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Basic sequences in (s)

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Abstract. A characterization is given of when a nuclear Fréchet space with basis is isomorphic to the subspace generated by a basic sequence in the space (s) of rapidly decreasing sequences. The characterization is in terms of a simple inequality which the basis must satisfy. Some consequences are derived and some information is given on subspaces of (s) which do not have bases.

In this paper we study subspaces of the nuclear Fréchet space (s) of rapidly decreasing sequences. Our main result is a theorem which gives a complete characterization under the assumption that the subspace has a basis. The condition is that there is a fundamental system of norms for the subspace such that the basis satisfies condition (d_3). This condition is the same as the condition of type d_1 of M. M. Dragilev [7] without the requirement that the basis be regular (Proposition 2).

Several consequences of this characterization are derived (Corollaries 1, 2, 3) and some open questions are mentioned.

In the last section, we give (without details of proof) some information on subspaces of (s) which do not possess a basis. In this case, although we know that such exist we do not have a complete characterization[1].

It would be interesting to know which other nuclear Fréchet spaces can have their basic sequences so simply characterized (of course, for the space ω-countable product of one-dimensional spaces, it is well known (see [4], Theorem 6) where this space is called (s)).

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Preliminaries. We shall denote by N the set of positive integers. Because of the profusion of indices there may be ambiguity between superscripts and exponents. To resolve this we shall, when it seems necessary, write q^k when k is a superscript and $(q)^k$ when k is an exponent.

If E is a nuclear Fréchet space, then a sequence $(\|\cdot\|_k)_{k\in N}$ of seminorms

[1] (*Added in proof*). Such a characterisation has recently been obtained by D. Vogt.

will be called a *fundamental system of seminorms* for E if $\|x\|_k \leqslant \|x\|_{k+1}$, $x \in E$ and $\operatorname{Lim}_n x_n = x$ in the topology of E iff $\lim_n \|x_n - x\|_k = 0$, $k \in N$. Two sequences of seminorms $(\|\cdot\|_k)$ and $(|\cdot|_k)$ are *equivalent* if

$$\forall j \exists k \text{ and } M > 0 \ni \|x\|_j \leqslant M|x|_k, \quad x \in E,$$

and

$$\forall k \exists l \text{ and } N > 0 \ni |x|_k \leqslant N\|x\|_l, \quad x \in E.$$

Thus, as is well known, the two systems are equivalent iff they are fundamental for the same topology.

It is well known and easy to show that a nuclear Fréchet space E has a continuous norm defined on it iff it has a fundamental system of seminorms, each of which is a norm.

A sequence (x_n) in a nuclear Fréchet space E is a *basis* if for each $x \in E$ there is a unique sequence (t_n) of scalars such that $x = \sum t_n x_n$. Clearly, if (x_n) is a basis and (d_n) is a sequence of non-zero scalars, then $(d_n x_n)$ is again a basis. The sequence $(d_n x_n)$ is called a *diagonal transformation of* (x_n). It follows from the theorem of A.S. Dynin and B.S. Mitiagin [11] that any permutation of a basis in a nuclear Fréchet space is again a basis.

Two bases (x_n), (y_n) for spaces E, F, respectively, are *equivalent* if there is an isomorphism $T: E \to F$ such that $Tx_n = y_n$, $n \in N$. The bases are *semi-equivalent* if one is equivalent to some permutation of the other.

A sequence in a nuclear Fréchet space E is a *basic sequence* if it is a basis for the closed subspace it generates. If (x_n) is a basis for E and $0 = p_0 < p_n < p_{n+1}$, $n \in N$ and

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} t_i x_i, \quad n \in N,$$

where (t_i) is a sequence of scalars and $y_n \neq 0$, $n \in N$, then (y_n) is called a *block basic sequence with respect to* (x_n). It is easy to check that it is always a basic sequence. Moreover, a diagonal transformation of a block basic sequence is again a block basic sequence with respect to the same basis.

The most important example of a nuclear Fréchet space with continuous norm is the Köthe space $K(a)$ determined by an infinite matrix $a = (a_n^k)_{k,n \in N}$ which satisfies

$$0 < a_n^k \leqslant a_n^{k+1}, \; n, k \in N \quad \text{and} \quad \sum_n \frac{a_n^k}{a_n^{k+1}} < \infty, \; k \in N.$$

Then

$$K(a) = \Big\{\xi = (\xi_n) \colon \|\xi\|_k = \sum_n |\xi_n| a_n^k < \infty, \; k \in N\Big\}$$

and the topology is determined by requiring that $(\|\cdot\|_k)$ be a fundamental system of norms. It is clear that, if e_n is the sequence of scalars which is 1 in the nth place and 0 elsewhere, then (e_n) is a basis for $K(a)$ called the *coordinate basis*. Thus $K(a)$ is a nuclear Fréchet space with basis and continuous norm. It is not hard to see that every such space is isomorphic to a Köthe space.

If $K(a)$ is a Köthe space and $1 \leqslant p \leqslant \infty$, then we obtain a fundamental system of norms $(|\cdot|_k)$ equivalent to $(\|\cdot\|_k)$ by letting $|\xi|_k$ be the l_p-norm of the sequence $(\xi_n a_n^k)_n$.

The space (s) of rapidly decreasing sequences is the Köthe space $K(a)$, where $a_n^k = n^k$. Thus (e^n) is a basis and a fundamental system of norms is given by $(\|\cdot\|_k)$, where

$$\|y\|_k = \sup_n |\xi_n| n^k, \quad k \in N, \quad \text{for} \quad y = \sum_n \xi_n e_n \in (s).$$

Since (s) has a continuous norm, it is obvious that any subspace of (s) also has a continuous norm.

Bases of type (d_3). A basis (x_n) in a nuclear Fréchet space with continuous norm E is said to be *regular* if there is a fundamental system of norms $(\|\cdot\|_k)$ such that

$$(\delta_0) \qquad \frac{\|x_n\|_{k+1}}{\|x_n\|_k} < \frac{\|x_{n+1}\|_{k+1}}{\|x_{n+1}\|_k}, \quad k, n \in N.$$

A basis is said to be of *type* d_1 if there is a fundamental system of norms which satisfies (δ_0) and also the condition

$$(\delta_1) \qquad \exists k \ni \forall j \; \exists l \ni \sup_n \frac{\|x_n\|_j^2}{\|x_n\|_k \|x_n\|_l} < \infty.$$

A basis is said to be of *type* d_2 if there is a fundamental system of norms which satisfies (δ_0) and also the condition

$$(\delta_2) \qquad \forall k \; \exists j \ni \forall l \; \sup_n \frac{\|x_n\|_k \|x_n\|_l}{\|x_n\|_j^2} < \infty.$$

These classifications were introduced by Dragilev [7] as generalizations of power bases in spaces of analytic functions on the whole plane (type d_1) or the open unit disk (type d_2).

For our purposes we will say that a basis (x_n) in a nuclear Fréchet space E with continuous norm is of *type* d_3 if there is a fundamental system of norms $(\|\cdot\|_k)$ such that

$$(\delta_3) \qquad \forall k \; \exists \varepsilon_k > 0 \ni \varepsilon_k \frac{\|x_n\|_{k+1}}{\|x_n\|_k} \leqslant \frac{\|x_n\|_{k+2}}{\|x_n\|_{k+1}}, \quad n \in N.$$

Notice that there is no assumption of regularity here. Notice also that in view of the definition of fundamental system it would be equivalent to require that (ε_k) be any preassigned sequence of positive numbers.

Condition (δ_0) very much depends on the choice of the fundamental system of norms whereas (δ_1) and (δ_2) do not. Condition (δ_3) is somewhere in the middle as indicated by the following proposition. On occasion we will say that a space is of a certain type when the choice of basis is clear or irrelevant.

PROPOSITION 1. *If (x_n) is a basis of type d_3 and $(|\cdot|_k)$ is any fundamental system of norms, then there is a subsequence $(|\cdot|_{k_l})_l$ such that condition (δ_3) holds.*

Proof. We may suppose that (δ_3) holds with the fundamental system of norms $(\|\cdot\|_k)$ and $\varepsilon_k = 1$. We may choose k_1 and M_1 such that $\|x_n\|_1 \leqslant M_1 |x_n|_{k_1}$, $n \in N$ and $k_2 > k_1$. Assume that $k_1 < k_2 < \ldots < k_l$ and $\delta_1, \ldots, \delta_{l-2}$ have been chosen such that

$$\delta_i \frac{|x_n|_{k_{i+1}}}{|x_n|_{k_i}} \leqslant \frac{|x_n|_{k_{i+2}}}{|x_n|_{k_{i+1}}}, \quad i = 1, 2, \ldots, l-2, \; n \in N.$$

Then we choose $k < j$ and $M, N > 0$ such that

$$\|x_n\|_k \leqslant M |x_n|_{k_{l-1}} \leqslant M |x_n|_{k_l} \leqslant N \|x_n\|_j, \quad n \in N.$$

Hence, for $n \in N$, we have

$$\frac{|x_n|_{k_l}}{|x_n|_{k_{l-1}}} \leqslant MN \frac{\|x_n\|_j}{\|x_n\|_k} = MN \frac{\|x_n\|_j}{\|x_n\|_{j-1}} \cdots \frac{\|x_n\|_{k+1}}{\|x_n\|_k}$$
$$\leqslant MN \frac{\|x_n\|_{2j-k}}{\|x_n\|_{2j-k-1}} \cdots \frac{\|x_n\|_{j+1}}{\|x_n\|_j} = MN \frac{\|x_n\|_{2j-k}}{\|x_n\|_j}$$
$$\leqslant N^2 \frac{\|x_n\|_{2j-k}}{|x_n|_{k_l}}.$$

It is then possible to complete the induction by choosing k_{l+1} and $R > 0$ such that $\|x_n\|_{2j-k} \leqslant R |x_n|_{k_{l+1}}$, $n \in N$ so that $\delta_{l-1} = 1/RN^2$. Thus (δ_3) is established.

Remark. It is obvious from the form of (δ_3) that this condition is unaffected if the basis (x_n) is subjected to a permutation and a diagonal transformation. Less on the surface is the fact that, if a space has a basis of type d_3, then every complemented basic sequence is also of type d_3. This is an immediate consequence of a theorem of C. Bessaga and Dragilev ([1], p. 315). Actually, we will show below that a much stronger statement is true (Corollary 1).

PROPOSITION 2. *A basis (x_n) in a nuclear Fréchet space is of type d_3 iff it has a fundamental system of norms for which the relation (δ_1) holds.*

Proof. We may first assume that

$$\|x_n\|_k^2 \leqslant \|x_n\|_1 \|x_n\|_{k+1}, \quad n, k \in N$$

so that

$$\|x_n\|_{k+1}^2 \leqslant \|x_n\|_1 \|x_n\|_{k+2} \leqslant \|x_n\|_k \|x_n\|_{k+2}$$

which gives (δ_3), with $\varepsilon_k = 1$.

Conversely, if the basis is of type d_3, then we may assume that (δ_3) holds with $\varepsilon_k = 1$ so that, for $n, k \in N$,

$$\frac{\|x_n\|_k}{\|x_n\|_1} = \frac{\|x_n\|_k}{\|x_n\|_{k-1}} \cdots \frac{\|x_n\|_2}{\|x_n\|_1} \leqslant \frac{\|x_n\|_{2k-1}}{\|x_n\|_{2k-2}} \cdots \frac{\|x_n\|_{k+1}}{\|x_n\|_k} = \frac{\|x_n\|_{2k-1}}{\|x_n\|_k}$$

which yields (δ_1).

The preceding result gives considerable information about the class of spaces of type d_3. For example, in addition to the spaces of type d_1 this class includes the metrizable G_∞ spaces of T. Terzioglu [13]. It also includes the spaces of class E_{13} of Dragilev [8] (which are the same as type d_1) but is disjoint from the seven other classes discussed in that paper. Indeed, it is immediate from Proposition 2 that no basis of type d_3 can have a subsequence of type d_2 (which implies that all but E_{23} of the remaining seven classes are disjoint from the class of spaces of type d_3). If we take any space of class E_{23}, then the basis has no subsequence of type d_2. On the other hand, it is regular so, by Proposition 2, it is of type d_3 iff it is of type d_1 which it is not. All of these properties of a space of class E_{23} are established in [8].

If one invokes the theorem of Bessaga and Dragilev mentioned above, then it follows immediately that if a space has a basis of type d_3 then no complemented basic sequence can be of type d_2. The same example of a space of class E_{23} shows that the converse is again false.

Main result. The following lemma is a special case of Lemma 2 ([9], p. 261) along with equation (3) of its proof. We omit the details of the proof.

LEMMA. *Let (a_n^k) be an infinite matrix of positive numbers such tha*

$$\frac{a_n^{k+1}}{a_n^k} < \frac{a_{n+1}^{k+1}}{a_{n+1}^k}, \quad n, k \in N.$$

Given numbers $t_1, \ldots, t_p$, we define, for $k \in N$,

$$q^k(t_1, \ldots, t_p) = \max\{q: \max_{1 \leqslant i \leqslant p} |t_i| a_i^k = |t_q| a_q^k\}.$$

Then if $0 < q^1 < \ldots < q^m \leqslant p$ are integers, it is possible to choose numbers $t_1, \ldots, t_p$ with $t_{q^1} \neq 0$ but otherwise arbitrary, $t_i = 0$ for $i \neq q^1, \ldots, q^m$ and

$$(1) \qquad |t_{q^k}| \frac{a_{q^k}^{k+1}}{a_{q^{k+1}}^{k+1}} < |t_{q^{k+1}}| < |t_{q^k}| \frac{a_{q^k}^{k}}{a_{q^{k+1}}^{k}}, \quad k = 1, 2, \ldots, m-1.$$

Moreover, if any such choice is made, then

$$q^k(t_1, \dots, t_p) = q^k, \quad k = 1, \dots, m.$$

THEOREM. *A nuclear Fréchet space E with a continuous norm and a basis (y_n) is isomorphic to a subspace of (s) iff the basis is of type d_3.*

Proof. First suppose that E is a subspace of (s) so we have the fundamental system of norms $(\|\cdot\|_k)$ defined above, and, for $x = \sum t_n e_n \in (s)$, we may define

$$q^k(x) = \max\{q\colon \|x\|_k = |t_q|(q)^k\}, \quad k \in \boldsymbol{N}.$$

We may also write

$$y_j = \sum_n \xi_n^j e_n, \; q_j^k = q^k(y_j) \quad \text{for } j, k \in \boldsymbol{N}.$$

Then, for any $j, k \in \boldsymbol{N}$, we have

$$\frac{\|y_j\|_{k+1}}{\|y_j\|_k} = \frac{|\xi^j_{q_j^{k+1}}|(q_j^{k+1})^{k+1}}{|\xi^j_{q_j^k}|(q_j^k)^k} \leqslant \frac{|\xi^j_{q_j^{k+1}}|(q_j^{k+1})^{k+1}}{|\xi^j_{q_j^{k+1}}|(q_j^{k+1})^k} = q_j^{k+1}$$

$$= \frac{|\xi^j_{q_j^{k+1}}|(q_j^{k+1})^{k+2}}{|\xi^j_{q_j^{k+1}}|(q_j^{k+1})^{k+1}} \leqslant \frac{|\xi^j_{q_j^{k+2}}|(q_j^{k+2})^{k+2}}{|\xi^j_{q_j^{k+1}}|(q_j^{k+1})^{k+1}} = \frac{\|y_j\|_{k+2}}{\|y_j\|_{k+1}}$$

so that (δ_3) is satisfied with $\varepsilon_k = 1$, $k \in \boldsymbol{N}$.

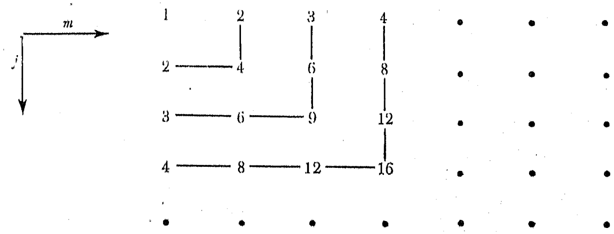
For the converse, we assume that (d_3) holds. Applying a result of Bessaga and A. Pełczyński ([3], Lemma 2), the invariance of (δ_3) under diagonal transformations, Proposition 1 and the fact that (ε_k) may be preassigned, we may assume that E has a fundamental system of seminorms $(|\cdot|_k)$ such that setting $c_j^k = |y_j|_k$ yields the relations

$$(2) \qquad c_j^1 = 1, \; j \in \boldsymbol{N} \quad \text{and} \quad j < \frac{c_j^{k+1}}{c_j^k} \leqslant \frac{1}{2}\,\frac{c_j^{k+2}}{c_j^{k+1}}, \; j, k \in \boldsymbol{N}.$$

Our next step is to observe that we can partition the basis (e_n) into countably many pairwise disjoint infinite subsequences, $(e_{(j,m)})_{(j,m)\in \boldsymbol{N}\times\boldsymbol{N}}$ in such a way that if we set $a^k_{(j,m)} = (jm)^k$ and write

$$\|y\|_k^* = \sup_{(j,m)} |\xi_{(j,m)}| a^k_{(j,m)} \quad \text{for} \quad y = \sum_{(j,m)} \xi_{j,m} e_{j,m} \in (s),$$

then $(\|\cdot\|_k^*)$ is a fundamental system of norms for (s). To see this, we first consider the following array of numbers jm, $j, m \in \boldsymbol{N}$:



If we linearly order this array by taking consecutively blocks consisting of perimeters of increasing squares (as indicated in the diagram) and then ordering within the block according to increasing magnitude, then the νth block has the following $2\nu-1$ terms:

$$\nu, \nu, 2\nu, 2\nu, 3\nu, 3\nu, \dots, (\nu-1)\nu, (\nu-1)\nu, \nu^2$$

which, in the linear order, occur at the places

$$(\nu-1)^2+1, (\nu-1)^2+2, \dots, \nu^2.$$

If we raise each term in the first list to the power $2k$, we get quantities correspondingly larger than if we raise each term in the second list to the power k. Conversely, each term in the first list is less than or equal to the corresponding term in the second list. It follows that the systems $(\|\cdot\|_k)$ and $(\|\cdot\|_k^*)$ define the same topology on (s). (Of course this is just the well-known argument showing that (s) is isomorphic to $(s)\otimes(s)$ ([12], p. 207).)

Now we fix $j \in \boldsymbol{N}$ and proceed to construct a strictly increasing sequence $(q_j^k)_k$ of positive integers such that

$$(3) \qquad j q_j^k < c_j^{k+1}/c_j^k < j q_j^{k+1}, \quad k \in \boldsymbol{N}.$$

To do this we first observe that it follows from (2) that $c_j^2/c_j^1 > j$ so that we may choose q_j^1 to be the largest positive integer such that $c_j^2/c_j^1 > j q_j^1$, and it follows that $q_j^1 \geqslant 1$.

Suppose then that we have chosen $q_j^1 < q_j^2 < \dots < q_j^{k+1}$ so that (3) holds and moreover, q_j^{k+1} is the smallest integer such that the right-hand inequality in (3) holds. Then, since $q_j^{k+1} > q_j^1 \geqslant 1$, it follows from (2) that

$$j q_j^{k+1} \leqslant 2j(q_j^{k+1}-1) < 2c_j^{k+1}/c_j^k \leqslant c_j^{k+2}/c_j^{k+1}.$$

This establishes the left-hand inequality in (3) for k replaced by $k+1$ and also shows that if q_j^{k+2} is chosen to be the smallest integer such that the right-hand inequality in (3) holds with k replaced by $k+1$, then $q_j^{k+2} > q_j^{k+1}$. This completes the definition of (q_j^k).

Let j still be fixed. We claim that there exists an $x_j \in (s)$ of the form

(4) $$x_j = \sum_{k=1}^{j} t_{q_j^k} e_{(j, q_j^k)}$$

such that

(5) $$\|x_j\|_k = c_j^k \quad \text{for} \quad k = 1, \ldots, j.$$

In fact, we may define the coefficients in (4) by

(6) $$t_{q_j^k} = c_j^k / (j q_j^k)^k \quad \text{for} \quad k = 1, \ldots, j.$$

The conditions (6) and (3) yield, for $k = 1, \ldots, j-1$,

$$|t_{q_j^k}| (j\ q_j^k)^{k+1} < |t_{q_j^{k+1}}| (j\ q_j^{k+1})^{k+1} < |t_{q_j^k}| (j\ q_j^k)^k j\ q_j^{k+1},$$

i.e., the coefficients in (4) defined by (6) satisfy condition (1) of the lemma with

$$a_n^k = (jn)^k, \quad p = q_j^j, \quad m = j, \quad q^i = q_j^i \text{ for } i = 1, \ldots, m.$$

Hence the lemma yields (5).

Finally, we note that (x_j) is a block basic sequence in (s) so it is a basis for the space it generates which is therefore a subspace of (s) isomorphic (by (5)) to E. This completes the proof of the theorem.

COROLLARY 1. *If E is a nuclear Fréchet space with a basis of type* d_3, *then every basic sequence in E is of type* d_3.

Proof. We embed E in (s) so that the subspace generated by a basic sequence is also embedded in (s).

The next corollary is a strengthening of the observations made at the end of the last section. It follows from those observations and Corollary 1.

COROLLARY 2. *If E is a nuclear Fréchet space with a basis of type* d_3, *then no basic sequence in E is of type* d_2.

Our last corollary follows from the actual construction in the proof of the theorem.

COROLLARY 3. *Every basic sequence in (s) is semi-equivalent to a block basic sequence with respect to a permutation of the natural basis.*

These results suggest the following questions.

PROBLEM 1. Is the converse of Corollary 1 true? That is, if E is a nuclear Fréchet space with a continuous norm and a basis (x_n), does the fact that no basic sequence in E is of type d_2 imply that (x_n) is of type d_3?

PROBLEM 2. Which nuclear Fréchet spaces E with continuous norm and basis (x_n) have the property that every basic sequence in E is semi-equivalent to a block basic sequence with respect to some permutation of (x_n)?

PROBLEM 3. Which basic sequences in (s) are semi-equivalent to a block basic sequence with respect to the basis (e_n)? (No permutation of (e_n) is permitted.)

Regarding Problem 3 we have some partial information. It is easy to check that in (s) a block basic sequence with respect to (e_n) must be regular and hence, by the theorem, of type d_1. The converse is false, even if we permit a permutation of the block basic sequence.

EXAMPLE. If $a_n^k = e^{n^k}$, $n, k \in N$ and $K(a)$ is the Köthe space determined by the matrix $a = (a_n^k)$, then the coordinate basis in $K(a)$ is of type d_1, but it is not semi-equivalent to a block basic sequence with respect to the basis (e_n) in (s).

Indeed, it is trivial to check that the basis is d_1. On the other hand, if it were semi-equivalent to a block basic sequence (y_n) with respect to (e_n) in (s), then since (e_n) in $K(a)$ and (y_n) are regular, it follows from the theorem of L. Crone and Robinson ([5], Lemma 2) that there is a sequence (d_n) of positive numbers such that $(d_n y_n)$ is equivalent to (e_n) (in $K(a)$). Thus we may write

$$d_n y_n = \sum_{i=p_{n-1}+1}^{p_n} t_i e_i, \quad 0 = p_0 < p_n < p_{n+1}, \ n \in N,$$

and

$$q_n^k = \max\{q : \max_{p_{n-1} < i \leqslant p_n} |t_i| i^k = |t_q| (q)^k\}.$$

From the equivalence, we obtain

(7) $$\forall j \exists\ k \text{ and } M_j > 0 \ni e^{n^j} \leqslant M_j |t_{q_n^k}| (q_n^k)^k \quad \text{for } n \in N,$$

(8) $$\forall k \exists l \text{ and } N_k > 0 \ni |t_{q_n^k}| (q_n^k)^k \leqslant N_k e^{n^l} \quad \text{for } n \in N.$$

Now for any positive integers $k_1 < k_2$, $k_3 < k_4$ and $n \in N$, we have

$$\left(\frac{|t_{q_n^{k_2}}| (q_n^{k_2})^{k_2}}{|t_{q_n^{k_1}}| (q_n^{k_1})^{k_1}} \right)^{\frac{1}{k_2 - k_1}} \leqslant q_n^{k_2} \leqslant p_n < q_{n+1}^{k_3} \leqslant \left(\frac{|t_{q_{n+1}^{k_4}}| (q_{n+1}^{k_4})^{k_4}}{|t_{q_{n+1}^{k_3}}| (q_{n+1}^{k_3})^{k_3}} \right)^{\frac{1}{k_4 - k_3}}.$$

We may consider that (7) defines a function $j \rightsquigarrow k(j)$ and (8) defined a function $k \rightsquigarrow l(k)$. We set $k_3 = k(1)$, $k_4 = k_3 + 1$, $l_1 = l(k_4)$, $k_1 = 1$, $l_2 = l(1)$, $j_1 = l_1 + l_2 + 1$ and $k_2 = k(j_1)$. We then obtain

$$\left(\frac{e^{n^{l_1 + l_2 + 1}}}{N_1 e^{n^{l_2}}} \right)^{\frac{1}{k_2 - 1}} \leqslant \frac{M_1 N_{k_4} e^{(n+1)^{l_1}}}{e^{n+1}}, \quad n \in N$$

which is not true for large n.

Subspaces of (s) without bases. Our theorem gives a complete characterization of subspaces of (s) with bases. Very little seems to be known about subspaces of (s) which do not have a basis. Using our theorem it is easy to show that certain examples of nuclear Fréchet spaces without bases which have been constructed ([2], [6]) cannot be embedded in (s). For instance, in the specific space constructed by P. Djakov and Mitiagin ([6], Equation (1.6)) we have the space

$$X = \Big\{x \in H:\ \|x\|_p^2 = \sum_{n=1}^{\infty} \|A_{np}x_n\|^2 < \infty,\ \forall p\Big\}.$$

Here

$$H = \Big\{x = (x_n):\ x_n = (x_n^0, x_n^1) \in H^2 \text{ and } \sum_n \|x_n\|^2 < \infty\Big\}.$$

H^2 is a two-dimensional Hilbert space and each A_{np} is an operator on H^2. The example is then determined by specifying the A_{np}.

Now, if one takes the subspace of X consisting of those $x = (x_n)$ in which each $x_n^1 = 0$, then this subspace has a basis and one can check that, if A_{np} is given by the equations (1.6) of [6], then the basis is not of type d_3 so the subspace, and hence x, cannot be embedded in (s). A similar argument applies to the example constructed by Bessaga [2].

The space (s) does, however, contain a subspace which has no basis. In fact, if one takes, in the notation of [6],

$$a_{np} = \begin{cases} 2^{n2^p}, & p \leqslant p_1, \\ 2^{n2^{p+p_2}}, & p > p_1, \end{cases} \qquad b_{np} = \begin{cases} 2^{n2^{p+p_1}}, & p \leqslant p_2, \\ 2^{n2^{p+p_3}}, & p > p_2, \end{cases}$$

then the space X of Djakov and Mitiagin has no basis but it can be embedded in (s). Instead of giving the details we refer to [10] in which a more general result is proved.

In trying to determine *all* subspaces of (s) a first step might be to answer the following question:

PROBLEM 4. Is it true that a nuclear Fréchet space E with continuous norm is isomorphic to a subspace of (s) if and only if every basic sequence in E generates a space isomorphic to a subspace of (s)?

This problem might also be interesting if (s) is replaced by some other nuclear Fréchet space.

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