hin impliziert $L(E) = B(E)$ die Normierbarkeit von E. Uss fragt nun in [6], ob aus der Tatsache, daß $\sigma(T)$ beschränkt ist für alle $T \in L(E)$ schon folgt, daß E Banach ist. Dies ist nicht der Fall.

Gegenbeispiel. Sei E ein unendlichdimensionaler Banachraum und E_s dieser Raum ausgestattet mit seiner schwachen Topologie. Da die Normtopologie zugleich die Mackey-Topologie ist, gilt $L(E) = L(E_s)$. Damit besitzt jeder Operator $T \in L(E_s)$ ein kompaktes Spektrum, ohne daß E_s deshalb normierbar wäre.

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On semi-Fredholm operators and the conjugate of a product of operators

by

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Abstract. The first part of this paper contains necessary and sufficient conditions for a linear operator between Banach spaces to be semi-Fredholm. In the second part it is shown that there is a duality between these conditions corresponding to the duality of Φ^-- and Φ^+-operators. In the third part these results are used to derive a necessary and sufficient condition for the equality $(TS)' = S'T'$.

1. Let X and Y be Banach spaces. For a linear operator $T\colon X \to Y$ with domain $D(T)$, range $R(T)$, and null space $N(T)$, let $\overline{T}$ denote the closure and $T'\colon Y' \to X'$ the conjugate of T, if defined. Let $B(X, Y)$ and $K(X, Y)$ denote the linear spaces of bounded and compact linear operators with domain X and range in Y, U_X the closed unit ball in X, $\Phi^-(X, Y)$ the set of semi-Fredholm operators with closed ranges and finite-dimensional null spaces, and $\Phi^+(X, Y)$ the set of semi-Fredholm operators the ranges of which have finite deficiency in Y.

First the Φ^- and Φ^+-operators will be characterized.

1. Theorem. *Let $T\colon X \to Y$ be a closable linear operator. The following statements are equivalent*:

(1) $\overline{T} \in \Phi^-(X, Y)$;

(2) *There are a Banach space Z, $C \in K(X, Z)$, and $a > 0$ such that*

$$\|x\| \leqslant a\|Tx\| + \|Cx\| \quad \text{for all } x \in D(T).$$

Proof. It is immediate that (2) is equivalent to

(2̄) There are a Banach space Z, $C \in K(X, Z)$, and $a > 0$ such that for $x \in D(\overline{T})$, $\|x\| \leqslant a\|\overline{T}x\| + \|Cx\|$.

Therefore T may be assumed to be closed.

Suppose $T \in \Phi^-(X, Y)$, $Z = X$, and let C be a projection from X onto $N(T)$. Since $N(T)$ is finite dimensional, C is compact. If $x \in D(T)$ and $u \in N(T)$, then $\|x\| - \|Cx\| \leqslant \|(I-C)(x-u)\|$. Employing the reduced minimum modulus $\gamma(T)$ ([5], p. 231) we have

$$\|x\| - \|Cx\| \leqslant \|I-C\|\gamma(T)^{-1}\|Tx\| = a\|Tx\|$$

with $a = \|I-C\|\gamma(T)^{-1} > 0$. This implies (2).

Suppose (2). Then for any bounded sequence (x_n) in $N(T)$ there exists a subsequence (y_k) such that (Cy_k) converges. Since $Ty_k = 0$, it follows from (2) that (y_k) converges. Thus $U_{N(T)}$ is relatively compact and therefore, $N(T)$ is finite dimensional. If $R(T)$ were not closed, then $\gamma(T) = 0$, which implies the existence of a bounded sequence (x_n) in $D(T)$ without accumulation point in $N(T)$ but $\lim Tx_n = 0$. For a subsequence (y_k), (Cy_k) converges. It follows from (2) that (y_k) converges to some $x \in X$. Since T is closed, $x \in N(T)$. This contradicts the choice of (x_n), hence $T \in \Phi^-(X, Y)$.

This characterization of Φ^--operators has frequently been employed for the treatment of differential operators. The following characterization Φ^+-operators seems to be new.

2. THEOREM. *Let T: $X \to Y$ be a closed linear operator. The following statements are equivalent:*

(1) $T \in \Phi^+(X, Y)$;

(2) *There are a Banach space Z, $C \in K(Z, Y)$, and $\alpha > 0$ such that*

$$U_Y \subseteq \alpha T U_X + C U_Z.$$

Proof. Suppose $T \in \Phi^+(X, Y)$. Choose $Z = Y$ and $C = I - P \in K(Z, Y)$ with a projection P from Y onto $R(T)$. Let $y \in U_Y$. It follows from $Y = R(T) \oplus R(C)$ that $y = Tx + Cy$ with $Tx = Py$. There is a $z \in D(T)$ such that $Tz = Tx$ and

$$\gamma(T)\|z\| \leqslant 2\|Tx\| \leqslant 2\|P\|.$$

Taking $\alpha = 2\|P\|\gamma(T)^{-1}$, it follows that $Tx = Tz \in \alpha T U_X$. Since $Cy \in C U_Y$, (2) is proved.

Suppose (2). For every $\varepsilon > 0$ there exists a finite-dimensional subspace M_ε of Y such that $CU_Z \subseteq \varepsilon U_Y + M_\varepsilon$ ([7], p. 146). Thus $U_Y \subseteq \alpha T U_X + \varepsilon U_Y + M_\varepsilon$. From this we shall derive $U_Y \subseteq R(T) + M_\varepsilon$ which implies $Y \subseteq R(T) + M_\varepsilon$, that is, $R(T)$ has finite deficiency. Now suppose $y \in U_Y$. Inductively we obtain sequences (x_k) in U_X, (y_k) in U_Y, and (w_k) in M_ε such that

$$y = \alpha T x_0 + \varepsilon y_1 + w_0$$

and

$$y_k = \alpha T x_k + \varepsilon y_{k+1} + w_k, \quad k = 1, 2, \ldots$$

For every n, this implies

$$y = \alpha T\Big(\sum_{k=0}^{n} \varepsilon^k x_k\Big) + \varepsilon^{n+1} y_{n+1} + \sum_{k=0}^{n} \varepsilon^k w_k.$$

Suppose $\varepsilon < 1$. Then $(\varepsilon^n y_n)$ converges to zero and $x = \sum_0^\infty \varepsilon^k x_k$ exists.

According to [6], there exists a norm $|\ |$ on Y such that T: $X \to (Y, |\ |)$ is continuous and $|y| \leqslant \|y\|$ for all $y \in Y$. Since

$$|w_k| \leqslant |y_k| + \varepsilon |y_{k+1}| + \alpha |Tx_k| \leqslant 1 + \varepsilon + |T|,$$

(w_k) is bounded. On M_ε $\|\ \|$ and $|\ |$ are equivalent, hence the series $\sum \varepsilon^k w_k$ converges to some $w \in M_\varepsilon$. Then $\sum \varepsilon^k Tx_k$ converges, too. Since T is closed, $x \in D(T)$ and $Tx = \sum \varepsilon^k x_k$. It follows that $y = \alpha Tx + w$, hence $U_Y \subseteq R(T) + M_\varepsilon$.

If (2) is valid for a closable linear operator T, then the inclusion is valid for $\overline{T}$. In this case $\overline{T} \in \Phi^+(X, Y)$.

2. The well-known duality of densely defined Φ^--and Φ^+- operators is reflected by a duality between the conditions (2) of Theorems 1 and 2. For this purpose we need the following result which is a special case of the theorem in [8].

3. LEMMA. *Let E and F be (real or complex) linear spaces, E^* and F^* their (algebraical) duals, S: $E \to F$ a linear map with domain E, ω: $F \to \mathbf{R}_+$ a seminorm, and $u^* \in E^*$. If $|\langle u, u^*\rangle| \leqslant \omega(Su)$ for all $u \in E$, there exists a $v^* \in F^*$ such that $|v^*| \leqslant \omega$ and $\langle u, u^*\rangle = \langle Su, v^*\rangle$ for all $u \in E$.*

For the next two theorems, let X, Y and Z be normed linear spaces and let X', Y', and Z' be their strong duals.

4. THEOREM. *Let A: $X \to Y$ be a densely defined linear operator,, $C \in B(X, Z)$, and $\alpha > 0$. The following statements are equivalent:*

(1) $\|x\| \leqslant \alpha\|Ax\| + \|Cx\|$ *for all* $x \in D(A)$;

(2) $U_{X'} \subseteq \alpha A' U_{Y'} + C' U_{Z'}$.

Proof. Suppose (1) and $x' \in U_{X'}$ and let $E = D(A)$, $F = Y \times Z$, $Sx = (\alpha Ax, Cx)$, $\omega(v, w) = \|v\| + \|w\|$, and $u^* = x'$. Then, for $x \in D(A)$, (1) reads $|\langle x, x'\rangle| \leqslant \omega(Sx)$ and Lemma 3 yields a $v^* = (y', z') \in F^*$ such that $|v^*| \leqslant \omega$. Hence, $v^* \in F'$ and $y' \in U_{Y'}$; $z' \in U_{Z'}$. Furthermore, for $x \in D(A)$

$$\langle x, x'\rangle = \langle Sx, v^*\rangle = \alpha\langle Ax, y'\rangle + \langle x, C'z'\rangle.$$

Then $y' \in D(A')$ and $x' = \alpha A'y' + C'z'$. Thus (2) follows.

Suppose (2). For each $x' \in U_{X'}$ there exist $y' \in U_{Y'} \cap D(A')$ and $z' \in U_{Z'}$ such that $x' = \alpha A'y' + C'z'$. Hence, if $x \in D(A)$, $|\langle x, x'\rangle| \leqslant \alpha\|Ax\| + \|Cx\|$ which implies (1).

5. THEOREM. *Let A: $X \to Y$ be a densely defined linear operator, $C \in B(Z, Y)$, and $\alpha > 0$. The following statements are equivalent:*

(1) $(1-\varepsilon) U_Y \subseteq \alpha A U_X + C U_Z$, $0 < \varepsilon < 1$;

(2) $\|y'\| \leqslant \alpha\|A'y'\| + \|C'y'\|$ *for all* $y' \in D(A')$.

Proof. Suppose (1), $0<\varepsilon<1$, and $y'\in D(A')$. Then

$$\begin{aligned}(1-\varepsilon)\|y'\| &= \sup\{|\langle y, y'\rangle|: y\in(1-\varepsilon)U_Y\}\\ &\leqslant \sup\{|\langle \alpha Ax+Cz, y'\rangle|: x\in U_X\cap D(A),\ z\in U_Z\}\\ &\leqslant \alpha\|A'y'\|+\|C'y'\|.\end{aligned}$$

Suppose (2). $U_Y\subseteq\overline{\alpha AU_X+CU_Z}$ implies (1). If this inclusion were false, there exist $y_0\in U_Y$, $y_0\notin\overline{\alpha AU_X+CU_Z}$ and $y_0'\in Y'$ such that

$$|\langle y_0, y_0'\rangle| > \sup\{|\langle \alpha Ax+Cz, y_0'\rangle|: x\in U_X\cap D(A),\ z\in U_Z\} = \eta.$$

This implies $\alpha|\langle Ax, y_0'\rangle| < |\langle y_0, y_0'\rangle|+\|C\|\|y_0'\|$ for $x\in U_X\cap D(A)$, hence $y_0'\in D(A')$. Now (2) implies $\|y_0'\|\leqslant\alpha\|A'y_0'\|+\|C'y_0'\|=\eta$ which contradicts $\eta<|\langle y_0, y_0'\rangle|\leqslant\|y_0'\|$.

For Banach spaces, A closed and C compact, the duality between Φ^-- and Φ^+-operators is obtained from Theorems 4 and 5 together with Theorems 1 and 2.

Let $A: X\to Z$ and $B: Y\to Z$ be linear operators. According to [3], B is called *A-co-continuous*, if there are non-negative constants α and β such that $BU_Y\subseteq AU_X+\beta U_Z$.

The proof of the following lemma is similar to that of Theorem 5.

6. Lemma ([3], 1.2.). *Let A and B be densely defined. Then B is A-co-continuous if and only if B' is A'-bounded, i.e. $D(A')\subseteq D(B')$ and there are non-negative constants α and β such that for all $z'\in D(A')$, $\|B'z'\|\leqslant\alpha\|A'z'\|+\beta\|z'\|$.*

3. The concept of relative co-continuity is used to examine the conjugate of a product of two operators. As is known, if S and T are densely defined linear operators, then merely $S'T'\subseteq(TS)'$, even if TS is densely defined.

$(TS)'=S'T'$ implies that T' is $(TS)'$-bounded ([4], V. 3.3.), for it follows that $D\big((TS)'\big)\subseteq D(T')$, the duals of normed linear spaces are complete and conjugate operators are closed. Lemma 6 then gives:

If $(TS)'=S'T'$, then T is TS-co-continuous.

The following example shows that the converse of this remark is not true. Let $\tau=\sum_0^n a_kD^k$ be a differential expression, where the coefficients are constant, let I be an interval, T_τ the maximal operator corresponding to τ in $L_2(I)$, and T_τ^R the restriction of T_τ to those $f\in D(T_\tau)$ which have compact supports in the interior of I. Then $T_{0,\tau}=\overline{T_\tau^R}$ is the minimal operator ([4], VI.2.1). Now let $S: L_2\to L_2\times L_2$ and $T: L_2\times L_2\to L_2$ be defined by

$$D(S)=D(T_\tau),\qquad Sf=(T_\tau f, f)$$

and

$$D(T)=\{(f,g): f,g\in D(T_\tau^R)\},\qquad T(f,g)=f.$$

As a projection T is continuous, thus, in particular, T is TS-co-continuous. Now $TS=T_\tau^R$ implies $(TS)'=(T_{0,\tau})'=T_{\tau^*}$ ([4], VI.2.3), and $S'T'=(T_\tau)'=T_{0,\tau^*}$. Let I be compact. Then $T_{\tau^*}\neq T_{0,\tau^*}$([4], VI.2.10.i) and in, this case $(TS)'\neq S'T'$.

Let X, Y, and Z be normed linear spaces, $S: X\to Y$ and $T: Y\to Z$ linear operators. According to [5], p. 166, a linear submanifold D of X is called a *core of S*, if $D\subseteq D(S)$ and $G(S)\subseteq\overline{G(S|_D)}$, where $G(S)$ denotes the graph of S.

7. Theorem. *Let S and T be densely defined and let $D(TS)$ be a core of S. Then TS is densely defined and $(TS)'=S'T'$ if and only if T is TS-co-continuous.*

Proof. It has already been shown that T is TS-co-continuous, if $(TS)'=S'T'$. Since $D(TS)$ is a core of S, $D(TS)$ is dense in $D(S)$. Thus TS is densely defined. By Lemma 6, T' is $(TS)'$-bounded, hence $D\big((TS)'\big)\subseteq D(T')$. Since $(TS)'$ is an extension of $S'T'$, it remains to show that $D\big((TS)'\big)\subseteq D(S'T')$. For this purpose suppose $z'\in D\big((TS)'\big)$, hence $z'\in D(T')$. For $x\in D(TS)$, $Sx\in D(T)$ and

$$\langle Sx, T'z'\rangle=\langle TSx, z'\rangle=\langle x, (TS)'z'\rangle.$$

Since $D(TS)$ is a core of S,

$$\langle Sx, T'z'\rangle=\langle x, (TS)'z'\rangle,\qquad x\in D(S).$$

It follows that $T'z'\in D(S')$, hence $z'\in D(S'T')$ and $D\big((TS)'\big)\subseteq D(S'T')$.

The following remarks serve the purpose of comparing Theorem 7 with known sufficient conditions for $(TS)'=S'T'$.

8. Lemma (see [2], Lemma 5). *Let X and Y be complete, let S be closed with closed range and let M be a linear submanifold of Y. Then $S^{-1}(M)$ is a core of S if and only if $\overline{R(S)\cap M}=R(S)$.*

Proof. Suppose $\overline{R(S)\cap M}=R(S)$. Let $(x, Sx)\in G(S)$. Since $Sx\in R(S)$ there exists a sequence (y_n) in $R(S)\cap M$ which converges to Sx. Since S is closed and has closed range, there exists a sequence (x_n) with limit x and $Sx_n=y_n$. Thus $x_n\in S^{-1}(M)$. (x_n, Sx_n) converges to (x, Sx). Now $S^{-1}(M)\subseteq D(S)$, therefore, $S^{-1}(M)$ is a core of S. Conversely, suppose that $S^{-1}(M)$ is a core of S. If $\overline{R(S)\cap M}\neq R(S)$, there exists $x_0\in D(S)$ such that $y_0=Sx_0\notin\overline{R(S)\cap M}$ and $y_0'\in Y'$ such that $\langle y_0, y_0'\rangle=1$ and $\langle y, y_0'\rangle=0$ whenever $y\in\overline{R(S)\cap M}$. Therefore, if $x\in S^{-1}(M)$, $\langle Sx, y_0'\rangle=0$. Since $S^{-1}(M)$ is a core of S, $\langle Sx, y_0'\rangle=0$ for all $x\in D(S)$. This contradicts $\langle Sx_0, y_0'\rangle=\langle y_0, y_0'\rangle=1$.

9. LEMMA. *Let X, Y and Z be complete, let T be closed and densely definied and $S \in \Phi^+(X, Y)$. Then $D(TS)$ is a core of S and T is TS-co-continuous.*

Proof. Since $R(S)$ has finite deficiency, $R(S) = \overline{R(S) \cap D(T)}$ ([4], IV.2.8). Since $R(S)$ is closed, $D(TS)$ is a core of S by Lemma 8. Theorem 2 yields $U_Y \subseteq aSU_X + CU_Y$, where C may be chosen to be finite dimensional with range in $D(T)$ ([4], IV.2.8). TC then has domain X and is closed, hence $TU_Y \subseteq aTSU_X + \|TC\| U_Z$ and therefore, T is TS-co-continuous.

Suppose that in Lemma 9 S is also densely defined. Then Theorem 7 implies $(TS)' = S'T'$. This has been proved in [9]. With similar conclusions [2], Theorem 6 is obtained. A partial converse of Theorem 7 and [1], Theorem 1, is given by

10. LEMMA. *If $(TS') = S'T'$ and $R(T') = Y'$, then $D(TS)$ is a core of S.*

Proof. $D = D(TS)$ is a core of S if and only if $S' = (S|_D)'$. Since $S' \subseteq (S|_D)'$, it remains to show that $D\big((S|_D)'\big) \subseteq D(S')$. If $y' \in D\big((S|_D)'\big)$, then there exists $x' \in D(T')$ such that $y' = T'x'$. For $x \in D$,

$$\langle x, (S|_D)'y' \rangle = \langle Sx, T'x' \rangle = \langle TSx, x' \rangle .$$

This implies $x' \in D\big((TS)'\big) = D(S'T')$, hence $y' = T'x' \in D(S')$.

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