Notes on orthogonal series III.

by

S. KACZMARZ (Lwów).

1. Let $\{\varphi_n(t)\}$ be an orthonormal system in the interval $\langle 0,1\rangle$. The system presents[1]) the *singularity* $k_p (1 \leqslant p < \infty)$, if there exists a function $f(t) \varepsilon L^p$ [$f(t)$ belongs to L^p if the function $|f(t)|^p$ is integrable over $\langle 0,1\rangle$[2])], such that $\sum_{n=1}^{\infty} |f_n|^p n^{p-2} = \infty$, where f_n are the coefficients of $f(t)$ with respect to the system $\{\varphi_n\}$. On the other hand, the *singularity* l_p $(1 \leqslant p < \infty)$ requires the existence of an orthogonal series $\sum_{n=1}^{\infty} a_n \varphi_n(t)$ with the properties: 1) $\sum_{n=1}^{\infty} |a_n|^p n^{p-2} < \infty$, 2) the series is not the development of a function belonging to L^p.

The purpose of this paper is to extend these definitions to the case $p = \infty$. We define namely the *singularity* k_∞ as the existence of a function $f(t) \varepsilon M$ (that is $f(t)$ is bounded almost everywhere), such that $\limsup_{n \to \infty} n|f_n| = \infty$. If there is a numerical sequence $\{a_n\}$ such that $n|a_n|$ is bounded and $\sum_{n=1}^{\infty} a_n \varphi_n(t)$ is not the development of a function belonging to M, we shall say that the system $\{\varphi_n\}$ presents the *singularity* l_∞.

[1]) See S. Kaczmarz und H. Steinhaus, Theorie der Orthogonalreihen, Monografje matematyczne t. VI, Warszawa-Lwów 1935 (referred in the sequence as OR) p. 237—238.

[2]) We write $\sum_{n=1}^{\infty} a_n \varphi_n(t) \varepsilon L^p$, if this series is the development of a function $f(t)$ belonging to L^p, otherwise $\sum_{n=1}^{\infty} a_n \varphi_n(t) \sim \varepsilon L^p$.

2. The following theorems show the relations between the singularities k and l.

Theorem 1. *If the system $\{\varphi_n\}$ presents the singularity l_∞, then the system presents also the singularity k_1.*

Suppose that the system does not present the singularity k_1. We have then for any function $f(t) \varepsilon L$ the relation $\sum_{n=1}^{\infty} \frac{|f_n|}{n} < \infty$.

Let $\{a_n\}$ be a sequence with the properties: $n|a_n| < A$, $\sum_{n=1}^{\infty} a_n \varphi_n(t) \sim \varepsilon M$. Then for any $f(t) \varepsilon L$ we have $\sum_{n=1}^{\infty} |a_n f_n| < \infty$, hence $\{a_n\}$ is the sequence of coefficients of a function belonging to M, contrary to the property of $\{a_n\}$ and the theorem is proved.

Theorem 2. *If the system $\{\varphi_n(t)\}$ presents the singularity k_1, $\varphi_n(t) \varepsilon M$, $\{\varphi_n(t)\}$ complete in M, then $\{\varphi_n(t)\}$ presents also the singularity l_∞.*

By the hypothesis there exists a function $f(t) \varepsilon L$, such that $\sum_{n=1}^{\infty} \frac{|f_n|}{n} = \infty$. Suppose that for any sequence $\{a_n\}$, such that $n|a_n| \leqslant 1$, we have $\sum_{n=1}^{\infty} a_n \varphi_n(t) \varepsilon M$; thus the sequence $\{\frac{1}{n}\}$ is a majorant[3]) for the space M. From the theorem [692] in OR[4]) follows that $\sum_{n=1}^{\infty} \frac{|f_n|}{n} < \infty$ for any $f(t) \varepsilon L$, contrary to the hypothesis. Consequently there exists a sequence $\{a_n\}$ such that $n|a_n| \leqslant 1$ and $\sum_{n=1}^{\infty} a_n \varphi_n(t) \sim \varepsilon M$.

Corollary. Under the assumptions: $\varphi_n \varepsilon M$, $\{\varphi_n\}$ complete in M, the singularities k_1 and l_∞ are equivalent.

Theorem 3. *If the system $\{\varphi_n\}$ presents the singularity l_1, then it presents also the singularity k_∞.*

Suppose that the system does not present the singularity k_∞. Then for any $f(t) \varepsilon M$ we have $n|f_n| < A$. Let $\{a_n\}$ be a se-

[3]) OR, p. 240.

[4]) OR, p. 240.

quence with the properties $\sum_{n=1}^{\infty}\frac{|a_n|}{n}<\infty$, $\sum_{n=1}^{\infty}a_n\varphi_n(t)\sim\varepsilon L$. The existence of such a sequence follows from the singularity l_1. Then we have $\sum_{n=1}^{\infty}|a_nf_n|<\infty$ for every $f(t)$ belonging to M, which [5] implies $\sum_{n=1}^{\infty}a_n\varphi_n(t)\,\varepsilon\,L$, contradictory to the singularity l_1, and so the system $\{\varphi_n\}$ presents the singularity k_∞.

Theorem 4. *The singularity k_∞ with the completness of $\{\varphi_n\}$ in the space L implies the existence of the singularity l_1.*

From the singularity k_∞ follows the existence of a function $f(t)\,\varepsilon\,M$, such that $\limsup_{n\to\infty} n|f_n|=\infty$. Consider the space X with elements $x=\{a_n\}$, such that $\sum_{n=1}^{\infty}\frac{|a_n|}{n}<\infty$. Define the norm of x by $\|x\|=\sum_{n=1}^{\infty}\frac{|a_n|}{n}$; then X is a space of the type B (that is vectorial, complete and normed). Suppose now that the system $\{\varphi_n\}$ does not present the singularity l_1. Thus every x is the sequence of coefficients of a function $g(t)$ belonging to L and therefore [6])

$$\lim_{n\to\infty}\int_0^1\Big|g(t)-\sum_{k=1}^{n}a_k\varphi_k(t)\Big|\,dt=0.$$

It follows that $\lim_{n\to\infty}\sum_{k=1}^{n}a_kf_k=\int_0^1 f(t)g(t)\,dt$ and $\sum_{n=1}^{\infty}|a_nf_n|<\infty$.

Hence for any $\{a_n\}$ with $\sum_{n=1}^{\infty}\frac{|a_n|}{n}<\infty$ we have

$$\sum_{n=1}^{\infty}\frac{|a_n|}{n}\,n|f_n|<\infty.$$

That implies $n|f_n|<A$, incompatible with the supposed property of $f(t)$ and so the theorem is proved.

Corollary. Under the assumption: $\{\varphi_n(t)\}$ complete in L, the singularities k_∞ and l_1 are equivalent.

[5]) OR, th. [646], p. 220.
[6]) OR, th. [673], p. 232.

3. The theorems proved above show the relations between singularities but they do not assure their existence. A sufficient condition for the existence gives the following

Theorem 5. *The singularity k_∞ exists, if $|\varphi_n(t)|\leqslant A$ for all n and almost all t.*

Take a sequence $\{a_n\}$ with the properties: $\sum_{n=1}^{\infty}\frac{|a_n|}{n}<\infty$, $\sum_{n=1}^{\infty}a_n^2=\infty$. Then on account of the boundedness of $\{\varphi_n\}$ we have $\sum_{n=1}^{\infty}a_n^2\varphi_n^2(t)=\infty$ on a set E of positive mesure [7]). This implies the existence of a sequence of indices $\{n_i\}$, such that [8])

$$\limsup_{k\to\infty}\int_0^1\Big|\sum_{i=1}^{k}a_{n_i}\varphi_{n_i}(t)\Big|\,dt=\infty.$$

Hence we can find a continuous function $g(t)$ with coefficients g_n satisfying the relation

$$\limsup_{k\to\infty}\sum_{i=1}^{k}g_{n_i}a_{n_i}=\infty.$$

We have therefore $\sum_{n=1}^{\infty}|a_ng_n|=\infty$; but the series $\sum_{n=1}^{\infty}\frac{|a_n|}{n}$ being convergent we have also $\limsup_{n\to\infty} n|g_n|=\infty$, which proves the existence of k_∞. Similarly we can prove:

Theorem 6. *The singularity k_∞ exists, if the system $\{\varphi_n\}$ is complete in L^2.*

A sufficient condition for the existence of the singularity l_∞ gives the

Theorem 7. *If $|\varphi_n(t)|\leqslant A$, then the system $\{\varphi_n\}$ presents the singularity l_∞.*

Consider [9]) a system $\{\psi_n(t)\}$, complete in M, containing the system $\{\varphi_n\}$. We can suppose that $\psi_n(t)\,\varepsilon\,M$ and that $\varphi_n(t)=\psi_{2n}(t)$, if the set of functions $\{\psi_n\}-\{\varphi_n\}$ is not finite. Then

[7]) OR, th. [512], p. 150.
[8]) OR, th. [676], p. 235.
[9]) OR, th. [614], p. 198.

we have[10])

$$\sum_{n=1}^{\infty} d_n |\psi_n(t)| \leqslant \alpha,$$

if the sequence $\{d_n\}$ is a majorant for $\{\psi_n\}$ in M.

Suppose now, that the system $\{\varphi_n\}$ does not present the singularity l_∞. Then $\sum_{n=1}^{\infty} a_n \varphi_n(t) \,\varepsilon\, M$ for any $\{a_n\}$, such that $n|a_n| \leqslant 1$. The sequence $\{\frac{1}{n}\}$ is therefore a majorant for $\{\varphi_n\}$ and the sequence $\{d_n\}$, where $d_{2n} = \frac{1}{n}$, $d_{2n+1} = 0$ is a majorant for $\{\psi_n\}$ hence, as mentioned above, $\sum_{n=1}^{\infty} \frac{|\varphi_n(t)|}{n} \leqslant \alpha$. The boundedness of $\{\varphi_n\}$ leads to the result $\sum_{n=1}^{\infty} \frac{1}{n} < \infty$. This contradiction implies the existence of l_∞. The proof in the case, when $\{\psi_n\} - \{\varphi_n\}$ is finite, is quite similar.

Corollary. From theorems 1 and 7 follows the singularity k_1 for $|\varphi_n(t)| \leqslant A$.

Theorem 8. *If* $|\varphi_n(t) \leqslant A$, *then the singularity* l_p *exists for* $1 \leqslant p < 2$.

Suppose, that for every $\{a_n\}$ such that $\sum_{n=1}^{\infty} |a_n|^p n^{p-2} < \infty$ we have $\sum_{n=1}^{\infty} a_n \varphi_n(t) \,\varepsilon\, L^p$. Take $a_n = 1$ for $n = k^\alpha$, $\alpha > \frac{1}{2-p}$ and $a_n = 0$ for other n. Then the series $\sum_{k=1}^{\infty} k^{\alpha(p-2)}$ is convergent and thus $\sum_{n=1}^{\infty} a_n \varphi_n(t) \,\varepsilon\, L^p$, contrary to the fact, that for bounded $\{\varphi_n\}$ we have $\lim_{n \to \infty} a_n = 0$.

[10]) OR, p. 242.

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