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Gaussian measures on L_p spaces $0 \leqslant p < \infty$

by

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Abstract. Using the correspondence between measures on L_p spaces, $0 \leqslant p < \infty$, and measurable processes with paths in L_p, given in [2], we prove that the independence of random elements with values in L_p as well as their distributions are determined by finite-dimensional distributions of the corresponding measurable processes. This result is applied to the investigation of Gaussian random elements with values in L_p, $0 \leqslant p < \infty$.

It is well known that if X_1, X_2 are independent real random variables such that X_1+X_2 and X_1-X_2 are independent, then X_1, X_2 are Gaussian. This property of real Gaussian random variables, proved already by Bernstein, has been used by Fréchet as one of two (equivalent) definitions of Gaussian random elements with values in a Banach space [5]. This definition allows us to consider Gaussian random elements in metric linear spaces which admit no nontrivial continuous linear functionals. The best known examples of such spaces are $L_p \equiv L_p(m)$ spaces, where m is the Lebesgue measure on [0, 1] and $0 \leqslant p < 1$. Of course, in such spaces the classical definition of Gaussian elements cannot be used. In this paper we investigate Gaussian random elements on L_p spaces, $0 \leqslant p < \infty$. For $p \geqslant 1$ these results were proved by Rajput [9].

Section 1 is preliminary. In Section 2 we prove two results. In Theorem 2.1 we prove that the support of a symmetric Gaussian measure defined on a linear metric space is a closed linear subspace. In Theorem 2.2 we give a short proof of the 0–1 law for Gaussian measures defined on complete separable metric linear spaces.

In Section 3 we consider measurable processes with paths in L_p. Theorem 1.1 proved in [2] gives the correspondence between measures on L_p and measurable processes with paths in L_p. We prove that the independence of random elements with values in L_p as well as their distributions are determined by finite-dimensional distributions of the corresponding measurable processes.

In Section 4 we apply the results of the preceding sections to prove results analogous to those obtained by Rajput [9] for $p \geqslant 1$. Theorem

4.1 has been proved in [2] (for $p = 0$ and the measure assumed to be finite) but this proof depends on a result of Rajput ([9], Theorem 3.2). The proof presented here is based on Theorem 3.1 in Section 3.

1. Preliminaries. Let E be a real separable metric linear space. By $\mathscr{B}(E)$ we denote the Borel σ-algebra in E. A mapping X defined on a probability space (Ω, Σ, P) with values in E is called a *random element* (r.e.) if it is measurable relative to the σ-algebras $\mathscr{B}(E)$ and Σ.

DEFINITION 1.1. We say that an r.e. X is *Gaussian* if for any independent r.e.'s X_1, X_2 with the same distribution as X, the r.e.'s X_1+X_2 and X_1-X_2 are independent.

A probability measure μ is *Gaussian* if it is the distribution of a Gaussian r.e. X. If ψ is a mapping from $E\times E$ into $E\times E$ defined as

$$\psi(x, y) = (x+y, x-y), \tag{1}$$

then we can equivalently state:

μ is *Gaussian* if and only if there are some probability measures ν_1, ν_2 such that

$$\mu\times\mu\big(\psi^{-1}(A)\big) = \nu_1\times\nu_2(A) \tag{2}$$

for every $A\in\mathscr{B}(E\times E)$.

If $\mathscr{B}(E)$ is generated by all continuous linear functionals on E, then this definition is consistent with the usual one: an r.e. X is *Gaussian* iff for every continuous linear functional f on E $f(X)$ is a Gaussian real random variable.

$(T, \mathscr{F}, m)$ will denote throughout this paper an arbitrary σ-finite measure space. Let S be the space of equivalence classes of all real-valued $\mathscr{F}$-measurable functions with convergence in measure m. Let S_0 be the subspace of S consisting of elements equivalent to all functions of the form 1_E, where $E\in\mathscr{F}$ and $m(E) < \infty$. In the sequel we always assume that S_0 is separable with respect to the topology induced from S. Let $L_0 = L_0(m)$ be the closed linear subspace in S generated by S_0. If $x\in L_0$, then we say that x is *strongly measurable*. By $L_p = L_p(m)$ we shall denote the set of all $x\in S$ whose pth power is m-integrable with the norm

$$\|x\|_p = \Big(\int |x(t)|^p m(dt)\Big)^r,$$

where $r = 1$ if $0 < p \leqslant 1$ and $r = 1/p$ if $p > 1$. It is clear that $L_p\subset L_0$, for $0 < p < \infty$.

When no confusion seems possible, we use the same notation for a function $\in L_p$ and the corresponding equivalence class. It is well known that L_p, $0\leqslant p<\infty$, is a real metric linear space (if $p\geqslant 1$ it is even a Banach space). By the assumption it follows that L_p is separable for $0\leqslant p < \infty$.

Let $\{\xi(t);\ t\in T\}$ be a stochastic process defined on a probability space (Ω, Σ, P); it is said to be *measurable* if the mapping ξ from $(\Omega\times T, \Sigma\times\mathscr{F}, P\times m)$ into R defined by $(\omega, t)\to\xi(\omega, t)$ is strongly measurable. It is not hard to prove that if $\{\xi(t);\ t\in T\}$ is measurable then $\xi(\omega,\cdot)\in L_0$ a.s. $[P]$. Now, let us suppose that $\xi(\omega,\cdot)\in L_p$ a.s. $[P]$. Let $\tilde\xi\colon \Omega\to L_p$ be a mapping defined as follows:

$$\tilde\xi(\omega) = \begin{cases}\xi(\omega,\cdot) & \text{if}\quad \xi(\omega,\cdot)\in L_p,\\ 0 & \text{if}\quad \xi(\omega,\cdot)\notin L_p.\end{cases}$$

By the measurability of ξ and separability of L_p, it follows that $\tilde\xi$ is a random element. The probability distribution of $\tilde\xi$ is denoted by μ_ξ and called the *measure induced by the process* ξ.

For the sake of brevity and convenience we use throughout this paper the following terminology, somewhat different from the classical one: a stochastic process $\{\xi(t);\ t\in T\}$ is called *Gaussian* if there is a $T_0\in\mathscr{F}$, $m(T_0) = 0$ such that, for every $t_1, \ldots, t_k\in T\setminus T_0$, $\langle\xi(t_1), \ldots, \xi(t_k)\rangle$ is Gaussian random vector.

The following theorem will be useful in the sequel:

THEOREM 1.1. *Let μ be a probability measure on $\big(L_p, \mathscr{B}(L_p)\big)$, $0\leqslant p<\infty$. Then there exists a measurable process $\{\xi(t);\ t\in T\}$ defined on $(\Omega, \Sigma, P) = \big(L_p, \mathscr{B}(L_p), \mu\big)$ and such that $\tilde\xi(x) = x$ a.s. $[P]$. In particular, $\mu_\xi = \mu$ and $\xi(\omega,\cdot)\in L_p$ a.s. (P). Moreover, if μ is Gaussian then ξ is Gaussian.*

This theorem has been proved in [2] if $p = 0$ and m is assumed to be finite. However, the proof remains valid for our general situation, with only inessential modifications. We shall also need the following "two-dimensional" version of this theorem:

THEOREM 1.1′. *Let μ be a measure on $\big(L_p\times L_p, \mathscr{B}(L_p\times L_p)\big)$, $0\leqslant p<\infty$. Then there exists a pair $\xi = (\xi_1, \xi_2)$ of measurable processes defined on $(\Omega, \Sigma, P) = \big(L_p\times L_p, \mathscr{B}(L_p\times L_p), \mu\big)$ and such that $\tilde\xi(x) = x$ a.s. $[P]$, where $\tilde\xi = (\tilde\xi_1, \tilde\xi_2)$.*

We need also

DEFINITION 1.2. Let $\{\xi(t);\ t\in T\}$ be a second-order process; then the *mean function* θ and the *covariance function* K of ξ are defined by $\theta(t) = E\xi(t)$ and $K(s, t) = E\big(\xi(t)-\theta(t)\big)\big(\xi(s)-\theta(s)\big)$.

2. Gaussian measures on metric linear spaces. Let E be a real separable metric linear space. By $\mathfrak{M}(E)$ we shall denote the set of all probability measures on E with the topology of weak convergence and with the operation of convolution.

By Definition 1.1 it immediately follows that the set of all Gaussian measures on E forms a closed (convolution) subsemigroup of $\mathfrak{M}(E)$.

Now, let $C(\mu)$ denote the support of μ. We prove the following

THEOREM 2.1. *Let μ be a symmetric Gaussian measure. Then the support of μ is a closed linear subspace of E.*

Proof. Since μ is symmetric, we have in (2) $\nu_1 = \nu_2 = \nu$. Let x, $y \in C(\mu)$ and let W_1, W_2 be some arbitrary open neighbourhoods of $x+y$ and $x-y$, respectively. By the continuity of ψ there are open neighbourhoods U and V of x and y, respectively, such that $\psi(U\times V) \subset W_1 \times W_2$. Hence

$$\nu\times\nu(W_1\times W_2) \geqslant \nu\times\nu\big(\psi(U\times V)\big) = \mu\times\mu(U\times V) > 0.$$

So, we have

$$x, y \in C(\mu) \Rightarrow x+y, x-y \in C(\nu).$$

Next, since $\psi^{-1}(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$, we obtain in the same way the reversed implication

$$x, y \in C(\nu) \Rightarrow \frac{x+y}{2}, \frac{x-y}{2} \in C(\mu).$$

Thus, if $x, y \in C(\mu)$ then, from the first implication we infer that $2x$, $2y \in C(\nu)$, and from the second that $x+y, x-y \in C(\mu)$. Further, $0 \in C(\mu)$, and so if $x \in C(\mu)$ then $x \in C(\nu)$ and, since $0 \in C(\nu)$, $x/2 \in C(\mu)$. Since $C(\mu)$ is closed, this proves that $C(\mu)$ is a closed linear subspace of E.

Next, we give a short proof of the $0-1$ law for Gaussian measures. We say that F is a *rational subspace of* E if $rx+sy \in F$ for any rational r, s, whenever $x, y \in F$.

THEOREM 2.2. *Let μ be a Gaussian measure on a complete separable metric linear space E. Let F be a completion measurable rational subspace of E. Then $\mu(F) = 0$ or $\mu(F) = 1$.*

Proof. First, let us observe that it suffices to prove our theorem if F is Borel-measurable. For, if $\mu(F) > 0$ then there exists a compact subset K of F of positive measure μ. Then, the rational subspace generated by K is Borel-measurable and of positive measure μ.

Thus, let us suppose that F is a Borel-measurable rational subspace of E such that $\mu(F) > 0$. Let π be the natural mapping from E onto E/F and let $\{x_\alpha;\ \alpha \in A\}$ be a fixed selector of $\{\pi^{-1}(y);\ y \in E/F\}$. Since $F-x_\alpha$ and $F-x_\beta$ for $\alpha \neq \beta$ are Borel-measurable and disjoint, we infer that there is an at most countable number of x_i, $x_i = x_{\alpha_i}$, such that $\mu(F-x_i) > 0$. Let H be the subgroup of E generated by $\{F-x_i;\ i = 1, 2, \ldots\}$. Then $\pi(H)$ is countable and torsion-free, since F is a rational subspace. Let μ' be the relativisation of μ to H. If we endow $\pi(H)$ with the discrete topology, then $\pi'\colon H \to \pi(H)$ is Borel-measurable. Hence $\mu'\circ\pi'$ $\big(\mu'\circ\pi'(A) = \mu(\pi^{-1}A)\big)$ is a Gaussian measure on $\pi(H)$ in the sense of Parthasarathy [8] (up to a multiplicative positive constant — see [6], Korollar 6.7) and therefore it must be degenerate ([8], p. 101). Hence, $\mu(F-x) = 0$, for $x \notin F$.

The rest of the proof is almost the same as in [4], p. 251; it is included here only for completeness.

By (2) it follows that $\nu_1 = \mu*\mu$, $\nu_2 = \mu*\tilde{\mu}$, where $\tilde{\mu}(A) = \mu(-A)$. Thus, again from (2) we obtain $\mu(F)^2 = \big(\mu*\mu(F)\big)\big(\mu*\tilde{\mu}(F)\big)$. Since

$$\mu*\tilde{\mu}(F) = \int \tilde{\mu}(F-x)\,d\mu(x) = \int \mu(F+x)\,d\mu(x) = \mu(F)^2$$

and also $\mu*\mu(F) = \mu(F)^2$, we have $\mu(F)^2 = \mu(F)^4$. Thus, $\mu(F) = 1$.

Remark 2.1. (i). One can also consider Gaussian measures (in the sense of Definition 1.1) on Polish groups (see [3] and [6]). The proof of Theorem 2.2 then yields the following result: if G is a Polish group such that the mapping $x \to 2x$ is a surjection and F is a Borel-measurable subgroup of G such that G/F is torsion-free, then $\mu(F) = 0$ or $\mu(F) = 1$.

(ii) Theorem 2.2 is proved (by a different method) in [4] for stable measures. However, if E does not have a sufficient number of continuous linear functionals, then it is not known whether any Gaussian measure is stable in the sense of [4].

3. Stochastic processes with paths in L_p, $0 \leqslant p < \infty$. In this section we shall prove the following

THEOREM 3.1. *Let ξ_i be some measurable processes with paths in L_p, $i = 1, 2$, and let μ_i denote the measure induced by ξ_i on L_p, $i = 1, 2$. Then $\mu_1 = \mu_2$ if and only if there is a $T_0 \in \mathscr{F}$, $m(T_0) = 0$ such that the corresponding finite-dimensional distributions of ξ_1 and ξ_2 based on points $\in T\setminus T_0$ are equal.*

THEOREM 3.2. *Let ξ_i be some measurable processes with paths in L_p, $i = 1, 2$. Let $\tilde{\xi}_i$ denote the random element with values in L_p induced by ξ_i, $i = 1, 2$. Then $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are independent if and only if there exists a $T_0 \in \mathscr{F}$, $m(T_0) = 0$ such that the random vectors*

$$\langle \xi_1(t_1), \ldots, \xi_1(t_k)\rangle, \quad \langle \xi_2(t_1), \ldots, \xi_2(t_k)\rangle$$

are independent for all $t_1, t_2, \ldots, t_k \in T\setminus T_0$.

For the sake of clarity and convenience we divide the proofs into two lemmas.

LEMMA 3.1. *Let $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2)$ be two pairs of measurable processes with paths in L_p. Let us assume that $\mu_\xi = \mu_\eta = \mu$. Then there exists a $T_0 \in \mathscr{F}$, $m(T_0) = 0$ such that the random vectors*

$$\langle \xi_1(t_1), \ldots, \xi_1(t_k), \xi_2(t_{k+1}), \ldots, \xi_2(t_m)\rangle,$$
$$\langle \eta_1(t_1), \ldots, \eta_1(t_k), \eta_2(t_{k+1}), \ldots, \eta_2(t_m)\rangle$$

have the same distribution, if $t_i \in T\setminus T_0$, $i = 1, \ldots, m$.

Proof. (i) Let $\xi = (\xi_1, \xi_2)$, $\zeta = (\zeta_1, \zeta_2)$ be two pairs of measurable processes such that $\tilde{\xi} = \tilde{\zeta}$ a.s. $[P]$. Then there exists a $T_0 \in \mathscr{F}$, $m(T_0) = 0$ such that $\xi(\cdot, t) = \zeta(\cdot, t)$ a.s. $[P]$, for every $t \in T \setminus T_0$. In particular, the corresponding finite-dimensional distributions of ξ and ζ based on points $\in T \setminus T_0$ are equal.

Indeed, $P\{\omega; \tilde{\xi}(\omega) \neq \tilde{\zeta}(\omega)\} = 0$. Thus, there exists an $N \in \Sigma$, $P(N) = 0$ such that

$$\xi_i(\omega, \cdot) = \zeta_i(\omega, \cdot) \text{ a.s. } [m] \quad \text{if } \omega \in \Omega \setminus N,\ i = 1, 2.$$

Then

$$P \times m\{(\omega, t);\ \xi_i(\omega, t) \neq \zeta_i(\omega, t);\ i = 1, 2\} = 0.$$

By Fubini's theorem there exists a $T_0 \in \mathscr{F}$, $m(T_0) = 0$, such that

$$P\{\omega;\ \xi_i(\omega, t) \neq \zeta_i(\omega, t), i = 1, 2\} = 0$$

for every $t \in T \setminus T_0$, which proves (i).

(ii) Let X_1, X_2 be two random elements with values in L_p. Let $\varphi = (\varphi_1, \varphi_2)$ be a pair of measurable processes with the properties as described in Theorem 1.1′, constructed for the measure $\mu = \mu_X$, where $X = (X_1, X_2)$.

Let $\zeta(\omega, t) = \varphi(X(\omega), t)$. Then it is easy to observe that $\tilde{\zeta} = X$ with probability one and that ζ has the same finite-dimensional distributions as φ.

(iii) Let ξ, η be two pairs of processes as described in this lemma. Let φ be a pair of processes constructed as in (ii). Let $\zeta_i^{(1)}(\omega, t) = \varphi_i(\tilde{\xi}(\omega), t)$; $\zeta_i^{(2)}(\omega, t) = \varphi_i(\tilde{\eta}(\omega), t)$, $i = 1, 2$. Then, by (ii), $\tilde{\zeta}^{(1)} = \tilde{\xi}$ and $\tilde{\zeta}^{(2)} = \tilde{\eta}$ a.s. $[P]$. By (i) it follows that there is a $T_0' \in \mathscr{F}$, $m(T_0') = 0$ such that $\zeta^{(1)}$ and ξ have the same finite-dimensional distributions based on points $\in T \setminus T_0'$, and $T_0'' \in \mathscr{F}$, $m(T_0'') = 0$, and such that $\zeta^{(2)}$ has the corresponding finite-dimensional distributions based on points $\in T \setminus T_0''$ equal to the distributions of η. Since $\zeta^{(i)}$, $i = 1, 2$, has the same finite-dimensional distributions as φ, ξ and η have the same finite-dimensional distributions based on points $\in T \setminus T_0$, where $T_0 = T_0' \cup T_0''$.

LEMMA 3.2. *Let $\{\xi_i, \eta_i;\ i = 1, 2, \ldots, k\}$ be a finite collection of non-negative measurable processes. Suppose that the random vectors*

$$\langle \xi_1(t_1^{(1)}), \ldots, \xi_1(t_1^{(j_1)});\ \xi_2(t_2^{(1)}), \ldots, \xi_2(t_2^{(j_2)});\ \ldots,\ \xi_k(t_k^{(1)}), \ldots, \xi_k(t_k^{(j_k)})\rangle,$$
$$\langle \eta_1(t_1^{(1)}), \ldots, \eta_1(t_1^{(j_1)});\ \eta_2(t_2^{(1)}), \ldots, \eta_2(t_2^{(j_2)});\ \ldots;\ \eta_k(t_k^{(1)}), \ldots, \eta_k(t_k^{(j_k)})\rangle$$

are independent [have the same distribution] for every finite subset $\{t_i^{(j)} \in T;\ j = 1, \ldots, j_i;\ i = 1, \ldots, k\}$ of T. Let

$$X_i = \int \xi_i(t)\, m(dt), \qquad Y_i = \int \eta_i(t)\, m(dt).$$

Suppose that $L_1(P)$ is separable. Then the random vectors

$$\langle X_1, X_2, \ldots, X_k\rangle, \qquad \langle Y_1, Y_2, \ldots, Y_k\rangle$$

are independent [have the same distribution].

Proof. (a) We first prove our lemma under some additional assumptions. Namely, suppose that

$$\int \|\xi_i(\cdot, t)\|_1 m(dt) < \infty, \qquad \int \|\eta_i(\cdot, t)\|_1 m(dt) < \infty, \qquad \text{for } i = 1, \ldots, k.$$

By these assumptions it follows that $\xi_i(\cdot, t), \eta_i(\cdot, t) \in L_1(P)$ a.s. $[m]$ so that ξ_i, η_i can be considered as Bochner's integrable mapping from T into $L_1(P)$, $i = 1, \ldots, k$. The Bochner integral of ξ_i, η_i will be denoted by (B)-$\int \xi_i(t)\ m(dt)$, (B)-$\int \eta_i(t)\ m(dt)$ and the sample path integrals by $\int \xi_i(t) m(dt)$ and $\int \eta_i(t)\, m(dt)$, $i = 1, \ldots, k$. It is easy to verify that

$$\text{(B)-}\int \xi_i(t)\, m(dt) = \int \xi_i(t)\, m(dt) \quad \text{a.s. } [P],$$
$$\text{(B)-}\int \eta_i(t)\, m(dt) = \int \eta_i(t)\ m(dt) \quad \text{a.s. } [P],$$

$i = 1, \ldots, k$. Using these equalities and the property of Bochner's integral, we shall construct sequences of m-simple functions $\varphi_i^{(n)}, \psi_i^{(n)}$ such that if $X_i^{(n)} = \text{(B)-}\int \varphi_i^{(n)}(t)\ m(dt)$, $Y_i^{(n)} = \text{(B)-}\int \psi_i^{(n)}(t)\, m(dt)$, $i = 1, \ldots, k$ then $X_i^{(n)} \to X_i$, $Y_i^{(n)} \to Y_i$ in $L_1(P)$, $i = 1, \ldots, k$, and the random vectors

$$\langle X_1^{(n)}, \ldots, X_k^{(n)}\rangle, \qquad \langle Y_1^{(n)}, \ldots, Y_k^{(n)}\rangle$$

are independent [have the same distribution]. The above will immediately imply the theorem.

Let $\{t_j\}$ be a sequence of elements of T such that

$$\overline{\{\xi_i(\cdot, t_j)\}_{j=1}^{\infty}} = \overline{\{\xi_i(\cdot, t);\ t \in T\}},$$
$$\overline{\{\eta_i(\cdot, t_j)\}_{j=1}^{\infty}} = \overline{\{\eta_i(\cdot, t);\ t \in T\}},$$

for $i = 1, \ldots, k$ ($\bar{A}$ denotes the $L_1(P)$ — closure of $A \subset L_1(P)$). Let $B_{j,i}^{(n)} = \{x \in L_1(P);\ \|x - \xi_i(\cdot, t_j)\|_1 < 1/n\}$, $D_{j,i}^{(n)} = \{x \in L_1(P);\ \|x - \eta_i(\cdot, t_j)\|_1 < 1/n\}$, $j = 1, 2, \ldots$, $i = 1, \ldots, k$, $n = 1, 2, \ldots$

Let

$$F_{j,i}^{(n)} = B_{j,i}^{(n)} \setminus \bigcup_{m=1}^{j-1} B_{m,i}^{(n)}, \qquad G_{j,i}^{(n)} = D_{j,i}^{(n)} \setminus \bigcup_{m=1}^{j-1} D_{m,i}^{(n)}$$

and

$$M_{j,i}^{(n)} = \{t \in T;\ \xi_i(\cdot, t) \in F_{j,i}^{(n)}\},$$
$$N_{j,i}^{(n)} = \{t \in T;\ \eta_i(\cdot, t) \in G_{j,i}^{(n)}\}.$$

Now, let

$$\varphi_i^{(n)}(t) = \begin{cases} \xi_i(\cdot, t_j) & \text{if} \quad t \in M_{j,i}^{(n)} \text{ and } \|\xi_i(\cdot, t_j)\|_1 \leqslant 2\|\xi_i(\cdot, t)\|_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_i^{(n)}(t) = \begin{cases} \eta_i(\cdot, t_j) & \text{if} \quad t \in N_{j,i}^{(n)} \text{ and } \|\eta_i(\cdot, t_j)\|_1 \leqslant 2\|\eta_i(\cdot, t)\|_1, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, \ldots, k$ and $n = 1, 2, \ldots$

It is easy to check that $\|\varphi_i^{(n)}(t) - \xi_i(t)\|_1 \to 0$, $\|\psi_i^{(n)}(t) - \eta_i(t)\|_1 \to 0$ a.s. $[m]$ if $n \to \infty$, $i = 1, \ldots, k$ and that $\|\varphi_i^{(n)}(t)\|_1 \leqslant 2\|\xi_i(\cdot, t)\|_1$, $\|\psi_i^{(n)}(t)\|_1 \leqslant 2\|\eta_i(\cdot, t)\|_1$ for $i = 1, \ldots, k$, $n = 1, 2, \ldots$ Thus, by the property of Bochner's integral,

$$(\text{B})\text{-}\int \xi_i(\cdot, t)\, m(dt) = s\text{-}\lim \left\{(B) - \int \varphi_i^{(n)}(t)\, m(dt)\right\},$$

$$(\text{B})\text{-}\int \eta_i(\cdot, t)\, m(dt) = s\text{-}\lim \left\{(B) - \int \psi_i^{(n)}(t)\, m(dt)\right\},$$

where "s-lim" denotes the $L_1(P)$-convergence. Let us write $X_i^{(n)} = (\text{B})\text{-}\int \varphi_i^{(n)}(t)\, m(dt)$, $Y_i^{(n)} = (\text{B})\text{-}\int \psi_i^{(n)}(t)\, m(dt)$. By the assumptions, it follows that the random vectors

$$\langle X_1^{(n)}, \ldots, X_k^{(n)}\rangle, \quad \langle Y_1^{(n)}, \ldots, Y_k^{(n)}\rangle,$$

are independent [have the same distribution]. Since $X_i = s\text{-}\lim X_i^{(n)}$, $Y_i = s\text{-}\lim Y_i^{(n)}$, the proof is completed.

(b) We complete the proof by dropping the assumption that $\int \|\xi_i(\cdot, t)\|_1 m(dt) < \infty$, $\int \|\eta_i(\cdot, t)\|_1 m(dt) < \infty$, $i = 1, \ldots, k$.

Let f be an element of $L_1(m)$ such that $f > 0$. Let $\xi_i^{(n)}(\omega, t) = \min\big(\xi_i(\omega, t), nf(t)\big)$, $\eta_i^{(n)}(\omega, t) = \min\big(\eta_i(\omega, t), nf(t)\big)$. It is easy to see that $\xi_i^{(n)}$, $\eta_i^{(n)}$ satisfy the assumptions described in (a). Moreover, $\xi_i^{(n)}(\omega, \cdot)$, $\eta_i^{(n)}(\omega, \cdot)$ are nondecreasing sequences convergent m-a.e. to $\xi_i(\omega, \cdot)$, $\eta_i(\omega, \cdot)$, respectively, for every fixed $\omega \in \Omega$. By the monotone convergence theorem, we have

$$\int \xi_i^{(n)}(\omega, t)\, m(dt) \to \int \xi_i(\omega, t)\, m(dt),$$

$$\int \eta_i^{(n)}(\omega, t)\, m(dt) \to \int \eta_i(\omega, t)\, m(dt)$$

P-a.e. for $i = 1, \ldots, k$. By part (a), it follows that the lemma is valid for $\xi_i^{(n)}$, $\eta_i^{(n)}$. Thus, the lemma is valid also for ξ_i, η_i. Thus the proof is complete.

Proof of Theorem 3.1. The necessity immediately follows from Lemma 3.1. We prove the sufficiency. Let $\xi = (\xi_1, \xi_2)$ be a pair of measurable processes with paths in L_p. Let us assume that there exists $T_0 \in \mathscr{F}$, $m(T_0) = 0$ such that the corresponding finite-dimensional distributions based on points $\in T \setminus T_0$ are the same. Using Theorem 1.1′, we can find a pair ξ_1', ξ_2' of measurable processes defined on $(\Omega, \Sigma, P) = \big(L_p \times L_p, \mathscr{B}(L_p \times L_p), \mu_\xi\big)$ such that $\mu_{\xi_i'} = \mu_{\xi_i}$, $i = 1, 2$. By the separability of L_p, it follows that $L_1(\Omega)$ is also separable, where $\Omega = L_p \times L_p$. Moreover, by Lemma 3.1, it follows that the corresponding finite-dimensional distributions of ξ_i and ξ_i' based on points $\in T \setminus T_0'$ are the same, where $T_0' \in \mathscr{F}$ and $m(T_0') = 0$, $i = 1, 2$. So, without loss of generality, we can assume that ξ_i, $i = 1, 2$, are defined on $(\Omega, \Sigma, P) = \big(L_p \times L_p, \mathscr{B}(L_p \times L_p), \mu_\xi\big)$. Of course, we can also assume that $T_0 = \emptyset$. Now, let ε_i be an arbitrary real positive number and let x_i be an arbitrary element of L_p, $i = 1, \ldots, k$. Let us write

$$\zeta_i = |\xi_1 - x_i|^p, \qquad \eta_i = |\xi_2 - x_i|^p \qquad \text{if } 0 < p < \infty,$$

$$\zeta_i = 1_{E_i}(\xi_1 - x_i), \qquad \eta_i = 1_{E_i}(\xi_2 - x_i) \qquad \text{if } p = 0,\ E_i = \{|y| > \varepsilon_i\},$$

$i = 1, \ldots, k$. Let $K_i = \{x;\ \|x - x_i\|_p \leqslant \varepsilon_i\}$, $i = 1, \ldots, k$, where $\|x\|_p$ is the usual L_p-norm if $0 < p < \infty$ and $\|x\|_0 = \inf\{a > 0;\ m\{t;\ |x(t)| > a\} \leqslant a\}$ (it is well known that $\|\cdot\|_0$ induces convergence in measure m). Then, by Lemma 3.2, we obtain

$$P\{\tilde\xi_1 \in \bigcap_{i=1}^k K_i\} = P\{\tilde\xi_2 \in \bigcap_{i=1}^k K_i\},$$

which means that

$$\mu_1\Big(\bigcap_{i=1}^k K_i\Big) = \mu_2\Big(\bigcap_{i=1}^k K_i\Big).$$

Since L_p is separable, we obtain $\mu_1 = \mu_2$, which completes the proof.

Proof of Theorem 3.2. *The necessity.* Suppose that $\xi = (\xi_1, \xi_2)$ is a pair of measurable processes with paths in L_p such that the induced random elements $\tilde\xi_1$ and $\tilde\xi_2$ are independent. Using Theorem 1.1′, we can construct a pair $\eta = (\eta_1, \eta_2)$ of measurable processes defined on $\big(L_p \times L_p, \mathscr{B}(L_p \times L_p), \mu_\xi\big)$ and such that $\mu_\xi = \mu_\eta = \mu$ and $\big(\eta_1(x), \eta_2(x)\big) = x$ a.s.$[\mu]$. By Lemma 3.1, it follows that there exists a $T_0 \in \mathscr{F}$, $m(T_0) = 0$ such that if $t_1, \ldots, t_m \in T \setminus T_0$ then the random vectors

$$\langle \xi_1(t_1), \ldots, \xi_1(t_k); \xi_2(t_{k+1}), \ldots, \xi_2(t_m)\rangle,$$

$$\langle \eta_1(t_1), \ldots, \eta_1(t_k); \eta_2(t_{k+1}), \ldots, \eta_2(t_m)\rangle$$

have the same distribution. Now, let $t_1, \ldots, t_k \in T \setminus T_0$ and let

$$Z_i = \langle \eta_i(t_1), \ldots, \eta_i(t_k)\rangle, \qquad i = 1, 2.$$

Let $A_1, A_2 \in \mathscr{B}(R^k)$ and let

$$B_i = \{x_i \in L_p;\ \langle \eta_i\big((x_1, x_2), t_1\big), \ldots, \eta_i\big((x_1, x_2), t_k\big)\rangle \in A_i\}, \qquad i = 1, 2.$$

By the construction, it follows that the random elements $\tilde\eta_1, \tilde\eta_2$ are independent. Hence

$$P\{x;\ \tilde\eta_1(x) \in B_1, \tilde\eta_2(x) \in B_2\} = P\{x;\ \tilde\eta_1(x) \in B_1\} P\{x;\ \tilde\eta_2(x) \in B_2\}.$$

On the other hand,

$$P\{x;\ \tilde{\eta}_1(x)\in B_1, \tilde{\eta}_2(x)\in B_2\} = P(B_1\times B_2) = P\{x;\ Z_1(x)\in A_1, Z_2(x)\in A_2\}$$

and

$$P\{x;\ \tilde{\eta}_i(x)\in B_i\} = P\{(x_1, x_2);\ x_i\in B_i\} = P\{x;\ Z_i(x)\in A_i\}, \qquad i = 1, 2.$$

These equalities show that Z_1, Z_2 are independent, which completes the proof of necessity.

The proof of *sufficiency* is similar to the proof of sufficiency in Theorem 3.1 and therefore it is omitted.

4. Gaussian measures on L_p spaces. Now let ξ be a measurable Gaussian process. Let us write $\theta(t) = E\xi(t)$ and

$$K(t, s) = E\big(\xi(t)-\theta(t)\big)\big(\xi(s)-\theta(s)\big).$$

THEOREM 4.1. *Let $\{\xi(t);\ t\in T\}$ be a measurable Gaussian process with paths in L_p. Let $\tilde{\xi}$ denote the random element induced on L_p by ξ. Then $\tilde{\xi}$ is a Gaussian random element.*

Proof. Let X_1, X_2 be two independent random elements defined on (Ω, Σ, P) with values in L_p with the same distributions as $\tilde{\xi}$. Let ζ_1, ζ_2 be two Gaussian measurable processes such that $\tilde{\zeta}_i = X_i$ a.s. $[P]$, $i = 1, 2$, constructed as in part (ii) of the proof of Lemma 3.1. By Theorem 3.2 it follows that there exists a $T_0\in\mathscr{F}$, $m(T_0) = 0$, such that

$$\langle\zeta_1(t_1), \ldots, \zeta_1(t_k)\rangle, \qquad \langle\zeta_2(t_1), \ldots, \zeta_2(t_k)\rangle$$

are independent Gaussian random vectors if $t_1, \ldots, t_k\in T\setminus T_0$. So, if we define

$$\xi_1(\omega, t) = \zeta_1(\omega, t)+\zeta_2(\omega, t), \qquad \xi_2(\omega, t) = \zeta_1(\omega, t)-\zeta_2(\omega, t),$$

then the random vectors

$$\langle\xi_1(t_2), \ldots, \xi_1(t_k)\rangle, \qquad \langle\xi_2(t_1), \ldots, \xi_2(t_k)\rangle$$

are independent if $t_1, \ldots, t_k\in T\setminus T_0$. Since, by the construction, $\tilde{\zeta}_i = X_i$ a.s. $[P]$, $i = 1, 2$, we have $\tilde{\xi}_1 = X_1+X_2$, $\tilde{\xi}_2 = X_1-X_2$ a.s. $[P]$. By Theorem 3.2, it follows that $\tilde{\xi}_1, \tilde{\xi}_2$ are independent and hence also X_1+X_2 and X_1-X_2 are independent. The proof is complete.

Remark 4.1. Let X_1, X_2 be independent symmetric Gaussian random elements with values in L_p with the same distribution. If s, t are arbitrary nonnegative reals such that $s^2+t^2 = 1$, then sX_1+tX_2 and tX_1-sX_2 are independent and have the same distribution as X_1. For, let (ξ_1, ξ_2) be a pair of measurable processes inducing on $L_p\times L_p$ the same distribution as (X_1, X_2). Then, by Theorem 3.1 and 3.2, there exists a $T_0\in\mathscr{F}$, $m(T_0) = 0$ such that

$$Z_1 = \langle\xi_1(t_1), \ldots, \xi_1(t_k)\rangle, \qquad Z_2 = \langle\xi_2(t_1), \ldots, \xi_2(t_k)\rangle$$

are independent symmetric Gaussian random vectors with the same distribution for every $t_1, \ldots, t_k\in T\setminus T_0$. Thus, sZ_1+tZ_2 and tZ_1-sZ_2 are independent and have the same distribution as Z_1. Using Theorem 3.1 and 3.2 again, we obtain the desired conclusion.

Now, we need a simple lemma (see [9], Lemma 3.1).

LEMMA 4.1. *Let ξ be a symmetric Gaussian random variable. Then, for all $a > 0$ and $b > 0$, we have*

$$E(|\xi|^a) = C(a, b)E(|\xi|^b)^{a/b},$$

where

$$C(a, b) = \Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)^{-(a/b)}\pi^{-((a-b)/2b)}.$$

THEOREM 4.2. *Let $\{\xi(t);\ t\in T\}$ be a measurable Gaussian process. Let θ and K be the mean and the covariance function of ξ, respectively. Then $\xi(\omega, \cdot)\in L_p$ a.s. $[P]$, $0 < p < \infty$, if and only if $\theta\in L_p$ and $K\in L_{p/2}$.*

Proof. Let us suppose that $\theta(t)\in L_p$ and $K(t, t)\in L_{p/2}$. Then $\{\xi(t)-\theta(t);\ t\in T\}$ is a symmetric Gaussian process with the covariance function belonging to $L_{p/2}$. Using Lemma 4.1 and Fubini's Theorem, we can easily obtain $\xi(\omega, \cdot)-\theta(\cdot)\in L_p$ a.s. $[P]$ (see [9], Proposition 3.4). Since $\theta\in L_p$, we have $\xi(\omega, \cdot)\in L_p$ a.s. $[P]$.

Conversely, let $\xi(\omega, \cdot)\in L_p$ a.s. $[P]$. Let η_1, η_2 be two copies of ξ defined on $(\Omega\times\Omega, \Sigma\times\Sigma, P\times P)$: $\eta_i(\omega) = \xi(\omega_i)$, where $\omega = (\omega_1, \omega_2)\in\Omega\times\Omega$. Then, for every $t_1, \ldots, t_k\in T$,

$$\langle\eta_1(t_1), \ldots, \eta_1(t_k)\rangle, \qquad \langle\eta_2(t_1), \ldots, \eta_2(t_k)\rangle$$

are independent. Let $\zeta(\omega, t) = \big(\eta_1(\omega, t)-\eta_2(\omega, t)\big)/\sqrt{2}$. Then ζ is a symmetric Gaussian measurable process with paths $\in L_p$ a.s. $[P]$.

Let μ be the distribution of $\tilde{\zeta}$ in L_p. Then, by Theorem 1 in [7], we have

$$\int \|x\|_p^r\,\mu(dx) < \infty$$

for every $r > 0$. Hence, by Fubini's Theorem and Lemma 4.1, we obtain

$$\int\{E\zeta(t)^2\}^{p/2}m(dt) = C(2, p)\int E|\zeta(t)|^p m(dt) = C(2, p)E\|\tilde{\zeta}(\omega)\|_p^r$$
$$= C(2, p)\int \|x\|_p^r\,\mu(dx),$$

where $r = 1$ if $0 < p \leqslant 1$ and $r = 1/p$ if $1 < p < \infty$. Let us observe that $E\zeta(t) = 0$ and that $K(t, s) = E\zeta(t)\zeta(s)$. Now, since $\{\xi(t)-\theta(t);\ t\in T\}$ is a symmetric Gaussian measurable process having the covariance function in $L_{p/2}$, by the first part of the proof we have $\xi(\omega, \cdot)-\theta(\cdot)\in L_p$ a.s. $[P]$. So, by the equality

$$\theta(t) = \xi(t)-\big(\xi(t)-\theta(t)\big),$$

we obtain $\theta\in L_p$.

Remark 4.2. Let X be a Gaussian random element with values in L_p. Then there exists a symmetric Gaussian random element Y and $y \in L_p$ such that $X = Y+y$ a.s.$[P]$. For, let ξ be a Gaussian measurable process such that $\tilde{\xi} = X$ a.s.$[P]$. Let $\theta(t) = E\xi(t)$. Then, by Theorem 4.2, $y = \theta \in L_p$ and $Y = \tilde{\xi} - \theta$ is a symmetric Gaussian random element satisfying the above statement.

COROLLARY 4.1. *Let μ be a Gaussian measure on L_p. Then the support of μ is the algebraic sum of an element of L_p and a closed linear subspace of L_p.*

Proof. This follows immediately from the above remark and Theorem 2.1.

By Theorem 2.2, we obtain

THEOREM 4.3. *Let μ be a Gaussian measure on L_p, $0 \leqslant p < \infty$, and let F be a completion measurable rational subspace of L_p. Then $\mu(F) = 0$ or $\mu(F) = 1$.*

By the same arguments as in [9], Corollary 4.1, we obtain

COROLLARY 4.2. *Let $\{\xi(t);\ t \in T\}$ be a measurable Gaussian process with the mean θ and the covariance function K and let f be a real $\mathscr{F}$-measurable function defined on T. Then either $f|\xi(\omega, \cdot)|^p \in L_1$ a.s. $[P]$ or $f|\xi(\omega, \cdot)|^p \notin L_1$ a.s. $[P]$, $0 < p < \infty$, according to whether $f|\theta|^p$ and $fK^{p/2}$ belong to L_1 or at least one of $f|\theta|^p$ and $fK^{p/2}$ does not belong to L_1.*

Remark 4.3. Let μ be a Gaussian measure on L_0. Then, by a similar method as in [2], Example, we can construct a linear Borel-measurable and one-to-one mapping $\Phi\colon D \to L_2(m)$ defined on a linear subspace D, $\mu(D) = 1$. Then the measure ν induced on L_2 by Φ: $\nu(A) = \mu(\Phi^{-1}A)$, $A \in \mathscr{B}(L_2)$, is Gaussian. Using this construction, the zero-one law for Gaussian measures on Banach spaces [1] and the fact that L_p is a Borel subset of L_0, we obtain the following result: let μ be a Gaussian measure on L_p and let G be a completion measurable subgroup of L_p; then $\mu(G) = 0$ or $\mu(G) = 1$.

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Remark 4.4. It is easy to observe that the results of Section 3 as well as Theorem 1.1 and Theorem 4.1 remain valid for spaces $N(L(\Omega, \Sigma, \mu))$ (see [10], Chapter I, § 3), in particular, for all Orlicz spaces.

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