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Banach–Lie algebras of compact operators

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Abstract. Let $B_c(X)$ be the Banach algebra of all compact endomorphisms of a Banach space X. It is proved that if a closed Lie subalgebra L of $B_c(X)$ is simple, then either L contains a finite-rank operator or all elements of L are quasinilpotent operators.

1. Introduction. A *normed* (*Banach*) *Lie algebra* (shortly, a B–L *algebra*) is a normed (Banach) space and a Lie algebra, with a Lie bracket $(a, b) \to [a, b]$, which is continuous in the norm topology of L.

For a Banach space X let $B(X)$, $B_c(X)$ and $B_0(X)$ denote the associative algebras of all continuous endomorphisms of X, all compact continuous endomorphisms of X and all finite-rank continuous endomorphisms of X, respectively. Since each associative Banach algebra has a natural Lie algebra structure under the Lie bracket $[a, b] = ab - ba$, we can regard $B(X)$ and $B_c(X)$ as B–L algebras and $B_0(X)$ as a normed Lie algebra.

Let L be a B–L algebra. For $a \in L$ define $\mathrm{Ad}_a \in B(L)$ by the formula $\mathrm{Ad}_a(b) = [a, b]$.

Definition. We introduce three classes of B–L algebras, saying that a B–L algebra L is:

(I) *simple* — if it has no proper closed ideals (i.e., if there exists no proper closed subspace of L invariant for all Ad_a with $a \in L$) and if $\dim L > 1$;

(II) *nilpotent* — if for each $a \in L$ the operator Ad_a is quasinilpotent (i.e., if the spectrum $\sigma(\mathrm{Ad}_a)$ is equal to $\{0\}$);

(III) *locally finite* — if there exists a dense subalgebra L_0 of L such that each finite subset of L_0 is contained in a finite-dimensional subalgebra.

In this note we exhibit some connections between classes (I), (II) and (III) in the case of B–L algebras of compact operators, i.e., closed Lie subalgebras of $B_c(X)$. In particular, our Theorem 4 implies that a simple B–L algebra of compact operators has to be locally finite or nilpotent. (We do not know whether the second possibility can occur.) The situation in the class of simple B–L algebras of compact operators is thus very similar to that of the case of separable L^* algebras (cf. [7], [8]).

We postpone a more detailed discussion of these topics as well as a discussion of some open questions to the last part of this note. We are indebted to several people for helpful discussions or suggestions: especially to C. Atkin, S. Kwapień, B. Mitiagin, I. Stewart and P. Wojtaszczyk.

2. Notation and preliminaries. All spaces and algebras we discussed in this paper are assumed to be over C — the field of complex numbers. Let X be a B-space and let $A \in B(X)$. By $\sigma(A)$ we denote the *spectrum of* A. For each complex number λ define

$$X_A^\lambda = \{x \in X:\ \|(A-\lambda)^n x\|^{1/n} \underset{n}{\to} 0\}.$$

X_A is a linear (in general not closed) subspace of X, not depending on the choice of norm in X, and $X_A \neq \{0\}$ implies that $\lambda \in \sigma(A)$. (Note that $\lambda \in \sigma(A)$ does not imply that $X_A^\lambda \neq \{0\}$.)

The following three propositions can easily be derived from the general spectral theory of bounded operators, as presented in [1], Chapter 7, and we omit the proofs.

PROPOSITION 1. *Let $A \in B(X)$ and let $\sigma(A) = W_1 \cup W_2$ be a decomposition of $\sigma(A)$ as the disjoint union of two closed subsets. There exists a unique direct sum decomposition*

$$X = X_1 \oplus X_2 \tag{1}$$

such that, for $i = 1, 2$, X_i is A-invariant, and, if A_i is the restriction of A to X_i, we have $\sigma(A_i) = W_i$.

In the following we shall refer to (1) as the decomposition of X associated with the decomposition $\sigma(A) = W_1 \cup W_2$.

PROPOSITION 2. *Let $A \in B(X)$ and $X = X_1 \oplus X_2$, where X_1 and X_2 are A-invariant. Let A_1 be the restriction of A to X_1.*

If $\lambda \notin \sigma(A_1)$, then $X_A^\lambda \subset X_2$.

Hint for the proof. There is a constant C such that for $y \in X$, $y = y_1 + y_2$ with $y_i \in X_i$ we have $\|y_1\| \leqslant C\|y\|$. In particular, for $x \in X_A^\lambda$, $x = x_1 + x_2$, we have $\|(A-\lambda)^n x_1\| \leqslant C\|(A-\lambda)^n x\|$.

PROPOSITION 3. *With the same notation as in Proposition 1 let λ_0 be an isolated point of $\sigma(A)$ and put $W_1 = \{\lambda_0\}$, $W_2 = \sigma(A)\setminus\{\lambda_0\}$. Then $X_1 = X_A^\lambda$. In particular, $X_A^\lambda \neq \{0\}$.*

Let $\mathscr{P}$ be the family of all polynomials of one complex variable. For $U \subset \boldsymbol{C}$ define $\mathrm{pc}(U)$ — *the polynomially convex cover* of U, by

$$\mathrm{pc}(U) = \{z \in \boldsymbol{C}:\ |P(z)| \leqslant \sup_{\lambda\in U} |P(\lambda)| \text{ for } P \in \mathscr{P}\}.$$

PROPOSITION 4. *Let $A \in B(X)$ and Y be a closed A-invariant subspace of X. Let A_Y be the restriction of A to Y. We have $\sigma(A_Y) \subset \mathrm{pc}(\sigma(A))$.*

Proof. Let $P \in \mathscr{P}$ and let $P(A)$, $P(A_Y)$ be the corresponding polynomials of A and A_Y, respectively. Since Y is $P(A)$-invariant and $P(A)$ restricted to Y is equal to $P(A_Y)$, we have

$$\begin{aligned}\sup_{\lambda\in\sigma(A)} |P(\lambda)| &= \sup_{\lambda\in\sigma(P(A))} |\lambda| = \lim_{n\to\infty} \|P(A)^n\|^{\frac{1}{n}}\\ &\geqslant \lim_{n\to\infty} \|P(A_Y)^n\|^{\frac{1}{n}} = \sup_{\lambda\in\sigma(P(A_Y))} |\lambda|\\ &= \sup_{\lambda\in\sigma(A_Y)} |P(\lambda)|,\end{aligned}$$

and this means that $\sigma(A_Y) \subset \mathrm{pc}(\sigma(A))$.

COROLLARY 5. *With the notation of Proposition 4, let $\sigma(A)$ be countable. Then $\sigma(A_Y) \subset \sigma(A)$.*

Proof. This is a consequence of Proposition 4 and the fact that, for U countable compact, $U = \mathrm{pc}(U)$ (cf. [3] Theorem 1.3.3).

COROLLARY 6. *Let L be a* B–L *algebra of compact operators. For $A \in L$ we have*

$$\sigma(\mathrm{Ad}_A) \subset \sigma(A) - \sigma(A) \tag{2}$$

and, in particular, $\sigma(Ad_A)$ is countable.

Proof. Let L be a closed Lie subalgebra of $B_c(X)$. For $A \in L$ define D_A: $B_c(X) \to B_c(X)$ by

$$D_A(B) = AB - BA \quad \text{for} \quad B \in B_c(X).$$

By a theorem of Rosenblum ([6]), $\sigma(D_A) \subset \sigma(A) - \sigma(A)$; hence $\sigma(D_A)$ is countable. Since Ad_A is the restriction of D_A to its invariant subspace L, by Corollary 5 we get (2).

3. Nilpotent algebras. Let $A \in B_c(X)$. We shall call A a *Volterra operator* if $\sigma(A) = \{0\}$.

DEFINITION 7. A B–L subalgebra of $B_c(X)$ is a *Volterra algebra* if each $A \in L$ is a Volterra operator.

PROPOSITION 8. *Let L_0 be a Lie subalgebra of $B_c(X)$ such that each $A \in L_0$ is a quasinilpotent operator. Then L, the uniform closure of L_0 in $B_c(X)$, is a nilpotent* B–L *subalgebra of $B_c(X)$. In particular, each Volterra algebra is nilpotent.*

Proof. Let $A_n \in L_0$ and $A = \lim_{n\to\infty} A_n$; then $\sigma(A) = \{0\}$ and, by Corollary 6, $\sigma(\mathrm{Ad}_A) = \{0\}$.

Proposition 8 is convenient for obtaining examples of Volterra algebras.

EXAMPLE 9. Let $e_n \in X$, $f_n \in X^*$ be a biorthogonal system, i.e., $f_m(e_n) = \delta_n^m$. Let L_0 be Lie algebra of finite-rank operators of the form

$$Ax = \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} a_{nk} f_k(x)\Big) e_n,$$

where $a_{nk} = 0$ for $n \geqslant k$, and only a finite number of a's does not vanish. One easily checks that $A^m = 0$ for m large enough, and hence, by Proposition 8, the uniform closure of L_0 is a Volterra algebra. In particular, if X is a Hilbert space and (e_n) is an orthonormal basis, we obtain Volterra algebras of operators having uppertriangular matrices with respect to (e_n).

EXAMPLE 10. Let L_0 be the set of all integral operators

$$(Af)(t) = \int_0^1 K(x, t) f(x)\, dx \tag{3}$$

acting in the space $L^2[0, 1]$, with kernels $K(x, t)$ being continuous functions on $[0, 1]\times[0, 1]$ and satisfying $K(x, t) = 0$ for $t+x \geqslant 1$. Since each operator of form (3) is a Volterra operator L, the closure of L_0 is a Volterra algebra.

Remark 11. Let $[L, L]$ denote the closed ideal generated by all $[A, B]$ with $A, B \in L$.

It is not difficult to check that in Example 9 $[L, L]$ is a proper ideal in L, but in Example 10 $[L, L] = L$.

EXAMPLE 12. Let $H = \bigoplus_{i=1}^{\infty} H_i$ be an orthogonal decomposition of a Hilbert space H with H_i finite-dimensional for $i = 1, 2, \ldots$ Let L_i be a nilpotent Lie algebra of endomorphisms of H_i and define L_∞ as the Lie subalgebra of $B_0(H)$ of all operators having the form $A = \bigoplus_{i=1}^{\infty} A_i$ with A_i belonging to H_i, and $A_i = 0$ for large i. Denote by L the completion of L_∞ with respect to the norm topology. L is nilpotent but not necessarily a Volterra algebra.

LEMMA 13. *Let $A, B \in B(X)$. If $\|\mathrm{Ad}_A^n(B)\|^{1/n} \underset{n}{\to} 0$, then for each λ the space X_A^λ is B-invariant.*

Proof. Let $x_0 \in X_A^\lambda$ and write

$$\alpha_j = \|\mathrm{Ad}_A^j(B)\| 2^j,$$
$$\beta_j = \|(A-\lambda)^j x_0\| 2^j,$$
$$\gamma_n = \max_{j \leqslant n} \alpha_{n-j+1}\beta_j .$$

Obviously, $\alpha_n^{1/n} \underset{n}{\to} 0$, $\beta_n^{1/n} \underset{n}{\to} 0$ and one easily checks that

$$\gamma_n^{1/n} \underset{n}{\to} 0 . \tag{4}$$

A simple inductive argument yields

$$[T, T^k B] = \sum_{j=0}^{k} \binom{k}{j} \big(\mathrm{Ad}_T^{k+1-j}(B)\big)\circ T^j \quad \text{for} \quad T \in B(X),$$

and thus

$$[T-\lambda, (T-\lambda)^k B] = \sum_{j=0}^{k} \binom{k}{j} \big(\mathrm{Ad}_T^{k+1-j}(B)\big)\circ (T-\lambda)^j, \tag{5}$$

but

$$(T-\lambda)^{n+1} B = \Big(\sum_{k=0}^{n} [(T-\lambda), (T-\lambda)^{n-k} B]\circ (T-\lambda)^k\Big) + B(T-\lambda)^{n+1}$$

and, using (5), we get for A and B

$$(A-\lambda)^{n+1} B = \Big(\sum_{k=0}^{n}\sum_{j=0}^{n-k} \binom{n-k}{j} \big(\mathrm{Ad}_A^{n+1-k-j}(B)\big)(A-\lambda)^{j+k}\Big) + B(A-\lambda)^{n+1};$$

hence

$$\|(A-\lambda)^{n+1}(Bx_0)\| \leqslant \Big(\sum_{k=0}^{n}\Big(\frac{1}{2}\Big)^k \sum_{j=0}^{n-k}\binom{n-k}{j} \alpha_{n-j-k+1}\beta_{j+k}\Big(\frac{1}{2}\Big)^{n-k-j+1}\Big(\frac{1}{2}\Big)^j\Big) + \|B\|\beta_{n+1}\Big(\frac{1}{2}\Big)^{n+1}$$

$$\leqslant \sum_{k=0}^{n}\Big(\frac{1}{2}\Big)^{k+1}\gamma_n \sum_{j=0}^{n-k}\binom{n-k}{j}\Big(\frac{1}{2}\Big)^{n-k-j}\Big(\frac{1}{2}\Big)^j + \|B\|\beta_{n+1}\Big(\frac{1}{2}\Big)^{n+1};$$

since

$$\sum_{j=0}^{n-k}\binom{n-k}{j}\Big(\frac{1}{2}\Big)^{n-k-j}\Big(\frac{1}{2}\Big)^j = 1,$$

we get

$$\|(A-\lambda)^{n+1}(Bx_0)\| \leqslant \gamma_n + \|B\|\beta_{n+1}(\tfrac{1}{2})^{n+1},$$

and by (4),

$$\|(A-\lambda)^n(Bx_0)\|^{1/n} \underset{n}{\to} 0, \quad \text{i.e.} \quad Bx_0 \in X_A^\lambda .$$

THEOREM 1. *Let L be a nilpotent* B–L *subalgebra of $B_c(X)$. Then either there exists a finite-dimensional L-invariant subspace of X or L is a Volterra algebra.*

Proof. If L is not a Volterra algebra, then there exists an $A \in L$ with $\sigma(A) \neq \{0\}$. Let $0 \neq \lambda_0 \in \sigma(A)$. Since A is compact, the space $X_A^{\lambda_0}$ is finite-dimensional and not $\{0\}$. For $B \in L$ we have $\|\mathrm{Ad}_A^n B\|^{1/n} \underset{n}{\to} 0$; hence, by Lemma 13, $X_A^{\lambda_0}$ is B-invariant. Thus $X_A^{\lambda_0}$ is L-invariant.

THEOREM 2. *Let L be a nilpotent* B–L *algebra of compact operators. Then either L has a proper closed ideal or L is a Volterra algebra.*

Proof. If L is not a Volterra algebra, then by Theorem 1, L has a non-trivial finite-dimensional representation ϱ; thus either L is finite dimensional (and hence it has a proper closed ideal) or the kernel of ϱ is a non-trivial ideal of L.

4. Locally finite algebras.

PROPOSITION 14. *Let L be a separable* B–L *algebra. L is locally finite iff there is an increasing sequence $L_1 \subset L_2 \subset L_3 \subset \ldots$ of finite-dimensional subalgebras of L such that $\bigcup_{n=1}^{\infty} L_n$ is dense in L.*

Proof. "If": Put $L_0 = \bigcup_{n=1}^{\infty} L_n$. Then L_0 satisfies the conditions in Definition (III) of the Introduction.

"Only if": Assume that L_0 is a dense subalgebra of L and each finite subset of L_0 is contained in a finite-dimensional subalgebra. Let $\{a_1, a_2, a_3, \ldots\}$ be a dense subset of L_0, and define L_n as the smallest subalgebra containing $\{a_1, \ldots, a_n\}$. L_n is finite-dimensional for $n = 1, 2, \ldots$, $L_n \subset L_{n+1}$, and $\bigcup_{n=1}^{\infty} L_n$ is a dense subalgebra of L.

Remark 15. Let L be a Lie subalgebra of $B_0(X)$. Each finite subset of L is contained in a finite-dimensional subalgebra of L.

Proof. See [4], p. 24.

COROLLARY 16. *Let L be a simple* B–L *algebra of operators. If L contains a finite-rank operator, then L is locally finite.*

Proof. Let L_0 be the ideal of finite-rank operators of L. Since L is simple, L_0 is dense in L, and, by Remark 15, L is locally finite.

THEOREM 3. *A* B–L *algebra of compact operators which is not nilpotent contains an operator of finite rank.*

Proof. Assume that L is not a nilpotent algebra, i.e., that there exists an $A \in L$ with $\sigma(\mathrm{Ad}_A) \neq \{0\}$. Since, by Corollary 6, $\sigma(\mathrm{Ad}_A)$ is countable compact, it contains a non-zero isolated point μ. Proposition 3 implies that there exists a $C \neq 0$, $C \in L$ such that

$$\lim_{n\to\infty} \|(\mathrm{Ad}_A - \mu)^n C\|^{1/n} = 0. \tag{6}$$

By (2), μ can be written (in finitely many ways, because 0 is the only accumulation point of $\sigma(A)$) in the form

$$\mu = \lambda_i - \lambda_i' \tag{7}$$

with λ_i, λ_i' belonging to $\sigma(A)$.

Let

$$W_1 = \{\lambda \in \sigma(A)\colon \lambda \neq 0 \text{ and } \exists\, \lambda' \in \sigma(A) \text{ such that } \lambda - \lambda' = \pm\mu\}.$$

W_1 is a finite subset of $\sigma(A)$ and $0 \notin W_1$. Putting $W_2 = \sigma(A) \setminus W_1$ we get a decomposition of $\sigma(A)$ into two disjoint closed subsets. Let

$$X = X_1 \oplus X_2 \tag{8}$$

be the associated direct sum decomposition of X, let P_i, $i = 1, 2$, be the projections defined by this decomposition, and let A_i, $i = 1, 2$, be the restrictions of A to X_i.

Let

$$B(X) = Z_{11} \oplus Z_{12} \oplus Z_{21} \oplus Z_{22} \tag{9}$$

be the direct sum decomposition of $B(X)$ associated with decomposition (8), with B_{ij}, the (i, j)th coordinate of $B \in B(X)$, given by

$$B_{i,j} = P_i B P_j, \quad i, j = 1, 2.$$

Since decomposition (8) is A-invariant, we have $A_{12} = A_{21} = 0$, and hence the spaces $Z_{i,j}$ are Ad_A-invariant for $i, j = 1, 2$. In particular, Ad_A restricted to Z_{22} is equal to Ad_{A_2}; hence, by Corollary 6 and Proposition 1, $\sigma(\mathrm{Ad}_{A_2}) \subset \sigma(A_2) - \sigma(A_2) = W_2 - W_2$. But the definition of W_1 and (2) imply that $\mu \notin \sigma(\mathrm{Ad}_{A_2})$; therefore, by Proposition 2, equality (6) implies that $C \in Z_{11} \oplus Z_{12} \oplus Z_{21}$, i.e., that

$$C = P_1 C P_1 + P_1 C P_2 + P_2 C P_1. \tag{10}$$

Now observe that since W_1 is a finite subset of $\sigma(A)$ not containing 0 and A is compact, the space X_1 is finite dimensional and hence $P_1 \in B_0(X)$. Thus (10) implies that C is a finite-rank operator.

Remark 17. We have proved in fact the following result on operators:

Let $A \in B_0(X)$, $B \in B(X)$. If $\|(\mathrm{Ad}_A - \mu)^n B\|^{1/n} \underset{n}{\to} 0$ for $\mu \neq 0$, then B is a finite-rank operator.

From Theorem 3 we deduce the following fact on simple B–L algebras of compact operators:

THEOREM 4. *Let L be a simple* B–L *algebra of compact operators. Either L is locally finite or L is a Volterra algebra.*

Proof. Assume that L is not a Volterra algebra. By Theorem 2, L is not nilpotent; hence, by Theorem 3, L contains a finite-rank operator and from Corollary 16 we infer that L is locally finite.

Let L be a Lie subalgebra of $B(X)$. By $\mathscr{L}(L)$ we shall denote the smallest closed associative subalgebra of $B(X)$ containing L.

LEMMA 18. *Let L be a finite-dimensional Volterra algebra. Then $\mathscr{L}(L)$ is also a Volterra algebra (i.e., each $B \in \mathscr{L}(L)$ is a quasinilpotent operator).*

Proof. Induction on $n = \dim L$. For $n = 1$ it is obvious. Assume that it is true for any Volterra algebra with a dimension less than n acting on any B-space X.

Let B be an element of L belonging to the centre of L (since L is finite dimensional, it is nilpotent in the classical sense, and hence it has a nontrivial centre). Suppose that there exists an $A \in \mathscr{L}(L)$ which is not quasinilpotent, i.e., that for some $\lambda \neq 0$ the space X_A^λ is not equal to $\{0\}$. Since $BC = CB$ for $C \in L$, we also have $AB = BA$, i.e.,

$$\mathrm{Ad}_A(B) = 0. \tag{11}$$

From (11) and Lemma 13 it follows that X_A^λ is B-invariant. Since X_A^λ is finite dimensional, the Volterra operator B is algebraically nilpotent when restricted to X_A^λ. Hence there exists an $x_0 \neq 0$, $x_0 \in X_A^\lambda$ such that $Bx_0 = 0$.

Let

$$Y = \{y \in X: By = 0\}.$$

Obviously Y is a closed and proper subspace of X. Take $y \in Y$ and $C \in L$; then $B(Cy) = CBy = 0$, and hence $Cy \in Y$ and Y is L-invariant.

Let L_Y be the quotient algebra of L given by restriction of the operators of L to the space Y. Since B restricted to Y is equal to 0, $\dim L_Y < n$. But $Y \cap X_A^\lambda$ contains x_0, and hence A restricted to Y is not quasinilpotent. Hence $\mathscr{L}(L_Y)$ is not a Volterra algebra and this contradicts the inductive assumption.

THEOREM 5. *Let L be a locally finite Volterra algebra; then $\mathscr{L}(L)$ is also a Volterra algebra.*

Proof. By Proposition 14, L is the closure of $\bigcup_{n=1}^{\infty} L_n$, where $L_1 \subset L_2 \subset \subset L_3 \subset \ldots$ is a chain of increasing finite-dimensional subalgebras of L. Obviously $\mathscr{L}(L)$ is the closure of $\bigcup_{n=1}^{\infty} \mathscr{L}(L_n)$ and since the set of Volterra operators is closed in $B(X)$, the theorem is a consequence of Lemma 17.

We recall the following

LOMONOSOV LEMMA (cf. [2], [5], p. 156). *Let $\mathscr{A}$ be an associative (not necessarily closed) subalgebra of $B(X)$. If there is no proper closed subspace of X invariant for all $A \in \mathscr{A}$, then for each $B \in B_c(X)$ there exists an $x_0 \in X$ and $A_0 \in \mathscr{A}$ such that $A_0 B x_0 = x_0$.*

In particular the Lomonosov Lemma implies

COROLLARY 19. *Let $\mathcal{L}$ be an associative subalgebra of $B_c(X)$ whose elements are Volterra operators. There exists a non-trivial closed $\mathcal{L}$-invariant subspace of X.*

COROLLARY 20. *Any locally finite Volterra algebra has a proper closed invariant subspace.*

Proof. By Theorem 5, $\mathscr{L}(L)$ satisfies the assumptions of Corollary 19.

During the preparation of this note for print, in [10] results coinciding with our Lemma 18, Theorem 5 and Corollary 19 were announced. Since the proofs of those results are not given in [10], for the sake of completeness we have included them in our note.

THEOREM 6. *Let L be a separable Volterra algebra containing a finite-rank operator. Then L has a proper closed ideal.*

Proof. We may restrict our attention to the case where the ideal of finite-rank operators is dense in L. If this is the case, then, by Remark 15, L is locally finite and, by Corollary 19, it has a proper invariant subspace. By standard arguments we get a maximal chain $V = \{X_\varphi\}_{\varphi \in \Phi}$ of L-invariant closed subspaces of X.

Assume first that the following condition is satisfied:

(o) For a $\varphi \in \Phi$ with $X_\varphi \neq \{0\}$ the representation of L given by the restriction of the operators of L to the space X_φ does not vanish.

Let $A_0 \in L$ and $\dim A_0(X) < \infty$. For $\varphi \in \Phi$ put $Y_\varphi = X_\varphi \cap A_0(X)$. Obviously $\{Y_\varphi\}_{\varphi \in \Phi}$ is a chain of finite-dimensional subspaces of X and, since $\bigcap_{\varphi \in \Phi} X_\varphi = \{0\}$, also $\bigcap_{\varphi \in \Phi} Y_\varphi = \{0\}$. Thus there is a φ_1 such that $Y_{\varphi_1} = \{0\}$, i.e., $A(X_{\varphi_1}) = \{0\}$. Hence the representation of L given by the restriction of the operators of L to the space X_{φ_1} has a non-trivial kernel, which is a proper closed ideal of L.

Assume now that the condition (o) is not satisfied. Let X_{φ_0} be the maximal subspace X_φ for which the restriction representation is trivial, and consider the representation ϱ of L induced in the space X/X_{φ_0}. Obviously for $\varphi \in \Phi$ the spaces $\tilde{X}_\varphi$, (images of X_φ under the canonical projection $\pi: X \to X/X_{\varphi_0}$), are $\varrho(L)$-invariant. If the family $\{\tilde{X}_\varphi\}_{\varphi \geqslant \varphi_0}$ satisfies the condition (o) with respect to the algebra $\varrho(L)$, then the previous part of our proof holds. So assume that for some φ_2 with $\varphi_2 > \varphi_0$ the representation given by the restriction of $\varrho(L)$ to $\tilde{X}_{\varphi_2}$ vanishes. This means that the algebra given by the restriction of $\varrho(L)$ to X_{φ_2} consists of operators satisfying the condition $A^2 = 0$, and hence it has a non-trivial centre. Thus the algebra L itself has a non-trivial closed ideal.

5. Remarks and problems. Schue in [7] and [8] has given an isomorphic classification of separable simple complex L^* algebras (an L^* algebra is a Hilbert–Lie algebra with an antilinear involution $*$ such that $\mathrm{Ad}_{a^*} = \mathrm{Ad}_a^*$). According to Schue's results, there are only three isomorphic types A, B and C of such algebras corresponding to the three infinite series (A_n), (B_n) and (C_n) (cf. [9], p. 304) of simple complex finite-dimensional Lie algebras. A, B and C can be realized as subalgebras of the L^* algebra of all Hilbert–Schmidt operators.

P. de la Harpe in [4], Chapter I, exhibited the fact that the algebras A, B and C are locally finite, and that most of their properties can be derived from similar properties of A_0, B_0 and C_0 — the ideals of the corresponding finite-rank operators. It would be interesting to get an intrinsic characterization of those ideals. Our proof of Theorem 3 suggests the following possibility: For a simple (not Volterra) algebra L of compact operators on a B-space X, let $L_0 = L \cap B_0(X)$ and let J_L be the ideal of L generated by all solutions T of the equations

$$(\mathrm{Ad}_A - \mu)^n T = 0 \tag{12}$$

with $\mu \neq 0$ and A belonging to L.

We have proved that $J_L \subset L_0$.

QUESTION 1. *Is J_L equal to L_0?*

Remark. Instead of (12) one may use equations of the form

$$W(\mathrm{Ad}_A)(T) = 0, \tag{13}$$

where W is a polynomial of one variable with complex coefficients and no zero root. More precisely: if $A \in B_c(X)$, $T \in B(X)$ and (13) holds, then T is a finite-rank operator. (This is an easy corollary of the result in Remark 17.)

We finally give some open questions:

QUESTION 2. *Is every simple* B–L *algebra of compact operators locally finite?*

QUESTION 3. *Has every Volterra algebra a proper closed ideal?*

QUESTION 3′. *The same, with the additional assumption that the algebra is locally finite.*

QUESTION 4. *Let L be a Volterra subalgebra of $B_c(X)$. Does there exist a proper closed L-invariant subspace of X?*

QUESTION 5. *With the same notation as in Question 4, is $\mathscr{L}(L)$ a Volterra algebra?*

Note that a positive answer to Question 5 implies a positive answer to Question 4, and in the same way 3 implies 2.

QUESTION 6. *Let L_0 be an algebraically simple Lie subalgebra of $B_0(X)$. Is the closure of L_0 in $B_c(X)$ a simple* B–L *algebra?*

QUESTION 7. *Let L_1 and L_2 be two algebraically isomorphic simple subalgebras of $B_c(X)$. Are L_1 and L_2 isomorphic?*

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