpha_i/(1-\varepsilon+k))f_i\| \leqslant 1$. Hence $\|\sum_{i=1}^{n}(\alpha_i/(1-\varepsilon+k))\,(f_i-g_i)\| \leqslant \varepsilon$ or

$$k = \|\sum_{i=1}^{n}\alpha_i(f_i-g_i)\| \leqslant \varepsilon(1-\varepsilon+k),$$

which implies that $k(1-\varepsilon) \leqslant \varepsilon(1-\varepsilon)$ and since $\varepsilon < 1$, this yields that $k \leqslant \varepsilon$.

Now if $\|\sum_{i=1}^{n}\alpha_i g_i\| = t \neq 0$, or $\|\sum_{i=1}^{n}((1-\varepsilon)/t)\,\alpha_i g_i\| \leqslant 1-\varepsilon$, then $\|\sum_{i=1}^{n}\alpha_i((1-\varepsilon)/t)\,(g_i-f_i)\| \leqslant \varepsilon$, or

$$\|\sum_{i=1}^{n}\alpha_i(g_i-f_i)\| \leqslant (\varepsilon/(1-\varepsilon))\,\|\sum_{i=1}^{n}\alpha_i g_i\| \qquad (\varepsilon).$$

If $\|\sum_{i=1}^{n}\alpha_i g_i\| = 0$, then $\sum_{i=1}^{n}\alpha_i g_i = 0$ and $\alpha_i = 0$ for every i and we see that (ε) holds in general.

Let η be any number > 0, and such that

$$(C(\mathfrak{M}_0)+\eta)\,(\varepsilon/(1-\varepsilon)) < 1.$$

We can find a projection E_0 of $\varDelta$ on $\mathfrak{M}$, such that $|E_0| \leqslant C(\mathfrak{M}_0) + \eta$. Hence if $f \in \varDelta$, then $f = g + h$, where $g = \sum_{i=1}^{n}\alpha_i g_i$ and $\|g\| \leqslant (C(\mathfrak{M}_0)+\eta)\|f\|$. Now we define a transformation T by the

equation $T(\sum_{i=1}^{n}\alpha_i g_i + h) = \sum_{i=1}^{n}\alpha_i f_i + h$. Then T is easily seen to be linear and to have domain $\varDelta$.

Furthemore by (ε)

$$\|Tf-f\| = \|\sum_{i=1}^{n}\alpha_i(g_i-f_i)\| \leqslant (\varepsilon/(1-\varepsilon))\,\|\sum_{i=1}^{n}\alpha_i g_i\|$$
$$\leqslant (\varepsilon/(1-\varepsilon))\,(C(\mathfrak{M}_0)+\eta)\,\|f\|.$$

Thus $\|Tf\| \leqslant \|Tf-f\| + \|f\| \leqslant (1+(\varepsilon/(1-\varepsilon))\,(C(\mathfrak{M}_0)+\eta)\,\|f\|$ and $\|Tf\| \geqslant \|f\| - \|f-Tf\| \geqslant (1-(\varepsilon/(1-\varepsilon))\,(C(\mathfrak{M}_0)+\eta)\|f\|$. Since $(\varepsilon/(1-\varepsilon))\,(C(\mathfrak{M}_0)+\eta) < 1$, this last equation implies that T^{-1} exists and $|T^{-1}| \leqslant 1/(1-(\varepsilon/(1-\varepsilon))\,(C(\mathfrak{M}_0)+\eta))$.

Now since E_0 is a projection on $\mathfrak{M}_0$, TE_0T^{-1} is a projection on $\mathfrak{M}$. For the domain of T^{-1} is $\varDelta$, since the range of T is $\varDelta$, as may be seen as follows. Let $f \in \varDelta$, and $k = (\varepsilon/(1+\varepsilon))\,(C(\mathfrak{M}_0)+\eta)$ then k is < 1. We first show that if $g = Tg'$, is in the range of T and $\|f-g\| \leqslant C$, then there exists an element g_1 in the range of T such that $\|f-g_1\| \leqslant kC$. For

$$\|f-T(g'-f+g)\| = \|f-Tg'-T(f-g)\| = \|f-g-T(f-g)\|$$
$$\leqslant k\|f-g\| \leqslant k\|C\|.$$

Since $\|f-Tf\| \leqslant k\|f\|$, applying the above process $n-1$ times yields that there exists a g_n in the range of T such that $\|f-g_n\| \leqslant k_n\|f\|$ for every n. Since k is < 1, this shows that the range of T is everywhere dense. Thus the domain of T^{-1} is everywhere dense. T is closed, hence T^{-1} is closed. Thus T^{-1} is closed bounded and has domain everywhere dense. Hence its domain is $\varDelta$. So we see that $TE_0T^{-1}\varDelta = TE_0\varDelta = T\mathfrak{M}_0 = \mathfrak{M}$ and $(TE_0T^{-1})^2 = TE_0T_0^{-1}TE_0T^{-1} = TE_0E_0T^{-1} = TE_0T^{-1}$. Thus TE_0T^{-1} is a projection on $\mathfrak{M}$.

So

$$C(\mathfrak{M}) \leqslant |TE_0T^{-1}| \leqslant (C(\mathfrak{M}_0)+\eta)\,\frac{1+(\varepsilon/(1-\varepsilon))\,(C(\mathfrak{M}_0)+\eta)}{1-(\varepsilon/(1-\varepsilon))\,(C(\mathfrak{M}_0)+\eta)}.$$

Since η may be taken arbitrarily small, we obtain

$$C(\mathfrak{M}_0) \leqslant C(\mathfrak{M}_0)\,\frac{1+(\varepsilon/(1-\varepsilon))\,C(\mathfrak{M}_0)}{1-(\varepsilon/(1-\varepsilon))\,C(\mathfrak{M}_0)}.$$

6. Let $\varDelta$ be m—dimensional and let $n \leqslant m$. We define $\overline{C}_n(\varDelta) =$ l. u. b. $(C(\mathfrak{M}),\ \mathfrak{M} \subset \varDelta,\ \mathfrak{M}\ n$—dimensional).

Lemma 6.1. $\bar{C}_n(L_p) \leqslant$ l. u. b. $(\bar{C}_n(l_{p,m});\ m < \infty)$, for $n < \infty$

Proof. Let l. u. b. $(\bar{C}_n(l_{p,m}),\ m < \infty) = k_n$. Now if $k_n = \infty$ the Lemma is obviously true. So we may suppose that k_n i $< \infty$. Now let ε be any number such that $0 < \varepsilon < 1$ anc $(\varepsilon/(1-\varepsilon))\, k_n < 1$. Let $\mathfrak{M}$ be any manifold of n dimensions in L_p Let $f_1, \ldots, f_n$ be n linearly independent elements of $\mathfrak{M}$. Nov since $\varepsilon < 1$, applying Lemmas 4.2 and 4.3 to $f_1, \ldots, f_n$ and $(\varepsilon$ yields the existence of an integer m and n linearly indepen- dent step—functions $h_1, \ldots, h_n$ with the properties enumerated ir Lemma 4.2. Let $\mathfrak{M}_0$ be the manifold determined by $h_1, \ldots, h_n$.

We now show that $C(\mathfrak{M}_0) \leqslant k_n$. Let y_i, $i = 1, \ldots, m$ be defined as follows: $y_i(t) = m^{1/p}$, for $(i-1)/m \leqslant t < i/m$, $y_i(t) = 0$ otherwise. Let $\mathfrak{N}$ be the closed linear manifold consisting of linear combinations of the $y_i(t)$. $\mathfrak{N}$ is readily seen to be iso- metrically isomorphic with $l_{p,m}$, also $\mathfrak{M}_0 \subset \mathfrak{N}$. Hence given an $\eta > 0$, we can find a projection E_0 of $\mathfrak{N}$ on $\mathfrak{M}$ such that $|E_0| \leqslant \bar{C}_n(l_{p,m}) + \eta \leqslant k_n + \eta$.

Now for any $f \in L_p$, let

$$Ff = \sum_{i=1}^{m} \int_0^1 f(t)\, y_i^{p-1}(t)\, dt\, y_i.$$

By a simple calculation one may verify that $|F| = 1$, and that F projects L_p on $\mathfrak{N}$. Consider $E_0 F$. The range of $E_0 F$ is $\mathfrak{M}_0$, since the range of F is $\mathfrak{N}$ and the range of E_0 on $\mathfrak{N}$ is $\mathfrak{M}_0$. Furthermore since $\mathfrak{M}_0$ is $\subset \mathfrak{N}$, if $f \in \mathfrak{M}_0$, $E_0 Ff = E_0 f = f$ and hence $E_0 F$ projects L_p on $\mathfrak{M}_0$. Thus $C(\mathfrak{M}_0)$ is $\leqslant |E_0 F| \leqslant |E_0|\,|F| = |E_0| \leqslant k_n + \eta$. This holds no matter how small η is taken, hence $C(\mathfrak{M}_0) \leqslant k_n$.

Since η was chosen in such a manner that $k_n(\varepsilon/(1-\varepsilon)) < 1$, we have that $C(\mathfrak{M}_0)(\varepsilon/(1-\varepsilon)) < 1$. Hence we may apply Lemma 5.1 to $\mathfrak{M}$, $\mathfrak{M}_0$ and η, and obtain that

$$C(\mathfrak{M}) \leqslant C(\mathfrak{M}_0)\, \frac{1 + C(\mathfrak{M}_0)(\varepsilon/(1-\varepsilon))}{1 - C(\mathfrak{M}_0)(\varepsilon/(1-\varepsilon))} \leqslant \frac{k_n(1 + k_n(\varepsilon/(1-\varepsilon)))}{1 - k_n(\varepsilon/(1-\varepsilon))}.$$

Inasmuch as this holds no matter how small ε is taken, we may conclude that $C(\mathfrak{M}) \leqslant k_n$. But $\mathfrak{M}$ was an arbitrary n—dimen- sional manifold in L_p, hence $\bar{C}_n(L_p) \leqslant k_n$.

7. Lemma 7.1. $\bar{C}(L_p) \leqslant$ l. u. b. $(\bar{C}(l_{p,n}),\ n < \infty)$.

Proof. Let l. u. b. $(\bar{C}(l_{p,n});\ n < \infty) = k$. If $k = \infty$, the statement is obvious. Hence we may assume that k is $< \infty$. Let k_n be as in Lemma 6.1. Then $k_n \leqslant k$.

Let $\mathfrak{M}$ be a closed linear manifold in L_p. Since L_p is sepa- rable, $\mathfrak{M}$ is separable. Hence there exists a sequence $\{f_n\}$ which is everywhere dense in $\mathfrak{M}$. Now if f_n is linearly dependent on $f_1, \ldots, f_{n-1}$, we drop it from the sequence. If the result is a finite sequence, $\mathfrak{M}$ has a finite dimensionality and Lemma 6.1 implies that $C(\mathfrak{M}) \leqslant k_n \leqslant k$. We may therefore confine our attention to infinite—dimensional $\mathfrak{M}$, i. e. those for which a sequence $\{f_n\}$ exists with each f_n linearly independent of the preceding $n-1$, and such that the linear combinations of the f_n are everywhere dense in $\mathfrak{M}$.

Let $\mathfrak{M}_n$ be the closed linear manifold determined by $f_1, \ldots, f_n$. It follows that $\mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \ldots \subset \mathfrak{M}_n \subset \ldots$. Furthermore, by the na- ture of the sequence $\{f_n\}$, given an $f \in \mathfrak{M}$, and an $\varepsilon > 0$, we can find an n such that if $m > n$, there exists a $g \in \mathfrak{M}_m$, with $\|f - g\| < \varepsilon$.

Let η be any number > 0. By Lemma 6.1, $C(\mathfrak{M}_m) \leqslant k_m \leqslant k$. Thus we can find a projection of L_p on $\mathfrak{M}_n$, E_n, such that $|E_n| \leqslant k + \eta$. Since the sequence $\{E_n\}$ is uniformly bounded, we can find a subsequence $\{E_{n_i}\}$ with $\lim_{i \to \infty} n_i = \infty$, which is weakly con- vergent to a transformation E with $|E| \leqslant k + \eta$.

Now if $f \in \mathfrak{M}$, we show that $E_n f \to f$. For given an $\varepsilon > 0$, we can find an n such that if $m > n$, there exists a $g_m \in \mathfrak{M}_m$ such that $\|f - g_n\| \leqslant \varepsilon/2(k+\eta)$. Now $f = g_m + (f - g_m)$, $E_m f = E_m g + E_m(f - g_m) = g_m + E_m(f - g_m)$. Hence $\|f - E_m f\| = \|f - g_m + E_m(f - g_m)\| \leqslant \|f - g_m\| + \|E_m(f - g_m)\| \leqslant \varepsilon/2(k+\eta) + (k+\eta)\, \|f - g_m\| \leqslant \varepsilon/2(k+\eta) + \varepsilon/2 \leqslant \varepsilon$. Thus $E_n f \to f$ strongly, which im- plies that $E_{n_i} f \to f$ strongly and hence weakly and $Ef = f$. Thus if $f \in \mathfrak{M}$, $Ef = f$ and so $E\Delta \supset \mathfrak{M}$.

Next, we show that $E\Delta \subset \mathfrak{M}$. For if h is not $\in \mathfrak{M}$, there exists a linear functional $F(f)$ on L_p, such that $F(h) = 1$, $F(f) = 0$, $f \in \mathfrak{M}$ (Cf. (*B*), Chap. IV, Lemma p. 57). Let g be any element of L_p. Then $E_n g$ is $\in \mathfrak{M}$, and $F(E_n g) = 0$. Hence by the defi- nition of weak convergence $F(Eg) = 0$, which implies that $Eg \neq h$. This is true for every g, hence h is not $\in E\Delta$ and $E\Delta \subset \mathfrak{M}$.

Thus $E\mathcal{A}=\mathfrak{M}$ and since if f is $\epsilon\,\mathfrak{M}$, $Ef=f$, $E^2=E$. Furthermore, since $|E|\leqslant k+\eta$, $C(\mathfrak{M})\leqslant k+\eta$. This last equation holds for any $\eta>0$, hence $C(\mathfrak{M})\leqslant k$. Since this is true for any $\mathfrak{M}\subset L_p$, $\bar{C}(L_p)\leqslant k$.

Theorem I and Lemma 7.1 yield

Theorem II. $\bar{C}(L_p)=\bar{C}(l_p)=$ l. u. b. $(\bar{C}(l_{p,n}),\ n<\infty)$.

Theorems I and II imply that the answer to (a) is the same as that of (b) and also to the question "Is l. u. b. $(\bar{C}(l_{p,n}),\ n<\infty)$ $<\infty$?" For if the answer to the last question is yes, then by Theorem II and the definitions of $\bar{C}(\mathcal{A})$ and $C(\mathfrak{M})$ the answer to (a) and (b) is yes. If on the other hand the answer is no, then $\bar{C}(L_p)=\bar{C}(l_p)=\infty$, and since by Theorem I there exists a closed linear manifold $\mathfrak{M}$, such that $C(\mathfrak{M})=\bar{C}(\mathcal{A})$, $\mathcal{A}=L_p$, or l_p, the definition of $C(\mathfrak{M})$ yields that the answer to both (a) and (b) is no.

8. We will show in this section that if the answer to problem (b) is yes, that of (c) is also yes. Let $k_p=\bar{C}(l_p)$, by hypothesis k_p is $<\infty$. Let $\mathfrak{M}$ be any closed linear manifold of l_p.

We use a result given in (*B*) Chap. XII, pp. 194—197, that there exists constants B_p and C_p such that to every infinite — dimensional manifold $\mathfrak{M}\subset l_p$, there exists an $\mathfrak{M}_0\subset\mathfrak{M}$, and a transformation T' with range $\mathfrak{M}_0$ such that, for every $f\,\epsilon\, l_p$,

$$B_p\|f\|\geqslant\|T'f\|\geqslant C_p\|f\|.$$

Let E be a projection of $\mathcal{A}$ on $\mathfrak{M}$. Let $\mathfrak{N}$ be the set of $f\,\epsilon\, l_p$ such that $Ef=0$. Let $\mathfrak{P}$ be the smallest closed linear manifold which contains $\mathfrak{N}$, $T'\mathfrak{N}$, $T'^2\mathfrak{N},\ldots$ which fact we denote by $\mathfrak{P}=\{\mathfrak{N},\,T'\mathfrak{N},\,T'^2\mathfrak{N},\ldots\}$. Obviously $T'\mathfrak{P}=\{T'\mathfrak{N},\,T'^2\mathfrak{N},\ldots\}$ and since $|E|<\infty$, $T'\mathfrak{P}=\mathfrak{P}.\mathfrak{M}_0=\mathfrak{P}.\mathfrak{M}$. Let F be a projection of l_p on $T'\mathfrak{P}$.

Now $F_0=1-E+FE$ is a projection on $\mathfrak{P}$. For $F_0^2=((1-E)+FE)((1-E)+FE)=(1-E)^2+(1-E)FE+FE(1-E)+FEFE$. Since $EF=F$, we have $(1-E)FE=(1-E)EFE=(E-E^2)FE=0$, $FE(1-E)=F(E-E^2)=0$, $FEFE=FFE=FE$ and hence $F_0^2=(1-E)^2+FE=1-E+FE=F_0$. Furthermore the range of F_0 is $\{\mathfrak{N},T'\mathfrak{P}\}=\mathfrak{P}$.

Let $T=T'F_0+(1-F_0)$. Since $\mathfrak{P}\supset T'\mathfrak{P}$, $T'F_0=F_0T'F_0$ and hence $T=F_0T'F_0+(1-F_0)$.

The range of T is $\mathfrak{M}$. For if h is $\epsilon\,\mathfrak{M}$, $h=(1-F_0)h+F_0h(1-F_0)h+((1-E)+FE)h$. Since $Eh=h$, $h=(1-F_0)h+(1-E)Eh+Fh=(1-F_0)h+Fh$. Since the range of F is $T'\mathfrak{P}$, $Fh=T'g=T'F_0g$ for some $g\,\epsilon\,\mathfrak{P}$. Also since g is $\epsilon\,\mathfrak{P}$, $(1-F_0)g=0$. Hence $h=(1-F_0)h+T'F_0g=(1-F_0)((1-F_0)h+g)+T'F_0((1-F_0)h+g)=((1-F_0)+T'F_0)((1-F_0)h+g)=T((1-F_0)h+g)$, or h is in the range of T. Hence the range of T includes $\mathfrak{M}$. But if h is in the range of T, $h=Tg$ for some g and since $1-F_0=(1-((1-E)+FE))=E-FE$ and $E(1-F_0)=E(E-FE)=E-EFE=E-FE=1-F_0$, $Eh=ETg=E(T'F_0+(1-F_0))g=ET'F_0g+E(1-F_0)g=T'F_0g+(1-F_0)g=Tg=h$, i. e. $h=Eh$. Hence h is $\epsilon\,\mathfrak{M}$, and the range of T is also included in $\mathfrak{M}$. This with our previous result shows that the range of T is $\mathfrak{M}$.

T^{-1} exists. For $Tf=0$ implies $(1-F_0)f+F_0T'F_0f=0$, hence $(1-F_0)f=0$ and $F_0T'F_0f=0=T'F_0f$. But T' has an nverse hence $F_0f=0$. Thus $f=(1-F_0)f+F_0f=0$ or $Tf=0$ mplies $f=0$. We have also shown in the preceding paragraph :hat for every h in $\mathfrak{M}$ which is the range of T, $h=Tf$, where $f=(1-F_0)h+g=(1-F_0)h+T'^{-1}Fh$. Thus $T^{-1}=(1-F_0)+T'^{-1}F$ defined on $\mathfrak{M}$.

Since F_0, F, T', T'^{-1} are all bounded, T and T^{-1} are oth bounded. Since the range of T is $\mathfrak{M}$, we see that $\mathfrak{M}$ is omorphic with l_p. Since $\mathfrak{M}$ was arbitrary, we have shown that ı affirmative answer to (b) implies an affirmative answer to (c).

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