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A characterization of Gaussian measures on Banach spaces

by

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Abstract. This paper is an outgrowth of my work [11] concerning decomposability semigroups of linear operators associated with probability measures on Euclidean tpaces. Its aim is to characterize Gaussian measures on separable Banach spaces in serms of decomposability properties.

The present paper is concerned with probability measures defined on Borel subsets of a separable real Banach space X. The set of all such measures under weak convergence can be metrized as a separable metric space ([7], p. 43). The adjoint space X^* in the X-topology is also separable. Given a linear operator A and a probability measure λ on X, we denote by $A\lambda$ the probability measure defined by the formula $A\lambda(S) = \lambda(A^{-1}(S))$ for all Borel subsets S of X. The decomposability semigroup $E(\lambda)$ consists of all linear operators A in X for which the equality

$$\lambda = A\lambda * \lambda_A \tag{1}$$

holds for a certain probability measure λ_A. Here $*$ denotes the convolution operation. Recently I proved that in the case of Euclidean spaces some probabilistic properties of measures can be expressed in terms of algebraic and topological properties of their decomposability semigroups ([10], [11]).

Now we quote the classical concept of decomposability of measures ([5], p. 78; [7], p. 64). Let $x \in X$. By δ_x we denote the probability measure concentrated at the point x. Measures not concentrated at a single point are called *nondegenerate*.

A measure λ is said to be *decomposable* if and only if there exist two nondegenerate measures μ and ν such that $\lambda = \mu * \nu$. In the contrary case, λ is said to be *indecomposable*. A measure α is said to be a *factor of* λ whenever $\lambda = \alpha * \beta$ for a certain probability measure β. We say that λ *has no indecomposable factor* whenever each nondegenerate factor of λ is decomposable.

The related concept of operator-decomposability has been introduced in [11]. Namely, λ is said to be *operator-decomposable* if and only if there exists a one-to-one operator A from $E(\lambda)$ such that the measure λ_A in decomposition (1) is nondegenerate. Then the measure $A\lambda$ is also non-degenerate, which shows that operator-decomposable measures are decomposable. Further, we say that λ *has no operator-indecomposable factors* whenever each non-degenerate factor of λ is operator-decomposable. Our aim is to examine probability measures on X without operator-indecomposable factors.

The characteristic functional $\hat{\lambda}$ is defined on X^* by means of the formula

$$\hat{\lambda}(x^*) = \int_X e^{ix^*(x)}\lambda(dx) \quad (x^* \in X^*).$$

For basic properties of characteristic functionals we refer to [2] (Chapter 5.3), [3] (Chapter 6.2), [6] and [12] (Chapter 4.1). In particular, $\hat{\lambda}$ determines λ,

(2) $$|\hat{\lambda}(x^*)| \leqslant 1 \quad (x^* \in X^*),$$

$\widehat{\mu * \nu} = \hat{\mu}\hat{\nu}$ and $\hat{\lambda}$ is continuous in the X-topology of X^*. Put

$$r_\lambda = \inf\{\|x^*\|:\ \hat{\lambda}(x^*) = 0\},$$

where the infimum of the empty set is assumed to be ∞. By the continuity of $\hat{\lambda}$ and the condition $\hat{\lambda}(0) = 1$ we have $r_\lambda > 0$. Moreover,

(3) $$r_\lambda \leqslant r_\mu$$

for all factors μ of λ. Let T be the countable product of unit intervals with the weak topology. By $dt_1 dt_2 \ldots$ we shall denote the integration with respect to the countable product of the Lebesgue measures on the unit intervals. Suppose that functionals $x_1^*, x_2^*, \ldots$ non-vanishing identically form a dense subset of X^* in the X-topology. Put

(4) $$y_n^* = \frac{x_n^*}{2^n\|x_n^*\|} \quad (n = 1, 2, \ldots).$$

Obviously, for every sequence $t_1, t_2, \ldots$ of real numbers from the unit interval the series $\sum_{n=1}^{\infty} t_n y_n^*$ strongly converges in X^* and $\|\sum_{n=1}^{\infty} t_n y_n^*\| \leqslant 1$. Given a probability measure λ on X for every r with $0 < r < r_\lambda$, we put

(5) $$m_r(\lambda) = -\int_T \log\left|\hat{\lambda}\left(r\sum_{n=1}^{\infty} t_n y_n^*\right)\right| dt_1 dt_2 \ldots$$

The functional m_r is finite, non-negative and

(6) $$m_r(\mu * \nu) = m_r(\mu) + m\ (\nu).$$

Moreover, m_r is invariant under shift transformation $\lambda \to \lambda * \delta_x$ and $m_r(\mu) \to m_r(\lambda)$ whenever $\mu \to \lambda$ and $r < r_\lambda$. The symmetrization λ^s of λ is defined by the formula $\lambda^s = \lambda * \tilde{\lambda}$, where $\tilde{\lambda}(S) = \lambda(-S)$ for all Borel subsets S of X. By (6), we have the formula

(7) $$m_r(\lambda^s) = 2m_r(\lambda).$$

Let us denote by D an arbitrary decomposition

$$\lambda = \lambda_1 * \lambda_2 * \ldots * \lambda_k$$

and put for $r < r_\lambda$

$$m_r(\lambda, D) = \max_{1 \leqslant j \leqslant k} m_r(\lambda_j).$$

We note that, by (3), $r_\lambda \leqslant r_{\lambda_j}$ $(j = 1, 2, \ldots, k)$. Further, let us denote by $M_r(\lambda)$ the greatest lower bound of $m_r(\lambda, D)$ for all possible decompositions D of λ. It is very easy to verify that

(8) $$M_r(\lambda) \leqslant \max_{1 \leqslant j \leqslant k} M_r(\mu_j)$$

whenever $\lambda = \mu_1 * \mu_2 * \ldots * \mu_k$.

Lemma 1. *Let $0 < r < r_\lambda$. Then $m_r(\lambda) = 0$ if and only if λ is concentrated at a single point.*

Proof. The formula $m_r(\delta_x) = 0$ is evident. To prove the converse implication, by (7) it suffices to show that the equality $m_r(\lambda^s) = 0$ yields $\lambda^s = \delta_0$. By (2), the equality $m_r(\lambda^s) = 0$ and the symmetry of λ^s imply

(9) $$\hat{\lambda}^s\left(\sum_{n=1}^{\infty} t_n y_n^*\right) = 1$$

for almost all elements $t_1, t_2, \ldots$ of T. By continuity we get formula (9) for all $t_1, t_2, \ldots$ in T. Hence, in particular, it follows that $\hat{\lambda}^s(ty_n^*) = 1$ $(n = 1, 2, \ldots;\ 0 \leqslant t \leqslant 1)$. Taking into account the well-known property of the characteristic function of number-valued random variables ([4], Chapter 14.2; [5], p. 108), we get the formula $\hat{\lambda}^s(ty_n^*) = 1$ for all real numbers t and positive integers n. Consequently, by (4), $\hat{\lambda}^s(x_n^*) = 1$ $(n = 1, 2, \ldots)$, which by the density of $x_1^*, x_2^*, \ldots$ in the X-topology of X^* yields $\hat{\lambda}^s(x^*) = 1$ for all $x^* \in X^*$. Thus $\lambda^s = \delta_0$, which completes the proof.

Lemma 2. *If λ has no indecomposable factor, then $M_r(\lambda) = 0$ $(0 < r < r_\lambda)$.*

Proof. The idea of the proof is similar to that in [5] (p. 115), where the case of number-valued random variables has been considered. It follows from the definition of $M_r(\lambda)$ that there exists a sequence of decompositions D_n, say $= \lambda_{n1} * \lambda_{n2} * \ldots * \lambda_{nk_n}$ $(n = 1, 2, \ldots)$, for which $m_r(\lambda, D_n)$

converges to $M_r(\lambda)$. Let $\mu_n = \lambda_{ns_n}(1 \leqslant s_n \leqslant k_n)$ be the factor of D_n for which $m_r(\mu_n) = m_r(\lambda, D_n)$, and write ν_n for the convolution of all other factors of D_n. Then $\lambda = \mu_n * \nu_n$ $(n = 1, 2, \ldots)$ and

$$\lim_{n\to\infty} m_r(\mu_n) = M_r(\lambda).$$

The sequences $\mu_1, \mu_2, \ldots$ and $\nu_1, \nu_2, \ldots$ are both shift compact ([7], p. 59). Since the functional m_r is invariant under shift transformation, we may assume without loss of generality that the sequences $\mu_1, \mu_2, \ldots$ and $\nu_1, \nu_2, \ldots$ both converge to μ and ν, respectively,

$$\lambda = \mu * \nu \tag{10}$$

and, by virtue of the continuity of m_r, $m_r(\mu) = M_r(\lambda)$. We show next by means of an indirect proof that $M_r(\lambda) \leqslant \frac{1}{2} m_r(\lambda)$. Let us therefore suppose that $M_r(\lambda) > \frac{1}{2} m_r(\lambda)$; it then follows from (6) and (10) that $m_r(\nu) < M_r(\lambda)$. Since μ is also decomposable, there exist two nondegenerate measures μ_1 and μ_2 such that $\mu = \mu_1 * \mu_2$. By Lemma 1, $m_r(\mu_1) > 0$ and $m_r(\mu_2) > 0$. Taking into account the formula $m_r(\mu_1) + m_r(\mu_2) = m_r(\mu) = M_r(\lambda)$, we have the inequality

$$M_r(\lambda) \leqslant \max\big(m_r(\mu_1), m_r(\mu_2), m_r(\nu)\big) < M_r(\lambda),$$

which gives a contradiction. Thus $M_r(\lambda) \leqslant \frac{1}{2} m_r(\lambda)$ for every probability measure λ without indecomposable factors.

Let D be an arbitrary decomposition $\lambda = \lambda_1 * \lambda_2 * \ldots * \lambda_n$ into non-degenerate factors. Since λ_j have no indecomposable factors, we have the inequality $M_r(\lambda_j) \leqslant \frac{1}{2} m_r(\lambda_j)$ $(j = 1, 2, \ldots, n)$. Consequently, by (8),

$$M_r(\lambda) \leqslant \tfrac{1}{2} \max_{1\leqslant j\leqslant n} m_r(\lambda_j),$$

which, by the definition of the functional M_r implies the inequality $M_r(\lambda) \leqslant \frac{1}{2} M_r(\lambda)$. Thus $M_r(\lambda) = 0$, which completes the proof.

A probability measure λ is said to be *infinitely divisible* whenever for every positive integer n there exists a probability measure λ_n such that $\lambda = \lambda_n^{*n}$, where the power is taken in the sense of convolution. For the theory of infinitely divisible probability measures on Banach spaces we refer to [8], [9] and [1]. In particular, if F is any bounded non-negative Borel measure on X vanishing at the origin, the Poisson measure $e(F)$ associated with F is defined as

$$e(F) = e^{-F(X)} \sum_{k=0}^{\infty} \frac{1}{k!} F^{*k},$$

where $F^{*0} = \delta_0$. The correspondence $F \leftrightarrow e(F)$ is one-to-one. Moreover

$$e(F) * e(G) = e(F+G) \tag{11}$$

and

$$Ae(F) = e(AF) \tag{12}$$

for any linear operator A in X.

Let H be a not necessarily bounded Borel measure on X vanishing at the origin. If there exists a representation $H = \sup_n F_n$, where F_n are bounded and the sequence $e(F_1), e(F_2), \ldots$ of associated Poisson measures is shift compact, then each limit point of translates $e(F_1), e(F_2), \ldots$ will be called a *generalized Poisson measure* and denoted by $\tilde{e}(H)$. It is clear that $\tilde{e}(H)$ is uniquely defined up to a shift transformation, i.e., for two limit points, say μ_1 and μ_2 of translates $e(F_1), e(F_2), \ldots$, there exists an element $x \in X$ such that $\mu_1 = \mu_2 * \delta_x$ ([8], p. 313). A. Tortrat proved in [8] p. 311 (see also [1], p. 22) the following analogue of the Lévy–Khinchine representation of infinitely divisible laws: each infinitely divisible measure λ on X has a unique representation $\lambda = \varrho * \tilde{e}(H)$, where ϱ is a symmetric Gaussian measure and $\tilde{e}(H)$ a generalized Poisson measure. By a *Gaussian measure on* X we mean such a measure ϱ that for every $x^* \in X^*$ the induced measure $x^*\varrho$ on the real line is Gaussian. The following Lemma is an extension of the Khinchine theorem ([5], p. 115) to the case of the Banach space.

LEMMA 3. *Probability measures without indecomposable factors are infinitely divisible.*

Proof. Let λ be a measure without indecomposable factors. By Lemma 2, $M_r(\lambda) = 0$ $(0 < r < r_\lambda)$. Consequently, there exists a sequence of decompositions D_n, say $\lambda = \lambda_{n1} * \lambda_{n2} * \ldots * \lambda_{nk_n}$ $(n = 1, 2, \ldots)$, for which

$$\lim_{n\to\infty} m_r(\lambda, D_n) = 0. \tag{13}$$

Let d be a distance function in the space of all probability measures on X. By x_{nj} we denote an element of X satisfying the condition

$$d(\lambda_{nj} * \delta_{x_{nj}}, \delta_0) \leqslant d_{nj} + \tfrac{1}{n}, \tag{14}$$

where

$$d_{nj} = \inf_{x\in X} d(\lambda_{nj} * \delta_x, \delta_0)$$

$(j = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots)$. Let $j_1, j_2, \ldots$ be an arbitrary sequence of indices satisfying the condition $1 \leqslant j_n \leqslant k_n$ $(n = 1, 2, \ldots)$. By Theorem 2.2 in [7], p. 59, the sequence of corresponding probability measures λ_{nj_n} is shift compact, i.e., the sequence $\lambda_{nj_n} * \delta_{y_n}$ is compact for suitably chosen elements y_n from X. Let μ be its limit point. Of course, μ is a factor of λ and, by (13), $m_r(\mu) = 0$. Thus by Lemma 1, μ is concentrated at a single point, which yields the relation $\lim_{n\to\infty} d_{nj_n} = 0$. Consequently,

$$\lim_{n\to\infty} \max_{1\leqslant j\leqslant k_n} d_{nj} = 0,$$

which, by (14), implies

$$\lim_{n\to\infty}\max_{1\leqslant j\leqslant k_n} d(\lambda_{nj}*\delta_{x_{nj}}, \delta_0) = 0.$$

In other words, the triangular array $\mu_{nj} = \lambda_{nj}*\delta_{x_{nj}}$ $(j = 1, 2, \ldots, k_n;$ $n = 1, 2, \ldots)$ satisfies the uniform infinitesimality condition and

$$\lambda = \mu_{n1}*\mu_{n2}*\ldots*\mu_{nk_n}*\delta_{x_n},$$

where $x_n = -\sum_{j=1}^{k_n} x_{nj}$ $(n = 1, 2, \ldots)$. Given a subspace Y of X with a finite codimension, we denote by P the natural mapping from X onto X/Y. Obviously, the triangular array $P\mu_{nj}$ $(j = 1, 2, \ldots, k_n; n = 1, 2, \ldots)$ satisfies the uniform infinitesimality condition. By the extended central limit theorem valid for finite-dimensional spaces the probability measure $P\lambda$ is infinitely divisible ([4], p. 310; [7], p. 199). Hence, it follows that λ itself is also infinitely divisible ([1], Chapter II, Theorem 2.1), which completes the proof.

LEMMA 4. *Let F be a bounded measure vanishing at the origin. If $e(F)$ has no indecomposable factor, then it is operator-indecomposable.*

Proof. Let A be a one-to-one operator from $E(e(F))$. Then

(15) $$AF(\{0\}) = 0.$$

Consider the decomposition

(16) $$e(F) = Ae(F)*\nu.$$

The measure ν has no indecomposable factor either and, by Lemma 3, is infinitely divisible. By (11), (12), (16) and the uniqueness of the Tortrat representation $\nu = \varrho*\tilde{e}(H)$, we have the formulas $\varrho = \delta_0$ and $H = F - AF$. Thus $AF(S) \leqslant F(S)$ for all Borel subsets S of $X\setminus\{0\}$. Moreover, by (15)

$$AF(X\setminus\{0\}) = F(X\setminus\{0\}),$$

which yields the equality $AF = F$. Consequently $\nu = \delta_0$, which shows that $e(F)$ is operator-indecomposable.

THEOREM. *A probability measure on a separable Banach space has no operator-indecomposable factor if and only if it is nondegenerate Gaussian.*

Proof. The sufficiency of the condition is obvious. Namely, it is a consequence of the Cramer decomposition theorem ([4], p. 271) and the formula $\mu = c_1I\mu*c_2I\mu*\delta_x$ for every Gaussian measure μ, where I is the unit operator, $c_1^2+c_2^2 = 1$ and x a suitable element of X. To prove the necessity we note that a measure λ without an operator-indecomposable factor has no indecomposable factor. Consequently, by Lemma 3, λ is infinitely divisible and admits the Tortrat representation $\lambda = \varrho*\tilde{e}(H)$, where ϱ is a symmetric Gaussian measure and $\tilde{e}(H)$ a generalized Poisson measure. Suppose that H does not vanish identically on $X\setminus\{0\}$. Then we can find a bounded measure $F \leqslant H$ having the same property and

vanishing at the origin. Evidently $e(F)$ is a factor of λ and, by Lemma 4, is operator-indecomposable, which gives a contradition. Thus H vanishes identically on $X\setminus\{0\}$. In other words, $\lambda = \varrho*\delta_c$, where ϱ is a nondegenerate Gaussian factor. The Theorem is thus proved.

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