The space $E$ of continuous piecewise linear functions on $I$ is a vector lattice, is dense in $C(I)$ by the Stone-Weierstrass theorem, but $C(I)$ is dense in $L_0(I)$, so $E$ is dense in $L_0(I)$. And clearly, if $r$ is a function and $rE \subseteq E_0$.

References


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Multipliers of $L_p^0$, II*

by

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Abstract. Let $X$ be an abelian group, the character group of a compact group $G$. For a subset $E$ of $X$ let $L_p^0(E)$ be the subspace of $E$-spectral functions in $L_p^0(G)$. We show that if $X$ is infinite and $p > 2$ is an even integer, then $E$ can be chosen so that not every multiplier of $L_p^0(E)$ extends to a multiplier of $L_p^0(G)$.

1. Let $G$ be a compact abelian group with character group $X$. For $1 \leq p \leq \infty$, let $L^p(G)$ be the usual Lebesgue space with respect to normalized Haar measure on $G$, and for $E \subseteq X$, let $L_p^0(E)$ be the translation invariant subspace of $L^p(G)$ consisting of those functions whose Fourier transforms vanish off of $E$. Let $M_p^0(E)$ denote the set of functions in $L^p(E)$ which are multipliers for the Fourier transform space of $L_p^0(E)$. That is, $M_p^0(E)$ is the set of functions $f \in L_p^0(E)$ such that for every $f \in L_p^0(E)$, there exists $g \in L_p^0(E)$ with $g(x) = f(x)$ for each $x \in E$. For $1 \leq p < \infty$, $M_p^0(E)$ can be identified with the space of operators on $L_p^0(E)$ which commute with translation by elements of $G$. Let $M_p^0 \subseteq M_p^0(E)$ denote the set of restrictions to $E$ of functions in $M_p^0 = M_p^0(E)$. Then, clearly, $M_p^0 \subseteq M_p^0(E)$. We are interested in the following questions:

(i) Does $M_p^0 \subseteq M_p^0(E)$?

(ii) For $1 \leq p_1 \leq 2 \leq p_2 < 2$ or $2 < p_1 \leq p_2 \leq \infty$, is $M_p^0 \subseteq M_p^0(E)$? (If $E = \mathbb{R}$ is replaced by $M_p^0 \subseteq M_p^0$, the answer is yes by the Riesz-Thorin theorem.)

Question (i) is posed for the circle group $T$ in [3], pp. 280–281, and has an affirmative answer for any $G$ if $p = 2$ (trivially) or if $p = \infty$ (see Proposition 1.2 below). On the other hand, in [12] the following theorem is proved.

**Theorem 1.1.** If $G$ is infinite and $1 \leq p < 2$, there exists $E \subseteq X$ for which $M_p^0 \subseteq M_p^0(E)$ is a proper subset of $M_p^0$.

The present paper is a sequel to [12], and our main result here, Theorem 4.1, will be an analogue of Theorem 1.1 for the case when $p > 2$ is an even integer. In Section 5 we will show that question (ii) sometimes has a negative answer.

* The results contained in this paper, together with those of [12], were announced in [11].
We start with an easy proposition which establishes a positive answer for question (i) when $p = \infty$. For unexplained definitions and notation, the reader may consult [8].

PROPOSITION 1.2. Let $G$ and $X$ be as above. Fix $E \subseteq X$ and $\varphi : M_p^E \rightarrow E$. There exists a multiplier $\Phi$ of $L^p(G)$ such that $\Phi|_E = \varphi$.

Proof. By [8], Theorem 35.9, it is enough to show that there exists $\mu : M(G)$ with $\hat{\mu}(\sigma) = \varphi(\sigma)$ if $\sigma \in E$. Since the translation invariant operator $T$ induced on $L^p(G)$ by the multiplier $\varphi$ is a bounded operator, it follows from the Hahn–Banach theorem that there exists $\mu : M(G)$ satisfying

$$\int_G \hat{f}(g^{-1}) \hat{\mu}(g) = T\hat{f}(1)$$

for $E$-polynomials $f$. (The symbol 1 stands for the identity element of $G$.)

But for such $f$, $T\hat{f}(1) = \sum_{\sigma \in E} \hat{f}(\sigma) \varphi(\sigma)$, so (1) implies that $\hat{\mu}(\sigma) = \varphi(\sigma)$ if $\sigma \in E$.

This proposition can easily be deduced from the more general results of [2], [7], or [13], but the proof for compact $G$ is so simple that we have included it for the sake of completeness.

We outline the rest of the paper: Sections 2 and 3 contain some preliminary results, Section 4 contains the proof of the main theorem (Theorem 4.1), and Section 5, as we have said, is concerned with question (ii).

2. Let $G$, $X$, and $E$ be as in Section 1. We start by introducing a number which measures the difficulty of interpolating functions in $M_p^E$ by functions in $M_{\varphi|_E}$.

DEFINITION 2.1. Let $K_p(E)$ be the infimum (possibly $\infty$) of the set $\{X > 0 : \text{for every } \varphi : M_p^E \text{ there exists } \Phi \in M^p \text{ with } \Phi|_E = \varphi \text{ and } X \leq \|\Phi\|_{M^p}\}$.

The usefulness for our purposes of the numbers $K_p(E)$ derives from Lemmas 2.5 and 2.7 below. In order to prove these lemmas, we introduce the spaces $A_p^E$. Fix $p$ with $1 < p < \infty$ and $q$ with $p^{-1} + q^{-1} = 1$.

DEFINITION 2.2. Let $A_p^E$ be the subspace of $C(G)$ consisting of those functions $f$ which have a representation

$$f = \sum_{i=1}^{\infty} f_i \ast g_i,$$

where $f_i \in L^p$, $g_i \in L^q$, and $\sum_{i=1}^{\infty} \|f_i\|_{L^p} \|g_i\|_{L^q} < \infty$.

For such a function $f$, define $\|f\|_{A_p^E}$ to be the infimum of the numbers

$$\sup_{\{f \ast g : g \in L^q\}} \|f\|_{L^p} \|g\|_{L^q},$$

where the inf is taken over all representations of $f$ as in (1).

With the norm $\|\cdot\|_{A_p^E}$, $A_p^E$ is a Banach space. We write $A_p^E$ for $A_p^E$ and recall that the spaces $A_p^E$ were originally defined in [6]. Our definition for $A_p^E$ is, of course, modelled after the one given for $A_p^D$ in [6].

The spaces $A_p^E$ are important because they are preduals of the spaces $M_p^E$.

LEMMA 2.3. If $M_p^E$ is the operator norm, then $M_p^E$ is isometrically isomorphic to the dual space of $A_p^E$. For trigonometric polynomials $f \in A_p^E$ and multipliers $\varphi : M_p^E$, the duality is given by $\langle f, \varphi \rangle = \sum_{\sigma \in E} f(\sigma) \varphi(\sigma)$.


To prove Lemma 2.5 we need the following slight variation of a result from [1].

LEMMA 2.4. For $i = 1, 2$, let $G_i$ be a compact abelian group with character group $\Gamma_i$, fix $E_i \subseteq \Gamma_i$, and let $\varphi_i$ be an element of $M_{\varphi_i}^E$. Then $\varphi_1 \ast \varphi_2$, considered as a function on $E_1 \times E_2 \subseteq \Gamma_1 \times \Gamma_2$, is an element of $M_{\varphi_1 \varphi_2}^{E_1 \times E_2}$. Further,

$$\|\varphi_1 \ast \varphi_2\|_{M_{\varphi_1 \varphi_2}^{E_1 \times E_2}} = \|\varphi_1\|_{M_{\varphi_1}^{E_1}} \|\varphi_2\|_{M_{\varphi_2}^{E_2}}.$$

Proof. See the proofs for [1], Chapitre III, Théorème 2, Lemme 1.

LEMMA 2.5. With notations as in Lemma 2.4, $K_p(\Gamma_1 \times \Gamma_2) \supseteq K_p(\Gamma_1) K_p(\Gamma_2)$.

Proof. From Lemma 2.3 and an elementary duality argument it follows that for any $E$

$$K_p(E) = \sup_{f \text{ an } E\text{-polynomial}} \|f\|_{A_p^E}.$$

Fix $\varepsilon > 0$ and, for $i = 1, 2$, let $f_i$ be an $E_i$-polynomial with $\|f_i\|_{E_i} = 1$, $\|f_i\|_{M_{\varphi_i}^{E_i}} \geq K_p(\Gamma_i) - \varepsilon$. Let us show first that

$$\|f_1 f_2\|_{M_{\varphi_1 \varphi_2}^{E_1 \times E_2}} \geq \|f_1\|_{M_{\varphi_1}^{E_1}} \|f_2\|_{M_{\varphi_2}^{E_2}}.$$

By Lemma 2.3, there exist $\varphi_i : M_{\varphi_i}^{E_i}$ with $\|\varphi_i\|_{M_{\varphi_i}^{E_i}} = 1$ and with

$$\sum_{\sigma \in \Gamma_i} f_i(\sigma) \varphi_i(\sigma) = \|f_i\|_{M_{\varphi_i}^{E_i}}.$$

By Lemma 2.4, $\|f_1 f_2\|_{M_{\varphi_1 \varphi_2}^{E_1 \times E_2}} = 1$, and so by Lemma 2.3 again

$$\|f_1 f_2\|_{M_{\varphi_1 \varphi_2}^{E_1 \times E_2}} \geq \langle f_1 f_2, \varphi_1 \varphi_2 \rangle = \sum_{(\sigma_1, \sigma_2) \in \Gamma_1 \times \Gamma_2} f_1(\sigma_1) f_2(\sigma_2) \varphi_1(\sigma_1) \varphi_2(\sigma_2)$$

$$= \|f_1\|_{M_{\varphi_1}^{E_1}} \|f_2\|_{M_{\varphi_2}^{E_2}}.$$
On the other hand, $A^p$ is a Banach algebra [9], and so $\|f\|_{A^p} \leq \|f\|_p \|f\|_p = 1$. Thus

$$K(E \times E) \leq \frac{1}{\|f\|_{A^p}} \mathcal{E} \leq \frac{1}{\|f\|_{A^p}} \mathcal{E} = (K_p(E) - \epsilon) \mathcal{E} = \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this establishes the lemma.

We shall need the next result to apply Lemma 2.5 to the integer group $Z$.

**Lemma 2.6.** Let $E$ be a finite subset of $\mathbb{Z}^n$ and fix $\epsilon > 0$. Let $\varphi$ be an element of $L^p(E)$. For each $m = 1, 2, \ldots$, let

$$F_m = \{n_1, n_2, \ldots, n_m \in \mathbb{Z} \mid (n_1, n_2) \in E\}$$

and define $\varphi_m$ on $F_m$ by $\varphi_m(n_1, n_2) = \varphi(n_1, n_2)$ (The function $\varphi_m$ is well defined for all large enough $m$.) Then there exists an integer $M$ such that for $m \geq M$,

$$\|\varphi_m - \varphi\|_{L^p(E)} < \epsilon.$$

Proof. If $E$ is finite, then we can choose $m$ such that $m \geq M$ and $f$ is an $E$-polynomial on $T^n$, then

$$(1 - \epsilon) \|f\|_{L^p(E)} \leq \|\sum_{(n_1, n_2) \in F_m} f(n_1, n_2)\|_{L^p(E)} \leq (1 + \epsilon) \|f\|_{L^p(E)}.$$

But it follows from [4], Lemmas 3.3 and 3.4, that, if $m$ is the Haar measure of the closed subgroup $(e^{i\theta}, e^{i\phi})$ of $T^n$, then the sequence $(\varphi_m)_{m=1}^\infty$ converges uniformly on compact subsets of $G(\mathbb{T}^n)$, and since the set $\{f \circ \pi E \mid \|f\|_{L^p(E)} = 1\}$ is compact in $G(\mathbb{T}^n)$, we see that $M$ can be chosen so that (1) holds for $m \geq M$.

**Lemma 2.7.** Let $E$ be a finite subset of $\mathbb{Z}$ and fix $\epsilon > 0$. There exists an integer $M$ such that for $m \geq M$ we have

$$K_p(E + mE) > \frac{1}{1 + \epsilon}.$$

Proof. Let $\varphi \in L^p(E \times E)$ be such that $\|\varphi\|_{L^p(E \times E)} = 1$ and

$$\|\varphi\|_{L^p(E \times E)} \geq \frac{1}{\|\varphi\|_{L^p(E \times E)}} = \mathcal{E} = \epsilon.$$

Let $M$ be as in the conclusion of Lemma 2.6 when in that lemma we take $F = E \times E$ and the present $\varphi$ and $\epsilon$. Then, using notation from Lemma 2.6, for $m \geq M$ we have

$$\|\varphi_m - \varphi\|_{L^p(E \times E)} < \epsilon.$$

Fix such an $m$ and let $h: \mathbb{Z}^2 \to \mathbb{Z}$ be the homomorphism $h(n_1, n_2) = n_1 + m n_2$. Suppose that $\varphi \in M^p$ is such that $\varphi \|_{L^p(E \times E)} = \varphi$. Then

$$\Phi \circ h^{-1} \in \mathcal{E},$$

so $\|\Phi \circ h^{-1}\|_{L^p(E \times E)} \geq K_p(E \times E)$ by (1). But $\|\varphi\|_{M^p} \geq \|\Phi \circ h\|_{M^p}$ by [10], Théorème 3, and $K_p(E \times E) \geq \mathcal{E}$ by Lemma 2.5. Thus if $\Phi \|_{L^p(E \times E)} = \mathcal{E}$, then $\|\Phi\|_{M^p} \geq K_p(E)^2$. Since $\|\varphi\|_{L^p(E \times E)} < 1 + \epsilon$ by (2), it follows that $K_p(E + mE) > \frac{1}{1 + \epsilon}$. Since $m \geq M$ was arbitrary, this proves the lemma.

3. In this section we establish that for certain values of $p$ and certain abelian groups $X$, one can choose finite sets $E \subseteq X$ having $K_p(E) > 1$. We begin by adopting some notation from [5].

Let $X \geq 0$ be an integer. For an abelian group $X$ and for $E \subseteq X$, $C(E, s)$ will denote the set of functions a defined on $E$ with values in $X$ such that $\sum a(n) = s$. For $a \in C(E, s)$, the set of all $s \in X$ such that $\sum a(n) = s$, $C(E, s, a)$ will be the set of all $x \in X$ for which $\prod x(n) = a$. The symbol $B(E, s, a)$ will denote the union of

$$\bigcup_{x \in X} C(E, s, a),$$

for an integer $p > 2$, $C_p(E, s)$ will be the set of functions in $C(E, s)$ whose values lie in $\{0, 1, \ldots, p - 1\}$, and $C_p(E, s, a)$, $B_p(E, s, a)$ are defined similarly.

For the remainder of this section $\varphi$ will denote a fixed prime $\geq 2$. For $m = 1, 2, \ldots, X_m$ will be the group $Z(\mathbb{Z})$, a subset $E$ of some $X_m$ will be called $\varphi$-independent if $B_p(E, x, s) = \{0\}$.

**Lemma 3.1.** For $m = 1, 2, \ldots, X_m$ is $\varphi$-independent for some $s = 2, 3, \ldots$ and if $\varphi \in L^p(E)$ is a function such that $\varphi(n) = 1$ for each $n \in E$, then $\|\varphi\|_{L^p(E)} = 1$.

Proof. For each $\varphi \in L^p(E, s)$, let $C(a) = \{\prod x(n) = a\}$. For $\varphi \in L^p(E, s)$, we can write

$$f^a = \prod_{a \in C(E, s, a)} \{C(a) \prod x(n) = a\} = \prod_{a \in C(E, s, a)} \{C(a) \prod x(n) = a\}.$$

Similarly, if $T$ denotes the translation invariant measure induced by the multiplier $\varphi$

$$(TY) = \prod_{a \in C(E, s, a)} \{C(a) \prod x(n) = a\}.$$

Since $\|\varphi\|_{L^p(E, s)} = \|\varphi\|_{L^p(E, s)}$ for any $g$, in order to prove that $T$ is an isometry, it is only necessary to show that $\prod \varphi(n) = a$, for fixed $y$, independent of $a \in C(E, s, y)$. But we can write $a \in C(E, s, y)$ as $b + c$, where $b \in C(E, s, 1)$, $c \in C(E, s, y)$, and $b(n) = a$ is a multiple of $q$ for each $n \in E$. Since $\varphi(n) = 1$ for $a \in E$, it follows that $\|\varphi(n) = a\|_{L^p(E, s)} = \|\varphi(n) = a\|_{L^p(E, s)}$. However, the $\varphi$-independence of $E$ implies that there is at most one $c \in C(E, s, y)$...
(see the proof of [5], Corollary 4.6), and this shows that \( \int \varphi(a)x^s \) is indeed independent of \( a \in C(E, s, y) \).

**Lemma 3.2.** For \( s = 2, 3, \ldots \) there exists \( E \subseteq X_{s-1} \) such that \( K_{x_2}(E) \geq 1 \).

**Proof.** Let \( X = X_{s-1} \). We will exhibit a \( g \)-independent set \( E \subseteq X \), a function \( \varphi \in \Phi(E) \) such that \( \varphi(x) = 1 \) if \( x \in E \), and an element \( x_2 \in X \setminus E \) such that the following holds: if \( E' = E \cup \{ x_2 \} \) and if \( \Phi \in \Phi(E') \) satisfies \( \Phi|_E = \varphi \), then \( \| \Phi \|_{E'}\|_{E''} \geq 1 \). By Lemma 3.1 and a compactness argument, this will establish the lemma.

We realize \( X \) as the set \( \{ 0, 1, \ldots, g-1 \}^{s-2} \) with coordinatewise addition modulo \( g \). We define

\[
\begin{align*}
&\mathbf{a}_s = (1, 1, \ldots, 1, q-1, \ldots, q-1, 0, 0, \ldots, 0), \\
&\mathbf{a}_s = (0, 1, 0, \ldots, 0), \\
&\mathbf{a}_s = (0, 1, 0, \ldots, 0), \\
&\mathbf{a}_{s-2} = (0, 0, \ldots, 0, 0, 0, \ldots, 0), \\
&\mathbf{a}_0 = (1, 1, \ldots, 1, 0, 0, \ldots, 0, 0, 0, \ldots, 0).
\end{align*}
\]

Let \( E = \{ \mathbf{a}_s \} \). It is easily checked that \( E \) is \( g \)-independent. Let \( \phi \neq 1 \) be a complex \( g \)th root of unity. Define \( \phi \in \Phi(E) \) by \( \varphi(x) = \phi \), \( \varphi(x) = 1 \) if \( x = j \), \( j = 2, \ldots, g^s - 2 \). With \( E' = E \cup \{ \mathbf{a}_0 \} \), assume that \( \Phi \in \Phi(E') \) satisfies \( \Phi|_E = \varphi \), \( \| \phi \|_{E'} \geq 1 \). We will derive a contradiction.

Let \( T \) be the translation invariant operator on the set of \( E' \)-polynomials corresponding to the multiplier function \( \Phi \). Consider the \( E' \)-polynomial \( f = \mathbf{f}(\mathbf{a}_s) \sum_{x \in E} \mathbf{a}_x \). As in the proof of Lemma 3.1,

\[
\begin{align*}
\mathbf{f}^* = \sum_{x \in E} \mathbf{a}_x \sum_{x \in E} \mathbf{f}(x)^{\mu_0} = \sum_{x \in E} \mathbf{a}_x \sum_{x \in E} \mathbf{f}(x)^{\mu_0} \mathbf{C}(x) \prod_{a \in \mathbb{C}} \mathbf{f}(a)^{\mu_0} \\
= \sum_{x \in E} \mathbf{a}_x \mathbf{C}(x) = \sum_{x \in E} \mathbf{a}_x \mathbf{C}(x) \prod_{a \in \mathbb{C}} \mathbf{f}(a)^{\mu_0}.
\end{align*}
\]

Similarly,

\[
\begin{align*}
(Tf)^* = \sum_{x \in E} \mathbf{a}_x \mathbf{f}(x)^{\mu_0} \mathbf{C}(x) \prod_{a \in \mathbb{C}} \mathbf{f}(a)^{\mu_0} \\
= \sum_{x \in E} \mathbf{a}_x \mathbf{f}(x)^{\mu_0} \mathbf{C}(x) \prod_{a \in \mathbb{C}} \mathbf{f}(a)^{\mu_0}.
\end{align*}
\]

The assumption \( \| \Phi \|_{E''} \geq 1 \) yields \( \| (Tf)^* \|_{E''} \leq \| f^* \|_{E'} \), which leads to

\[
\begin{align*}
0 \leq \sum_{x \in E} \left[ \left( \sum_{x \in E} \mathbf{f}(x) \mathbf{S}_y(x) \right) \left( \sum_{x \in E} \mathbf{f}(x) \mathbf{S}_y(x) \right) \right] \\
- \left( \sum_{x \in E} \mathbf{f}(x) \mathbf{C}(x) \prod_{a \in \mathbb{C}} \mathbf{f}(a)^{\mu_0} \mathbf{S}_y(x) \right) \left( \sum_{x \in E} \mathbf{f}(x) \mathbf{C}(x) \prod_{a \in \mathbb{C}} \mathbf{f}(a)^{\mu_0} \mathbf{S}_y(x) \right) \\
- \sum_{j \leq c} \sum_{j \leq c} \mathbf{f}(x) \mathbf{C}(x) \prod_{a \in \mathbb{C}} \mathbf{f}(a)^{\mu_0} \mathbf{S}_y(x) \mathbf{S}_y(x) \\
- 2 \Re \sum_{j \leq c} \sum_{j \leq c} \mathbf{f}(x) \mathbf{C}(x) \prod_{a \in \mathbb{C}} \mathbf{f}(a)^{\mu_0} \mathbf{S}_y(x) \mathbf{S}_y(x)
\end{align*}
\]

and this holds for any choice of \( f(x) \).

As in the proof of Lemma 3.1, it follows that \( \prod_{a \in \mathbb{C}} \varphi(a)^{\mu_0} \) is independent of \( \mathbf{a} \in C(E', s, y) \) as long as \( \mathbf{a}(\mathbf{x}_0) = 0 \). Thus \( \mathbf{S}_y(x) = |\mathbf{S}_y(x)| \) and so

\[
E(y) = 2 \Re \sum_{x \in E} \mathbf{S}_y(x) |\mathbf{S}_y(x)| |\mathbf{S}_y(x) - \overline{\mathbf{S}_y(x)}| + O(\mathbf{f}(\mathbf{a}_0)^{\mu_0}).
\]

Hence (1) yields

\[
0 \leq 2 \Re \sum_{x \in E} \mathbf{S}_y(x) |\mathbf{S}_y(x)| |\mathbf{S}_y(x) - \overline{\mathbf{S}_y(x)}| + O(\mathbf{f}(\mathbf{a}_0)^{\mu_0}).
\]

Since \( \mathbf{f}(\mathbf{a}_0) \) is arbitrary, this implies that

\[
\begin{align*}
0 &= \sum_{x \in E} \mathbf{S}_y(x) |\mathbf{S}_y(x)| |\mathbf{S}_y(x) - \overline{\mathbf{S}_y(x)}| \\
&= \sum_{x \in E} \mathbf{S}_y(x) |\mathbf{S}_y(x)| |\mathbf{S}_y(x) - \overline{\mathbf{S}_y(x)}|.
\end{align*}
\]

Since \( \prod_{a \in \mathbb{C}} \varphi(a)^{\mu_0} \) is independent of \( a \in C(E, s-1, y) \), and since \( \prod_{a \in \mathbb{C}} \varphi(a)^{\mu_0} \) is independent of \( b \in C(E, s-1, y) \), (2) becomes

\[
0 = \sum_{x \in E} \mathbf{S}_y(x) |\mathbf{S}_y(x)| |\mathbf{S}_y(x) - \overline{\mathbf{S}_y(x)}|.
\]

Since \( \prod_{a \in \mathbb{C}} \varphi(a)^{\mu_0} \) is independent of \( a \in C(E, s-1, y) \), and since \( \prod_{a \in \mathbb{C}} \varphi(a)^{\mu_0} \) is independent of \( b \in C(E, s-1, y) \), (2) becomes

\[
\begin{align*}
0 &= \sum_{x \in E} \mathbf{S}_y(x) |\mathbf{S}_y(x)| |\mathbf{S}_y(x) - \overline{\mathbf{S}_y(x)}|.
\end{align*}
\]
where, for a given $y$, $\alpha$ and $\beta$ are any fixed elements of $C(E, s, y)$ and $C(E, s-1, y\cdot x_n^{-1})$, respectively. Since $|\Psi(E)| \leq 1$, the real part of each summand in the RHS of (3) is $\geq 0$. Thus, to violate (3) and obtain our contradiction, we need only show that one of these summands has a strictly positive real part. There are two cases to consider, $\Phi(E) = 1$ and $\Phi(E) \neq 1$.

If $\Phi(E) = 1$, we examine the term of (3) corresponding to

\[ y = \left( \frac{1}{s}, \frac{1}{s}, \ldots, \frac{1}{s} \right). \]

Then $y = a_0, a_\alpha, a_{\alpha+1}, \ldots, a_{\alpha-1}$ and $\gamma_n^{-1} = a_0, a_\alpha, a_{\alpha+1}, \ldots, a_{\alpha-1}$. Let $\eta \in C(E, s, y)$ be defined by $\eta(a_0) = 1$ if $i = 1$, $3s+1, \ldots, 3s-2$, $a_0(a_0) = 0$ otherwise, and let $\beta \in C(E, s, \gamma_n^{-1})$ be defined by $\beta(a_0) = 1$ if $i = s+2, s+3, \ldots, 2s$, $\beta(a_0) = 0$ otherwise. The term corresponding to $y$ is

\[ \left( \sum_{x \in \Gamma(E, x_n^{-1})} C(a) \right) \left( \sum_{x \in \Gamma(E, x_n^{-1})} C(x) \right) \left( 1 - \Phi(E) \prod_{x \in \Gamma(E, x_n^{-1})} \varphi(x) \right). \]

The first two factors are strictly positive, since $\eta \in C(E, s, y)$, $\beta \in C(E, s, \gamma_n^{-1})$, respectively, $\Phi(E) = 1$, if we use the definitions of $\varphi$, $\eta$, $\beta$, we see that the last factor is $(1 - \omega)$. Thus (4) has strictly positive real part, a contradiction.

If $\Phi(E) \neq 1$, then we examine the term corresponding to

\[ y = \left( \frac{1}{s}, \frac{1}{s}, \ldots, \frac{1}{s} \right). \]

Using $y = a_0, a_\alpha, a_{\alpha+1}, \ldots, a_{\alpha-1}$ and $\gamma_n^{-1} = a_0, a_\alpha, a_{\alpha+1}, \ldots, a_{\alpha-1}$ to define $\eta \in C(E, s, y)$, $\beta \in C(E, s, \gamma_n^{-1})$, respectively, we reach a similar contradiction. This concludes the proof of the lemma.

Next we prove an analogue of Lemma 3.2 for the group $Z$.

**Lemma 3.3.** Fix $s = 2, 3, \ldots$. There exists a finite set $E \subseteq Z$ such that $K_{x_n}(E) > 1$.

**Proof.** Let $E = \{0, 1, \ldots, s \}$. For each $n \in Z$, the set $B(E, s, n)$ contains at most one element, so an argument like that in the proof of Lemma 3.1 shows that if $\varphi \in \Phi(E)$ has $|\varphi(n)| = 1$ for each $n \in E$, then $|\varphi|_{\varphi(E)} = 1$. Thus, as in the proof of Lemma 3.2, it suffices to exhibit $\varphi \in \Phi(E)$ with $|\varphi| = 1$ and a finite $E' \supseteq E$ such that if $\Phi(E)$ satisfies $|\varphi|_{\varphi(E)} = 1$, then $|\Phi|_{\varphi(E)} > 1$.

Let $E' = \{-1, 0, 1, s-1\}$ and define $\varphi(0) = 1, \varphi(1) = 1, \varphi(s-1) = -1$. Suppose, to get a contradiction, that there exists $\Phi(E)$ with $|\varphi| = 1$ and $|\Phi|_{\varphi(E)} = 1$. Considering the $E'$-polynomial $f(\varphi) = \bar{f}(1) - 1 \cdot \sum_{\alpha = 1}^{s-1} \varphi^{(\alpha)}$ and bearing in mind the fact that $\text{card}(B(E, s, n)) \leq 1$ for each $n \in Z$, we find that the argument of Lemma 3.2 yields

\[ \left( \sum_{x \in \Gamma(E, x_n^{-1})} C(a) \right) \left( \sum_{x \in \Gamma(E, x_n^{-1})} C(x) \right) \left( 1 - \Phi(E) \prod_{x \in \Gamma(E, x_n^{-1})} \varphi(x) \right). \]

where, for a given $n, a$ and $b$ are the unique elements of $C(E, s, n)$, $C(E, s-1, n+1)$, respectively. Since $|\Phi(E)| < 1$, it again suffices to show that one of the summands of the RHS of (3) has strictly positive real part. If $\Phi(E) = 1$, the term corresponding to $n = s$ will do: take $a(0) = 0, a(1) = 1, a(s+1) = 0$ and $b(0) = s-2, b(1) = 0, b(s+1) = 1$. If $\Phi(E) = 1$, choose $n = -1$ and take $a(0) = s-1, a(1) = 1, a(s+1) = 0, b(0) = s-3, b(1) = 2, b(s+1) = 0$.

**4. Our object now is to prove the following theorem.**

**Theorem 4.1.** Let $G$ be an infinite compact abelian group with character group $X$. Fix $s = 2, 3, \ldots$. There exists $E \subseteq X$ for which $\mathcal{M}^E|_E$ is a proper subset of $\mathcal{M}^E|_E$.

We begin by observing that it is enough to find $E \subseteq X$ of the form

\[ E = \bigcup_{i \in I} E_i, \]

where

\[ (a) \quad K_{x_n}(E_i) \to \infty, \]

\[ (b) \quad \sup_{x \in X} |x_n|_{\mathcal{M}^{E_i}} = S < \infty. \]

\[ (x_{E_i}) \quad \text{is the characteristic function of } E_i. \]

If (a) holds, then for every large positive number $M$ there exists some $i$ and some $v_i \in \mathcal{M}^{E_i}$ such that $|v_i|_{\mathcal{M}^{E_i}} \leq 1$ and

\[ |\varphi|_{\varphi(E_i)} \leq M \quad \text{if } \varphi|_{E_i} = \varphi_i. \]

Since $|x_n|_{\mathcal{M}^{E_i}} = S$ by (b) and since $|v_i|_{\mathcal{M}^{E_i}} < 1$, it follows that $|x_n|_{\mathcal{M}^{E_i}} \leq S$, where $v_i$ is extended to $E$ by $v_i|_{E \setminus E_i} = 0$. Thus (1) implies $K_{x_n}(E_i) \geq M|S|$. Since this holds for every $M$, $K_{x_n}(E) \to \infty$ and so $\mathcal{M}^{E_i} \sim \mathcal{M}^{E}$. Next we note that in view of (2), Theorems 2.1 and 2.2, it is sufficient to carry this out for the case $X = \mathcal{M}^E Z(q)$ for some prime $q$, $Z = \mathcal{M}^E Z(q^n)$ for some prime $q$, and $Z = \mathcal{M}^E Z(q^n)$, where $(q_n)^{m_n} = 1$ is an increasing sequence of primes.

Let $q = 2$ be a fixed prime. We will consider first the case of $X = \mathcal{M}^E Z(q^n)$. From Lemmas 3.2 and 2.5 it follows that there exists an increasing sequence of integers $\{n_i\}_{i=1}^\infty$ and a sequence $\{E_i\}_{i=1}^\infty$ with $E_i \subseteq Z(q^{n_i})$ and $K_{x_n}(E_i) \to \infty$. Then $X = \mathcal{M}^E Z(q^{n_i})$ and we can consider
each $E_i$ as a subset of $X$ and define $E = \bigcup_{i=1}^{\infty} E_i$. Now (a) is already satisfied by, say, [5], Theorem 2.1, and (b) follows from the fact that $E_i = E \cap Z(q)_{n_i}$ since $\|x|_{n_i}\|_{q_i} = 1$ in any space $M^{(p)}(1 \leq p \leq \infty)$.

Next we consider the case $X = Z$. From Lemmas 3.3 and 2.7 we can deduce the existence of an increasing sequence $(n_i)_{n=1}^{\infty}$ of integers and a sequence $(E_i)_{n=1}^{\infty}$ with $E_i \subseteq [n_i, n_{i+1})$, and $K_p(E_i) \to \infty$. Letting $E = \bigcup_{i=1}^{\infty} E_i$, we have $E_i = E \cap [n_i, n_{i+1})$. Thus (b) follows from the uniform boundedness of the norms of the $X(q_i, n_{i+1})$ in $M^{(p)}$.

To treat the cases $X = Z(q^{(p)})$, $X = \bigcup_{n=1}^{\infty} Z(q_n)$, we need to two lemmas on cyclic groups.

**Lemma 4.2** Let $(n_i)_{n=1}^{\infty}$ be an increasing sequence of positive integers. Let $G_n$ be the character group of $Z(n_i)$, considered as a subgroup of $T$, and fix $p (1 \leq p < \infty)$, a positive integer $N$, and $\varepsilon > 0$. Then there exists a positive integer $K$ such that for $k \geq K$ and any trigonometric polynomial of the form $f(x) = \sum_{n=0}^{N} f(n)e^{ixn}$, we have

$$
(1-\varepsilon)\|f\|_{L^p(G_n)} \leq \|f\|_{L^p(G_n)} \leq (1+\varepsilon)\|f\|_{L^p(G_n)}.
$$

**Proof.** Let $\lambda_n$ be the normalized Haar measure on $G_n$ and let $\lambda$ be Haar measure on $T$. Then $\lambda_n \to \lambda$ weak-$*$, so $\lambda_n \to \lambda$ uniformly on compact subsets of $O(T)$. Since the set $\{f: f(x) = \sum_{n=0}^{N} f(n)e^{ixn}, \|f\|_{L^p(G_n)} = 1\}$ is compact in $O(T)$, the lemma follows.

Our next lemma requires some additional notation. Let $E \subseteq Z$ be a finite set of nonnegative integers, and let $m$ be a nonnegative integer so large that $E \subseteq [0, m-1]$. Then, for $1 < p < \infty$, $K_p(E, m)$ will denote the number $K_p(E)$ when $E$ is considered as a subset of the group $Z(m) = \{0, 1, \ldots, m-1\}$, while $K_p(E, Z)$ will denote $K_p(E)$ when $E$ is considered as a subset of the group $Z$.

**Lemma 4.3.** For $1 < p < \infty$, there exists a constant $\delta_o > 0$ such that the following holds: for every increasing sequence $(n_i)_{n=1}^{\infty}$ of positive integers, there exists $K$ such that for $k \geq K$,

$$\delta_o K_p(E, Z) \leq K_p(E, n_i).$$

**Proof.** Let $C_p < \infty$ be such that $\|x|_{n_i}\|_{M^{(p)}} \leq C_p$, for $N = 1, 2, \ldots$. Fix $N$ such that $E \subseteq [0, N]$ and $K$ be as in the conclusion of Lemma 4.2 (where we take $\varepsilon = \frac{1}{4}$ and the present $N$). Let $E \subseteq M^{(p)}$ be such that $\|\Phi\|_{M^{(p)}} = 1$ and $\|\Phi\|_{M^{(p)}} \geq K_p(E, Z)$ if $\Phi(x) = 0$. For $k \geq K$ it follows from the conclusion of Lemma 4.2 that the norm of $\Phi$ in $M^{(p)}$ when $E$ is considered as a subset of $Z(n_i)$, is $\leq 3$. But if $\Phi \mid M^{(p)}_{G_n}$ is such that $\Phi|_E = \Phi$, then

$$\|\Phi\|_{M^{(p)}} \geq K_p(E, Z) \quad 3C_p.$$

(If $\|\Phi\|_{M^{(p)}_{G_n}}$ denotes the norm of $\Phi|_{G_n}$ in $M^{(p)}_{G_n}$ with $[0, N]$ considered as a subset of the group $X \times Z(n_i)$, then the inequality

$$\|\Phi\|_{M^{(p)}_{G_n}} \geq 3C_p \|\Phi\|_{M^{(p)}_{G_n}}$$

follows from the conclusion of Lemma 4.2. But $C_p \|\Phi\|_{M^{(p)}_{G_n}} \geq \|\Phi|_{G_n}\|_{M^{(p)}_{G_n}}$ since $\|x|_{n_i}\|_{M^{(p)}} < C_p$, and $\|\Phi|_{G_n}\|_{M^{(p)}_{G_n}} \geq K_p(E, Z)$, since $\Phi = \Phi$.)

Thus we may take $\delta_o = \frac{1}{3C_p}$ in the conclusion of the present lemma.

Now it is easy to treat the cases $X = Z(q^{(p)})$, $X = \bigcup_{n=1}^{\infty} Z(q_n)$ of Theorem 4.1. We start by recalling that for $s = 2, 3, \ldots$

(2) there exists a sequence $(E_i)_{i=1}^{\infty}$ of finite sets of nonnegative integers such that $K_p(E_i) \to \infty$.

Fix a prime $q$. The group $Z(q^{(p)})$ is an increasing union of finite cyclic groups, so $Z(q^{(p)}) = \bigcup_{n=1}^{\infty} H_q$. By (2) and Lemma 4.3, there exists an increasing sequence $(n_i)_{i=1}^{\infty}$ of positive integers and a sequence $(E_i)_{i=1}^{\infty}$ of sets with $E_i \subseteq H_{n_i+1} \subseteq H_{n_i}$ such that $K_p(E_i) \to \infty$ and such that $E_i$ is contained in a coset of $C_i$ of $H_{n_i}$ in $H_{n_i+1}$. If $E = \bigcup_{i=1}^{\infty} E_i$, it follows from $E_i = E \cap E_i$ that $\|x|_{n_i}\|_{M^{(p)}} = 1$. That is, the $E_i$ can be picked so that (a) and (b) at the beginning of the section are satisfied. This proves Theorem 4.1 when $X = Z(q^{(p)})$.

Now suppose that $(n_i)_{i=1}^{\infty}$ is an increasing sequence of primes. Reasoning as above, there exist subsequences $(n_i)_{i=1}^{\infty}$ and $(E_i)_{i=1}^{\infty}$ with $K_p(E_i) \to \infty$ and with $E_i \subseteq Z(q_{n_i})$. Considering each $E_i$ as a subset of $\bigcup_{i=1}^{\infty} Z(q_{n_i})$ and letting $E = \bigcup_{i=1}^{\infty} E_i$, we see that (a) and (b) are again satisfied, for $E_i = E \cap Z(q_{n_i})$. This completes the proof of Theorem 4.1.

We note that the only ingredients of the above proof which depend on the assumption that $p > 2$ is an even integer are Lemmas 3.2 and 3.3, which constitute the computational part of the proof.
5. In this section we establish the following theorem.

**Theorem 5.1.** For \( n = 3, 4, \ldots \), there exists a subset \( E \) of the group \( \mathbb{Z}^* \) and a function \( f \in \mathbb{M}_E^p \) such that \( f \) is in \( \mathbb{M}_E^p \) for only a finite number of \( p \) (recall \( 2, 2n-2 \)), i.e., \( f \) is in no space \( \mathbb{M}_E^p \) for \( p \neq 2 \).

We will need the following lemma.

**Lemma 5.2.** Fix \( n = 3, 4, \ldots \). Let \( \sigma \) be a primitive \( n \)-th root of unity and choose a complex number \( \lambda \) with \(|\lambda| = 1, \chi \not\equiv E\), and \(|1+i\lambda| > |1+i\lambda^3| = 1, i = 1, \ldots, n-1\). The function of a real variable

\[
F_p(\theta) = \prod_{i=1}^{n-1} (1 + \lambda^i \omega^j)^p
\]

can have \( \frac{d}{d\theta} F_p(\theta) |_{\theta = 0} = 0 \) for only a finite number of \( p \equiv (2, 2n-2) \), say \( p = p_1, \ldots, p_r \).

Proof. Since distinct multiplicative functions on a group are linearly independent \([8], \text{Lemma 29.41}\), it is not the case that

\[(1) \quad |1+i\lambda^i \chi(1) + |1+i\lambda^i \chi(1)| + \ldots + |1+i\lambda^{n-1} \chi(1)| = 0\]

holds for each \( \sigma \not\equiv E \). But the LHS of (1) defines an entire function of \( \sigma \), so (1) can hold for only a finite number of \( \sigma \) in any finite interval. Since \( \frac{d}{d\theta} F_p(\theta) |_{\theta = 0} = 0 \) implies that (1) holds for \( \sigma = p - 2 \), this proves the lemma.

Now fix \( n = 3, 4, \ldots \) and regard \( Z(n) = \{0, 1, \ldots, n-1\} \) as the character group of the subgroup \( G_n \) of \( T \) consisting of the \( n \)-th roots of unity. Let \( E \subseteq Z(n) \) be the set \( \{0, 1\} \). A computation shows that if the Haar measure on \( G_n \) is normalized, then for any \( E \)-polynomial on \( G_n \) we have

\[
|f|_{L^1(G_n)} = |f(0)|^{n-1} + |f(1)|^{n-1} + \sum_{i=1}^{n-1} |f(i)|^{n-1} \geq \left( \frac{n(n-1)}{2} \right)^{1/p} \left( \frac{2}{n} \right)^{n-1} \left( \frac{1}{n} \right)^{p-1} \geq \left( \frac{n(n-1)}{2} \right)^{1/p} \left( \frac{2}{n} \right)^{n-1} \left( \frac{1}{n} \right)^{p-1} \geq 1.
\]

Thus \( |f|_{L^1(G_n)} \) depends only on \( |f(0)|, |f(1)| \), and so each function \( f \) defined on \( E \) by \( \hat{f}(0) = 1, \hat{f}(1) = e^{i\theta} \) for \( \theta \in E \) has \( \|f\|_{L^1(G_n)} = 1 \).

If, on the other hand, \( \|f\|_{L^1(G_n)} \leq 1 \) for some value of \( p \), then it must be the case that for any number \( \hat{f}(0), \hat{f}(1) \), we have the inequality

\[
(2) \quad \left( \frac{1}{n} \sum_{j=1}^{n-1} |\hat{f}(0) + \hat{f}(1) e^{i\theta} \omega^j|^{p} \right)^{1/p} \leq \left( \frac{1}{n} \sum_{j=1}^{n-1} |\hat{f}(0) + \hat{f}(1) \omega^j|^{p} \right)^{1/p},
\]

where \( \omega \) is a primitive \( n \)-th root of unity. This is so because the RHS of (2) is the norm in \( L^p(G_n) \) of the \( E \)-polynomial having Fourier coefficients \( \hat{f}(0) \) and \( \hat{f}(1) \). Let \( \{\hat{f}_j\}_{j=1}^{n-1} \) be a sequence of (positive) numbers such that \( \hat{f}_j > 0 \). Taking \( \hat{f}(0) = 1, \hat{f}(1) = \lambda \) in (2), it follows from Lemma 5.2 that if \( p \equiv (2, 2n-2) \), then there exists some \( \theta \) such that \( \hat{f}(0) > \hat{f}(1) \).

Now let \( E \subseteq \mathbb{Z}^* \) be the set \( \{0, 1\} \). That is, \( E \) is the set of all \( \sigma = (\sigma_1, \sigma_2, \ldots) \in \mathbb{Z}^* \) such that \( \sigma_i \in E \), \( i = 1, 2, \ldots \). Let \( \{\hat{f}_j\}_{j=1}^{n-1} \) be a sequence in each \( \hat{f}_j \) and each \( \hat{f}_j \) occur infinitely often, and define \( \varphi \) by the formula

\[
\varphi(\sigma_1, \sigma_2, \ldots) = \prod_{i=1}^{\infty} \varphi_i(\sigma_i).
\]

(This makes sense because \( \sigma_n = 0 \) for all large \( n \), \( \varphi_0(0) = 1 \). Then it follows easily from Lemma 2.4 and the equalities \( \|\varphi_0\|_{V^0(G_n)} = 1 \), \( \hat{f}(0) = 1, \hat{f}(1) = \lambda \), \( \|\varphi_0\|_{V^0(G_n)} = 1 \). But if \( p \equiv (2, 2n-2) \), then \( \|\varphi\|_{L^p(G_n)} \geq 1 + \varepsilon \) for some \( \varepsilon > 0 \) and an infinite number of indices \( i \). Together with Lemma 2.4 imply that \( \varphi \in \mathbb{M}_E^p \). This completes the proof of Theorem 5.1.

References

Gaussian measures on $L^p$ spaces $0 \leq p < \infty$

by

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Abstract. Using the correspondence between measures on $L^p$ spaces, $0 < p < \infty$, and measurable processes with paths in $L^p$, given in [2], we prove that the independence of random elements with values in $L^p$ as well as their distributions are determined by finite-dimensional distributions of the corresponding measurable processes. This result is applied to the investigation of Gaussian random elements with values in $L^p$, $0 < p < \infty$.

It is well known that if $X_1, X_2$ are independent real random variables such that $X_1 + X_2$ and $X_1 - X_2$ are independent, then $X_1, X_2$ are Gaussian. This property of real Gaussian random variables, proved already by Bernstein, has been used by Fréchet as one of two (equivalent) definitions of Gaussian random elements with values in a Banach space [5]. This definition allows us to consider Gaussian random elements in metric linear spaces which admit no non-trivial continuous linear functionals. The best known examples of such spaces are $L^p = L^p(m)$ spaces, where $m$ is the Lebesgue measure on $[0, 1]$ and $0 \leq p < 1$. Of course, in such spaces the classical definition of Gaussian elements cannot be used. In this paper we investigate Gaussian random elements on $L^p$ spaces, $0 \leq p < \infty$. For $p > 1$ these results were proved by Rajput [8].

Section 1 is preliminary. In Section 2 we prove two results. In Theorem 2.1 we prove that the support of a symmetric Gaussian measure defined on a linear metric space is a closed linear subspace. In Theorem 2.2 we give a short proof of the 0–1 law for Gaussian measures defined on complete separable metric linear spaces.

In Section 3 we consider measurable processes with paths in $L^p$. Theorem 1.1 proved in [2] gives the correspondence between measures on $L^p$ and measurable processes with paths in $L^p$. We prove that the independence of random elements with values in $L^p$ as well as their distributions are determined by finite-dimensional distributions of the corresponding measurable processes.

In Section 4 we apply the results of the preceding sections to prove results analogous to those obtained by Rajput [8] for $p \geq 1$. Theorem