The space E_0 of continuous piecewise linear functions on I is a vector lattice, is dense in $C(I)$ by the Stone–Weierstrass theorem, but $C(I)$ is dense in $L_\varrho(I)$, so E_0 is dense in $L_\varrho(I)$. And clearly, τ is constant if τ is a function and $\tau E_0 \subseteq E_0$.

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Multipliers of L_E^p, II*

by

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Abstract. Let X be an abelian group, the character group of a compact group G. For a subset E of X let L_E^p be the subspace of E-spectral functions in $L^p(G)$. We show that if X is infinite and $p > 2$ is an even integer, then E can be chosen so that not every multiplier of $\widehat{L_E^p}$ extends to a multiplier of $\widehat{L^p}(G)$.

1. Let G be a compact abelian group with character group X. For $1 \leqslant p \leqslant \infty$, let $L^p(G)$ be the usual Lebesgue space with respect to normalized Haar measure on G, and for $E \subseteq X$, let L_E^p be the translation invariant subspace of $L^p(G)$ consisting of those functions whose Fourier transforms vanish off of E. Let M_E^p denote the set of functions in $l^\infty(E)$ which are multipliers for the Fourier transform space of L_E^p. Thus $\varphi \in M_E^p$ if and only if for every $f \in L_E^p$ there exists $g \in L_E^p$ with $\hat{g}(x) = \varphi(x)\hat{f}(x)$ for each $x \in E$. For $1 \leqslant p < \infty$, M_E^p can be identified with the space of operators on L_E^p which commute with translation by elements of G. Let $M^p|_E$ denote the set of restrictions to E of functions in $M^p (= M_X^p)$. Then, clearly, $M^p|_E \subseteq M_E^p$. We are interested in the following questions:

(i) Does $M^p|_E = M_E^p$?

(ii) For $1 \leqslant p_1 < p_2 < 2$ or $2 < p_2 < p_1 \leqslant \infty$, is $M_E^{p_1} \subseteq M_E^{p_2}$? (If $M_E^{p_i}$ is replaced by $M^{p_i}|_E$, $i = 1, 2$, the answer is yes, by the Riesz–Thorin theorem.)

Question (i) is posed for the circle group T in [3], pp. 280–281, and has an affirmative answer for any G if $p = 2$ (trivially) or if $p = \infty$ (see Proposition 1.2 below). On the other hand, in [12] the following theorem is proved.

THEOREM 1.1. *If G is infinite and $1 \leqslant p < 2$, there exists $E \subseteq X$ for which $M^p|_E$ is a proper subset of M_E^p.*

The present paper is a sequel to [12], and our main result here, Theorem 4.1, will be an analogue of Theorem 1.1 for the case when $p > 2$ is an even integer. In Section 5 we will show that question (ii) sometimes has a negative answer.

* The results contained in this paper, together with those of [12], were announced in [11].

We start with an easy proposition which establishes a positive answer for question (i) when $p = \infty$. For unexplained definitions and notation, the reader may consult [8].

PROPOSITION 1.2. *Let G and X be as above. Fix $E \subseteq X$ and $\varphi \in M_E^\infty$. There exists a multiplier Φ of $\widehat{L^\infty}(G)$ such that $\Phi|_E = \varphi$.*

Proof. By [8], Theorem 35.9, it is enough to show that there exists $\mu \in M(G)$ with $\hat{\mu}(x) = \varphi(x)$ if $x \in E$. Since the translation invariant operator T induced on L_E^∞ by the multiplier φ is a bounded operator, it follows from the Hahn–Banach theorem that there exists $\mu \in M(G)$ satisfying

$$\int_G f(g^{-1})\,d\mu(g) = Tf(1) \tag{1}$$

for E-polynomials f. (The symbol 1 stands for the identity element of G.) But for such f, $Tf(1) = \sum_{x \in E} \hat{f}(x)\varphi(x)$, so (1) implies that $\hat{\mu}(x) = \varphi(x)$ if $x \in E$.

This proposition can easily be deduced from the more general results of [2], [7], or [13], but the proof for compact G is so simple that we have included it for the sake of completeness.

We outline the rest of the paper: Sections 2 and 3 contain some preliminary results, Section 4 contains the proof of the main theorem (Theorem 4.1), and Section 5, as we have said, is concerned with our question (ii).

2. Let G, X, and E be as in Section 1. We start by introducing a number which measures the difficulty of interpolating functions in M_E^p by functions in $M^p|_E$.

DEFINITION 2.1. Let $K_p(E)$ be the infimum (possibly ∞) of the set $\{K > 0\colon$ for every $\varphi \in M_E^p$ there exists $\Phi \in M^p$ with $\Phi|_E = \varphi$ and $\|\Phi\|_{M^p} \leqslant K\|\varphi\|_{M_E^p}\}$.

The usefulness for our purposes of the numbers $K_p(E)$ derives from Lemmas 2.5 and 2.7 below. In order to prove these lemmas, we introduce the spaces A_E^p. Fix p with $1 < p < \infty$ and q with $p^{-1} + q^{-1} = 1$.

DEFINITION 2.2. Let A_E^p be the subspace of $C(G)$ consisting of those functions f which have a representation

$$f = \sum_{i=1}^\infty f_i * g_i, \text{ where } f_i \in L_E^p,\ g_i \in L^q, \quad \text{and} \quad \sum_{i=1}^\infty \|f_i\|_{L^p}\|g_i\|_{L^q} < \infty. \tag{1}$$

For such a function f, define $\|f\|_{A_E^p}$ to be the infimum of the numbers $\sum_{i=1}^\infty \|f_i\|_{L^p}\|g_i\|_{L^q}$, where the inf is taken over all representations of f as in (1).

With the norm $\|\cdot\|_{A_E^p}$, A_E^p is a Banach space. We write A^p for A_X^p and recall that the spaces A^p were originally defined in [6]. Our definition for A_E^p is, of course, modelled after the one given for A^p in [6].

The spaces A_E^p are important because they are preduals of the spaces M_E^p.

LEMMA 2.3. *If M_E^p is given the operator norm, then M_E^p is isometrically isomorphic to the dual space of A_E^p. For trigonometric polynomials $f \in A_E^p$ and multipliers $\varphi \in M_E^p$, the duality is given by $\langle f, \varphi\rangle \mapsto \sum_{x \in E} \hat{f}(x)\varphi(x)$.*

Proof. The proof given for A^p in [6] works here as well.

To prove Lemma 2.5 we need the following slight variation of a result from [1].

LEMMA 2.4. *For $i = 1, 2$, let G_i be a compact abelian group with character group X_i, fix $E_i \subseteq X_i$, and let φ_i be an element of $M_{E_i}^p$. Then $\varphi_1 \cdot \varphi_2$, considered as a function on $E_1 \times E_2 \subseteq X_1 \times X_2$, is an element of $M_{E_1 \times E_2}^p$. Further,*

$$\|\varphi_1 \cdot \varphi_2\|_{M_{E_1 \times E_2}^p} = \|\varphi_1\|_{M_{E_1}^p} \cdot \|\varphi_2\|_{M_{E_2}^p}.$$

Proof. See the proofs for [1], Chapitre III, Théorème 2, Lemme 1.

LEMMA 2.5. *With notations as in Lemma 2.4, $K_p(E_1 \times E_2) \geqslant K_p(E_1)K_p(E_2)$.*

Proof. From Lemma 2.3 and an elementary duality argument it follows that for any E

$$K_p(E) = \sup_{\substack{f \neq 0 \\ f \text{ an } E\text{-polynomial}}} \frac{\|f\|_{A_E^p}}{\|f\|_{A^p}}.$$

Fix $\varepsilon > 0$ and, for $i = 1, 2$, let f_i be an E_i-polynomial with $\|f_i\|_{A^p} = 1$, $\|f_i\|_{A_{E_i}^p} \geqslant K_p(E_i) - \varepsilon$. Let us show first that

$$\|f_1 f_2\|_{A_{E_1 \times E_2}^p} \geqslant \|f_1\|_{A_{E_1}^p} \|f_2\|_{A_{E_2}^p}.$$

By Lemma 2.3, there exist $\varphi_i \in M_{E_i}^p$ with $\|\varphi_i\|_{M_{E_i}^p} = 1$ and with

$$\sum_{x_i \in E_i} \hat{f}(x_i)\varphi(x_i) = \|f_i\|_{A_{E_i}^p}.$$

By Lemma 2.4, $\|\varphi_1\varphi_2\|_{M_{E_1 \times E_2}^p} = 1$, and so by Lemma 2.3 again

$$\|f_1 f_2\|_{A_{E_1 \times E_2}^p} \geqslant \langle f_1 f_2, \varphi_1\varphi_2\rangle = \sum_{(x_1, x_2) \in E_1 \times E_2} \hat{f}_1(x_1)\hat{f}_2(x_2)\varphi_1(x_1)\varphi_2(x_2)$$
$$= \|f_1\|_{A_{E_1}^p} \|f_2\|_{A_{E_2}^p}.$$

On the other hand, A^p is a Banach algebra [9], and so $\|f_1f_2\|_{A^p} \leqslant \|f_1\|_{A^p}\|f_2\|_{A^p} = 1$. Thus

$$K(E_1\times E_2) \geqslant \frac{\|f_1f_2\|_{A^p_{E_1\times E_2}}}{\|f_1f_2\|_{A^p}} \geqslant \|f_1\|_{A^p_{E_1}}\|f_2\|_{A^p_{E_2}} \geqslant (K_p(E_1)-\varepsilon)(K_p(E_2)-\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, this establishes the lemma.

We shall need the next result to apply Lemma 2.5 to the integer group Z.

Lemma 2.6. *Let F be a finite subset of Z^2 and fix $\varepsilon > 0$. Let φ be an element of $l^\infty(F)$. For each $m = 1, 2, \ldots$, set*

$$F_m = \{n_1+mn_2 \in Z\colon (n_1, n_2) \in F\}$$

and define φ_m on F_m by $\varphi_m(n_1+mn_2) = \varphi(n_1, n_2)$. (The function φ_m is well defined for all large enough m.) Then there exists an integer M such that for $m \geqslant M$,

$$\big|\|\varphi_m\|_{M^p_{F_m}} - \|\varphi\|_{M^p_F}\big| < \varepsilon.$$

Proof. It suffices to show that for any $\varepsilon_1 > 0$ there exists M_1 such that if $m \geqslant M_1$ and f is any F-polynomial on T^2, then

(1) $$(1-\varepsilon_1)\|f\|_{L^p(T^2)} \leqslant \Big\|\sum_{(n_1,n_2)\in F}\hat f(n_1, n_2)e^{i(n_1+mn_2)\theta}\Big\|_{L^p(T)} \leqslant (1+\varepsilon_1)\|f\|_{L^p(T^2)}.$$

But it follows from [4], Lemmas 3.3 and 3.4, that, if λ_m is the Haar measure of the closed subgroup $\{(e^{i\theta}, e^{im\theta})\colon e^{i\theta} \in T\}$ of T^2, then the sequence $\{\lambda_m\}_{m=1}^\infty$ converges weak-* to the Haar measure λ of T^2. Since then $\lambda_m \to \lambda$ uniformly on compact subsets of $C(T^2)$, and since the set $\{|f|^p\colon f \in L^p_F, \|f\|_{L^p} = 1\}$ is compact in $C(T^2)$, we see that M_1 can be chosen so that (1) holds for each $m \geqslant M_1$.

Lemma 2.7. *Let E be a finite subset of Z and fix $\varepsilon > 0$ There exists an integer M such that for $m \geqslant M$ we have*

$$K_p(E+mE) > \frac{K_p(E)^2}{1+\varepsilon}.$$

Proof. Let $\varphi \in l^\infty(E\times E)$ be such that $\|\varphi\|_{M^p_{E\times E}} = 1$ and

(1) $$\|\psi\|_{M^p_{Z^2}} \geqslant K_p(E\times E) \text{ if } \psi \in M^p_{Z^2} \text{ satisfies } \psi|_{E\times E} = \varphi.$$

Let M be as in the conclusion of Lemma 2.6 when in that lemma we take $F = E\times E$ and the present φ and ε. Then, using notation from Lemma 2.6, for $m \geqslant M$ we have

(2) $$\|\varphi_m\|_{M^p_{E+mE}} < 1+\varepsilon.$$

Fix such an m and let $h\colon Z^2 \to Z$ be the homomorphism $h(n_1, n_2) = n_1+mn_2$. Suppose that $\Phi \in M^p_Z$ is such that $\Phi|_{E+mE} = \varphi_m$. Then $\Phi\circ h|_{E\times E} = \varphi$, so $\|\Phi\circ h\|_{M^p_{Z^2}} \geqslant K_p(E\times E)$ by (1). But $\|\Phi\|_{M^p_Z} \geqslant \|\Phi\circ h\|_{M^p_{Z^2}}$ by [10], Théorème 3, and $K_p(E\times E) \geqslant K^p(E)^2$ by Lemma 2.5. Thus if $\Phi|_{E+mE} = \varphi_m$, then $\|\Phi\|_{M^p_Z} \geqslant K_p(E)^2$. Since $\|\varphi_m\|_{M^p_{E+mE}} < 1+\varepsilon$ by (2), it follows that $K_p(E+mE) > \dfrac{K_p(E)^2}{1+\varepsilon}$. Since $m \geqslant M$ was arbitrary, this proves the lemma.

3. In this section we establish that for certain values of p and certain abelian groups X, one can choose finite sets $E \subseteq X$ having $K_p(E) > 1$. We begin by adopting some notation from [5].

Let $s \geqslant 0$ be an integer. For an abelian group X and for $E \subseteq X$, $C(E, s)$ will denote the set of functions a defined on E with values in N such that $\sum_{x\in E} a(x) = s$. For $x_0 \in X$, $C(E, s, x_0)$ will be the set of $a \in C(E, s)$ for which $\prod_{x\in E} x^{a(x)} = x_0$. The symbol $B(E, s, x_0)$ will denote the union $\bigcup_{t=1}^{s} C(E, t, x_0)$. For an integer $p \geqslant 2$, $C_p(E, s)$ will be the set of functions in $C(E, s)$ whose values lie in $\{0, 1, \ldots, p-1\}$, and $C_p(E, s, x_0)$, $B_p(E, s, x_0)$ are defined similarly.

For the remainder of this section q will denote a fixed prime $\geqslant 2$. For $m = 1, 2, \ldots, X_m$ will be the group $Z(q)^m$, and a subset E of some X_m will be called *qs-independent* if $B_q(E, qs, 1) = \{0\}$.

Lemma 3.1. *For $m = 1, 2, \ldots$, if $E \subseteq X_m$ is qs-independent for some $s = 2, 3, \ldots$ and if $\varphi \in l^\infty(E)$ is a function such that $\varphi^q(x) = 1$ for each $x \in E$, then $\|\varphi\|_{M^{2s}_E} = 1$.*

Proof. For each $a \in C(E, s)$, let $C(a) = s!\big(\prod_{x\in E} a(x)!\big)^{-1}$. Let $f = \sum_{x\in E}\hat f(x)x$ be an arbitrary E-polynomial. According to [8], 29.5, we can write

$$f^s = \sum_{a\in C(E,s)} C(a)\prod_{x\in E}[\hat f(x)x]^{a(x)} = \sum_{y\in X_m} y\Big(\sum_{a\in C(E,s,y)} C(a)\prod_{x\in E}\hat f(x)^{a(x)}\Big).$$

Similarly, if T denotes the translation invariant operator induced by the multiplier φ,

$$(Tf)^s = \sum_{y\in X_m} y\Big(\sum_{a\in C(E,s,y)} C(a)\prod_{x\in E}[\hat f(x)\varphi(x)]^{a(x)}\Big).$$

Since $\|g\|^{2s}_{L^{2s}} = \|g^s\|^2_{L^2}$ for any g, in order to prove that T is actually an isometry, it is only necessary to show that $\prod_{x\in E}\varphi(x)^{a(x)}$ is, for fixed y, independent of $a \in C(E, s, y)$. But we can write $a \in C(E, s, y)$ as $b+c$, where $b \in B(E, s, 1)$, $c \in B_q(E, s, y)$, and $b(x)$ is a multiple of q for each $x \in E$. Since $\varphi^q(x) = 1$ if $x \in E$, it follows that $\prod_{x\in E}\varphi(x)^{a(x)} = \prod_{x\in E}\varphi(x)^{c(x)}$. However, the qs-independence of E implies that there is at most one $c \in B_q(E, s, y)$

(see the proof of [5], Corollary 4.6), and this shows that $\prod_{x\in E}\varphi(x)^{a(x)}$ is indeed independent of $a\in C(E,s,y)$.

LEMMA 3.2. *Fix $s=2,3,\ldots$ There exists $E\subseteq X_{4s-3}$ such that $K_{2s}(E)>1$.*

Proof. Let $X=X_{4s-3}$. We will exhibit a qs-independent set $E\subseteq X$, a function $\varphi\in l^\infty(E)$ such that $\varphi^q(x)=1$ if $x\in E$, and an element $x_0\in X\setminus E$ such that the following holds: if $E'=E\cup\{x_0\}$ and if $\Phi\in l^\infty(E')$ satisfies $\Phi|_E=\varphi$, then $\|\Phi\|_{M^{2s}_{E'}}>1$. By Lemma 3.1 and a compactness argument, this will establish the lemma.

We realize X as the set $\{0,1,\ldots,q-1\}^{4s-3}$ with coordinatewise addition modulo q. We define

$$x_1=(\underbrace{1,1,\ldots,1}_{2s-1},\ \underbrace{q-1,q-1,\ldots,q-1}_{2s-2}),$$

$$x_2=(1,0,0,\ldots,0),$$

$$x_3=(0,1,0,0,\ldots,0),$$

$$\ldots\ldots\ldots\ldots$$

$$x_{4s-2}=(0,0,\ldots,0,1),$$

$$x_0=(\underbrace{1,1,\ldots,1}_{s},\ \underbrace{0,0,\ldots,0}_{s-1},\ \underbrace{q-1,q-1,\ldots,q-1}_{s-1},\ \underbrace{0,0,\ldots,0}_{s-1}).$$

Let $E=\{x_1,\ldots,x_{4s-2}\}$. It is easily checked that E is qs-independent. Let $w\neq 1$ be a complex qth root of unity. Define $\varphi\in l^\infty(E)$ by $\varphi(x_1)=w$, $\varphi(x_j)=1$ if $j=2,\ldots,4s-2$. With $E'=E\cup\{x_0\}$ assume that $\Phi\in l^\infty(E')$ satisfies $\Phi|_E=\varphi$, $\|\Phi\|_{M^{2s}_{E'}}=1$. We will derive a contradiction.

Let T be the translation invariant operator on the set of E'-polynomials corresponding to the multiplier function Φ. Consider the E'-polynomial $f=\hat f(x_0)x_0+\sum_{x\in E}x$. As in the proof of Lemma 3.1,

$$f^s=\sum_{y\in X}y\Big(\sum_{a\in C(E',s,y)}\prod_{x\in E'}\hat f(x)^{a(x)}\Big)=\sum_{y\in X}\Big(\sum_{j=0}^{s}\hat f(x_0)^j\sum_{\substack{a\in C(E',s,y)\\ a(x_0)=j}}C(a)\prod_{x\in E}\hat f(x)^{a(x)}\Big)$$

$$=\sum_{y\in X}\Big(\sum_{j=0}^{s}\hat f(x_0)^j\sum_{\substack{a\in C(E',s,y)\\ a(x_0)=j}}C(a)\Big)=\sum_{y\in X}\Big(\sum_{j=0}^{s}\hat f(x_0)^jS_j(y)\Big).$$

Similarly,

$$(Tf)^s=\sum_{y\in X}\Big(\sum_{j=0}^{s}[\hat f(x_0)\Phi(x_0)]^j\sum_{\substack{a\in C(E',s,y)\\ a(x_0)=j}}C(a)\prod_{x\in E}\varphi(x)^{a(x)}\Big)$$

$$=\sum_{y\in X}\Big(\sum_{j=0}^{s}[\hat f(x_0)\Phi(x_0)]^jS_j'(y)\Big).$$

The assumption $\|\Phi\|_{M^{2s}_{E'}}=1$ yields $\|(Tf)^s\|^2_{L^2}\leqslant\|f^s\|^2_{L^2}$, which leads to

$$(1)\qquad 0\leqslant\sum_{y\in X}\Big\{\Big(\sum_{j=0}^{s}\hat f(x_0)^jS_j(y)\Big)\overline{\Big(\sum_{j=0}^{s}\hat f(x_0)^jS_j(y)\Big)}-$$

$$-\Big(\sum_{j=0}^{s}[\hat f(x_0)\Phi(x_0)]^jS_j'(y)\Big)\overline{\Big(\sum_{j=0}^{s}[\hat f(x_0)\Phi(x_0)]^jS_j'(y)\Big)}\Big\}$$

$$=\sum_{y\in X}\Big\{\sum_{j=0}^{s}|\hat f(x_0)|^{2j}|S_j(y)|^2+2\operatorname{Re}\sum_{\substack{j,k=0\\ j<k}}^{s}\hat f(x_0)^j\overline{\hat f(x_0)^k}S_j(y)\overline{S_k(y)}-$$

$$-\sum_{j=0}^{s}|\hat f(x_0)\Phi(x_0)|^{2j}|S_j'(y)|^2-2\operatorname{Re}\sum_{\substack{j,k=0\\ j<k}}^{s}\big(\hat f(x_0)\Phi(x_0)\big)^j\overline{\big(\hat f(x_0)\Phi(x_0)\big)^k}S_j'(y)\overline{S_k'(y)}\Big\}$$

$$=\sum_{y\in X}\{E(y)\},$$

and this holds for any choice of $\hat f(x_0)$.

As in the proof of Lemma 3.1, it follows that $\prod_{x\in E}\varphi(x)^{a(x)}$ is independent of $a\in C(E',s,y)$ so long as $a(x_0)=0$. Thus $|S_0(y)|=|S_0'(y)|$ and so

$$E(y)=2\operatorname{Re}\overline{\hat f(x_0)}\big(S_0(y)\overline{S_1(y)}-\overline{\Phi(x_0)}S_0'(y)\overline{S_1'(y)}\big)+O\big(|\hat f(x_0)|^2\big).$$

Hence (1) yields

$$0\leqslant 2\operatorname{Re}\Big\{\overline{\hat f(x_0)}\Big[\sum_{y\in X}\big(S_0(y)\overline{S_1(y)}-\overline{\Phi(x_0)}S_0'(y)\overline{S_1'(y)}\big)\Big]\Big\}+O\big(|\hat f(x_0)|^2\big).$$

Since $\hat f(x_0)$ is arbitrary, this implies that

$$(2)\qquad 0=\sum_{y\in X}\big(S_0(y)\overline{S_1(y)}-\overline{\Phi(x_0)}S_0'(y)\overline{S_1'(y)}\big)$$

$$=s\sum_{y\in X}\Big[\Big(\sum_{a\in C(E,s,y)}C(a)\Big)\Big(\sum_{b\in C(E,s-1,yx_0^{-1})}C(b)\Big)-$$

$$-\overline{\Phi(x_0)}\Big(\sum_{a\in C(E,s,y)}C(a)\prod_{x\in E}\varphi(x)^{a(x)}\Big)\overline{\Big(\sum_{b\in C(E,s-1,yx_0^{-1})}C(b)\prod_{x\in E}\varphi(x)^{b(x)}\Big)}\Big].$$

Since $\prod_{x\in E}\varphi(x)^{a(x)}$ is independent of $a\in C(E,s,y)$, and since $\prod_{x\in E}\varphi(x)^{b(x)}$ is independent of $b\in C(E,s-1,yx_0^{-1})$, (2) becomes

$$(3)\qquad 0=\sum_{y\in X}\Big[\Big(\sum_{a\in C(E,s,y)}C(a)\Big)\Big(\sum_{b\in C(E,s-1,yx_0^{-1})}C(b)\Big)\Big(1-\overline{\Phi(x_0)}\prod_{x\in E}\varphi(x)^{\overline{a}(x)-\overline{b}(x)}\Big)\Big],$$

where, for a given y, $\bar a$ and $\bar b$ are any fixed elements of $C(E, s, y)$ and $C(E, s-1, yx_0^{-1})$, respectively. Since $|\Phi(x_0)| \leqslant 1$, the real part of each summand in the RHS of (3) is $\geqslant 0$. Thus, to violate (3) and obtain our contradiction, we need only show that one of these summands has a strictly positive real part. There are two cases to consider, $\Phi(x_0) = 1$ and $\Phi(x_0) \neq 1$.

If $\Phi(x_0) = 1$, we examine the term of (3) corresponding to

$$y = (\underbrace{1, 1, \ldots, 1}_{2s-1},\ \underbrace{q-1, q-1, \ldots, q-1}_{s-1},\ \underbrace{0, 0, \ldots, 0}_{s-1}).$$

Then $y = x_1 x_{3s} x_{3s+1} \ldots x_{4s-2}$ and $yx_0^{-1} = x_{s+2} x_{s+3} \ldots x_{2s}$. Let $\bar{\bar a} \in C(E, s, y)$ be defined by $\bar{\bar a}(x_i) = 1$ if $i = 1, 3s, 3s+1, \ldots, 4s-2$, $\bar{\bar a}(x_i) = 0$ otherwise, and let $\bar{\bar b} \in C(E, s-1, yx_0^{-1})$ be defined by $\bar{\bar b}(x_i) = 1$ if $i = s+2, s+3, \ldots, 2s$, $\bar{\bar b}(x_i) = 0$ otherwise. The term corresponding to y is

$$(4)\qquad \Big(\sum_{a \in C(E,s,y)} C(a)\Big)\Big(\sum_{b \in C(E,s-1,yx_0^{-1})} C(b)\Big)\Big(1-\Phi(x_0)\prod_{x \in E}\varphi(x)^{\bar{\bar a}(x)-\bar{\bar b}(x)}\Big).$$

The first two factors are strictly positive, since $\bar{\bar a} \in C(E, s, y)$, $\bar{\bar b} \in C(E, s-1, yx_0^{-1})$. Since $\Phi(x_0) = 1$, if we use the definitions of φ, $\bar{\bar a}$, $\bar{\bar b}$, we see that the last factor is $(1-w)$. Thus (4) has strictly positive real part, a contradiction.

If $\Phi(x_0) \neq 1$, then we examine the term corresponding to

$$y = (\underbrace{1, 1, \ldots, 1}_{s},\ \underbrace{0, 0, \ldots, 0}_{3s-3}).$$

Using $y = x_2 x_3 \ldots x_{s+1}$, $yx_0^{-1} = x_{2s+1} x_{2s+2} \ldots x_{3s-1}$ to define $\bar{\bar a} \in C(E, s, y)$, $\bar{\bar b} \in C(E, s-1, yx_0^{-1})$, respectively, we reach a similar contradiction. This concludes the proof of the lemma.

Next we prove an analogue of Lemma 3.2 for the group Z.

LEMMA 3.3. *Fix $s = 2, 3, \ldots$ There exists a finite set $E \subseteq Z$ such that $K_{2s}(E) > 1$.*

Proof. Let $E = \{0, 1, s+1\}$. For each $n \in Z$, the set $B(E, s, n)$ contains at most one element, so an argument like that in the proof of Lemma 3.1 shows that if $\varphi \in l^\infty(E)$ has $|\varphi(n)| = 1$ for each $n \in E$, then $\|\varphi\|_{M_E^{2s}} = 1$. Thus, as in the proof of Lemma 3.2, it suffices to exhibit $\varphi \in l^\infty(E)$ with $|\varphi| = 1$ and a finite $E' \supseteq E$ such that if $\Phi \in l^\infty(E')$ satisfies $\Phi|_E = \varphi$, then $\|\Phi\|_{M_{E'}^{2s}} > 1$.

Let $E' = \{-1, 0, 1, s+1\}$ and define $\varphi(0) = 1$, $\varphi(1) = 1$, $\varphi(s+1) = -1$. Suppose, to get a contradiction, that there exists $\Phi \in l^\infty(E')$ with $\Phi|_E = \varphi$ and $\|\Phi\|_{M_{E'}^{2s}} = 1$. Considering the E'-polynomial $f(e^{i\theta}) = \hat f(-1)e^{-i\theta} + \sum_{m \in E} e^{im\theta}$ and bearing in mind the fact that $\operatorname{card}(B(E, s, n)) \leqslant 1$ for each $n \in Z$, we find that the argument of Lemma 3.2 yields

$$(3')\qquad 0 = \sum_{\substack{n \in Z \\ C(E,s,n) \neq \varphi,\ C(E,s-1,n+1) \neq \varphi}} C(a)C(b)\Big(1-\Phi(-1)\prod_{m \in E}\varphi(m)^{a(m)+b(m)}\Big),$$

where, for a given n, a and b are the unique elements of $C(E, s, n)$, $C(E, s-1, n+1)$, respectively. Since $|\Phi(-1)| \leqslant 1$, it again suffices to show that one of the summands of the RHS of (3′) has strictly positive real part. If $\Phi(-1) = 1$, the term corresponding to $n = s$ will do: take $a(0) = 0$, $a(1) = s$, $a(s+1) = 0$ and $b(0) = s-2$, $b(1) = 0$, $b(s+1) = 1$. If $\Phi(-1) \neq 1$, choose $n = 1$ and take $a(0) = s-1$, $a(1) = 1$, $a(s+1) = 0$, $b(0) = s-3$, $b(1) = 2$, $b(s+1) = 0$.

4. Our object now is to prove the following theorem.

THEOREM 4.1. *Let G be an infinite compact abelian group with character group X. Fix $s = 2, 3, \ldots$ There exists $E \subseteq X$ for which $M^{2s}|_E$ is a proper subset of M_E^{2s}.*

We begin by observing that it is enough to find $E \subseteq X$ of the form $E = \bigcup_{i=1}^{\infty} E_i$, where

(a) $K_{2s}(E_i) \to \infty$,

(b) $\sup_i \|\chi_{E_i}\|_{M_E^{2s}} = S < \infty$. ($\chi_{E_i}$ is the characteristic function of E_i.)

(If (a) holds, then for every large positive number M there exists some i and some $\varphi_i \in M_E^{2s}$ such that $\|\varphi_i\|_{M_{E_i}^{2s}} \leqslant 1$ and

$$(1)\qquad \|\Phi_i\|_{M^{2s}} \geqslant M \quad \text{if} \quad \Phi_i|_{E_i} = \varphi_i.$$

Since $\|\chi_{E_i}\|_{M_E^{2s}} \leqslant S$ by (b) and since $\|\varphi_i\|_{M_{E_i}^{2s}} \leqslant 1$, it follows that $\|\varphi_i\|_{M_E^{2s}} \leqslant S$, where φ_i is extended to E by $\varphi_i|_{E \setminus E_i} = 0$. Thus (1) implies $K_{2s}(E) \geqslant M/S$. Since this holds for every M, $K_{2s}(E) = \infty$ and so $M^{2s}|_E \neq M_E^{2s}$).

Next we note that in view of [5], Theorems 2.1 and 2.3, it is sufficient to carry this out for the cases $X = \overset{\infty}{\underset{n=1}{P^*}} Z(q)$ for some prime q, $X = Z$, $X = Z(q^\infty)$ for some prime q, and $X = \overset{\infty}{\underset{n=1}{P^*}} Z(q_n)$, where $\{q_n\}_{n=1}^\infty$ is an increasing sequence of primes.

Let $q \geqslant 2$ be a fixed prime. We will consider first the case of $X = \overset{\infty}{\underset{n=1}{P^*}} Z(q)$. From Lemmas 3.2 and 2.5 it follows that there exists an increasing sequence of integers $\{n_i\}_{i=1}^\infty$ and a sequence $\{E_i\}_{i=1}^\infty$ with $E_i \subseteq Z(q)^{n_i}$ and $K_{2s}(E_i) \to \infty$. Then $X = \overset{\infty}{\underset{i=1}{P^*}} Z(q)^{n_i}$ and we can consider each E_i as a subset of X and define $E = \bigcup_{i=1}^{\infty} E_i$. Now (a) is already satisfied (by, say, [5], Theorem 2.1), and (b) follows from the fact that $E_i = E \cap Z(q)^{n_i}$, since $\|\chi_{Z(q)^{n_i}}\| = 1$ in any space $M^p (1 \leqslant p \leqslant \infty)$.

Next we consider the case $X = Z$. From Lemmas 3.3 and 2.7 we can deduce the existence of an increasing sequence $\{n_i\}_{i=1}^{\infty}$ of integers and a sequence $\{E_i\}_{i=1}^{\infty}$ with $E_i \subseteq [n_i, n_{i+1}-1]$ and $K_{2s}(E_i) \to \infty$. Letting $E = \bigcup_{i=1}^{\infty} E_i$, we have $E_i = E \cap [n_i, n_{i+1}-1]$. Thus (b) follows from the uniform boundedness of the norms of the $\chi_{[n_i, n_{i+1}-1]}$ in M^{2s}.

To treat the cases $X = Z(q^\infty)$, $X = \overset{\infty}{\underset{n=1}{P^*}} Z(q_n)$, we need two lemmas on cyclic groups.

Lemma 4.2 *Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers. Let G_k be the character group of $Z(n_k)$, considered as a subgroup of T, and fix p $(1 \leqslant p < \infty)$, a positive integer N, and $\varepsilon > 0$. Then there exists a positive integer K such that for $k \geqslant K$ and any trigonometric polynomial of the form $f(e^{i\theta}) = \sum_{n=0}^{N} \hat{f}(n) e^{in\theta}$, we have*

$$(1-\varepsilon)\|f\|_{L^p(T)} \leqslant \|f|_{G_k}\|_{L^p(G_k)} \leqslant (1+\varepsilon)\|f\|_{L^p(T)}.$$

Proof. Let λ_k be the normalized Haar measure on G_k and let λ be Haar measure on T. Then $\lambda_k \to \lambda$ weak-*, so $\lambda_k \to \lambda$ uniformly on compact subsets of $C(T)$. Since the set $\{|f|^p\colon f(e^{i\theta}) = \sum_{n=0}^{N} \hat{f}(n) e^{in\theta},\ \|f\|_{L^p(T)} = 1\}$ is compact in $C(T)$, the lemma follows.

Our next lemma requires some additional notation. Let $E \subseteq Z$ be a finite set of nonnegative integers, and let m be a nonnegative integer so large that $E \subseteq [0, m-1]$. Then, for $1 < p < \infty$, $K_p(E, m)$ will denote the number $K_p(E)$ when E is considered as a subset of the group $Z(m) = \{1, 2, \ldots, m-1\}$, while $K_p(E, Z)$ will denote $K_p(E)$ when E is considered as a subset of the group Z.

Lemma 4.3. *For $1 < p < \infty$, there exists a constant $\delta_p > 0$ such that the following holds: for every increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers, there exists K such that for $k \geqslant K$,*

$$\delta_p K_p(E, Z) \leqslant K_p(E, n_k).$$

Proof. Let $C_p < \infty$ be such that $\|\chi_{[0,N]}\|_{M_Z^p} \leqslant C_p$ for $N = 1, 2, \ldots$ Fix N such that $E \subseteq [0, N]$ and let K be as in the conclusion of Lemma 4.2 (where we take $\varepsilon = \frac{1}{2}$ and the present N). Let $\varphi \in M_E^p$ be such that $\|\varphi\|_{M_E^p} = 1$ and $\|\Phi\|_{M_Z^p} \geqslant K_p(E, Z)$ if $\Phi|_E = \varphi$. For $k \geqslant K$ it follows from the conclusion of Lemma 4.2 that the norm of φ in M_E^p when E is considered as a subset of $Z(n_k)$, is $\leqslant 3$. But if $\Phi \in M_{Z(n_k)}^p$ is such that $\Phi|_E = \varphi$, then

$$\|\Phi\|_{M_{Z(n_k)}^p} \geqslant \frac{K_p(E, Z)}{3C_p}.$$

(If $\|\Phi\|_{M_{[0,N]}^p(X)}$ denotes the norm of $\Phi|_{[0,N]}$ in $M_{[0,N]}^p$ with $[0, N]$ considered as a subset of the group X $(X = Z, Z(n_k))$, then the inequality

$$\|\Phi\|_{M_{[0,N]}^p(Z(n_k))} \geqslant \tfrac{1}{3}\|\Phi\|_{M_{[0,N]}^p(Z)}$$

follows from the conclusion of Lemma 4.2. But $C_p\|\Phi\|_{M_{[0,N]}^p(Z)} \geqslant \|\Phi\chi_{[0,N]}\|_{M_Z^p}$ since $\|\chi_{[0,N]}\|_{M_Z^p} \leqslant C_p$, and $\|\Phi\chi_{[0,N]}\|_{M_Z^p} \geqslant K_p(E, Z)$, since $\Phi\chi_{[0,N]}|_E = \varphi$.)

Thus we may take $\delta_p = \dfrac{1}{9C_p}$ in the conclusion of the present lemma.

Now it is easy to treat the cases $X = Z(q^\infty)$, $X = \overset{\infty}{\underset{n=1}{P^*}} Z(q_n)$ of Theorem 4.1. We start by recalling that for $s = 2, 3, \ldots$

(2) there exists a sequence $\{E_n\}_{n=1}^{\infty}$ of finite sets of nonnegative integers such that $K_{2s}(E_n) \to \infty$.

Fix a prime q. The group $Z(q^\infty)$ is an increasing union of finite cyclic groups, say $Z(q^\infty) = \bigcup_{n=1}^{\infty} H_n$. By (2) and Lemma 4.3, there exists an increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers and a sequence $\{E_i\}_{i=1}^{\infty}$ of sets with $E_i \subseteq H_{n_{i+1}} \setminus H_{n_i}$ such that $K_{2s}(E_i) \to \infty$ and such that E_i is contained in a coset C_i of H_{n_i} in $H_{n_{i+1}}$. If $E = \bigcup_{i=1}^{\infty} E_i$, it follows from $E_i = E \cap C_i$ that $\|\chi_{E_i}\|_{M_E^{2s}} = 1$. That is, the E_i can be picked so that (a) and (b) at the beginning of the section are satisfied. This proves Theorem 4.1 when $X = Z(q^\infty)$.

Now suppose that $\{q_n\}_{n=1}^{\infty}$ is an increasing sequence of primes. Reasoning as above, there exist subsequences $\{q_{n_i}\}_{i=1}^{\infty}$ and $\{E_i\}_{i=1}^{\infty}$ with $K_{2s}(E_i) \to \infty$ and with $E_i \subseteq Z(q_{n_i})$. Considering each E_i as a subset of $\overset{\infty}{\underset{n=1}{P^*}} Z(q_n)$ and letting $E = \bigcup_{i=1}^{\infty} E_i$, we see that (a) and (b) are again satisfied, for $E_i = E \cap Z(q_{n_i})$. This completes the proof of Theorem 4.1.

We note that the only ingredients of the above proof which depend on the assumption that $p > 2$ is an even integer are Lemmas 3.2 and 3.3, which constitute the computational part of the proof.

5. In this section we establish the following theorem.

THEOREM 5.1. *Fix $n = 3, 4, \ldots$ There exists a subset E of the group $\overset{\infty}{\underset{j=1}{P^*}} Z(n)$ and a function $\varphi \in M_E^{2n-2}$ such that φ is in M_E^p for only a finite number of $p \in (2, 2n-2)$. Thus φ is in no space $M^p|_E$ for $p \neq 2$.*

We will need the following lemma.

LEMMA 5.2. *Fix $n = 3, 4, \ldots$ Let ω be a primitive n-th root of unity and choose a complex number λ with $|\lambda| = 1$, $\lambda \notin R$, and $|1+\lambda| > |1+\lambda\omega^j|$, $j = 1, \ldots, n-1$. The function of a real variable*

$$F_p(\theta) = \sum_{j=0}^{n-1} |1+\lambda e^{i\theta}\omega^j|^p$$

can have $\frac{d}{d\theta} F_p(\theta)\big|_{\theta=0} = 0$ for only a finite number of $p \in (2, 2n-2)$, say $p = p_1, \ldots, p_l$.

Proof. Since distinct multiplicative functions on a group are linearly independent ([8], Lemma 29.41), it is not the case that

$$(1) \qquad |1+\lambda|^x \operatorname{Im}\lambda + |1+\lambda\omega|^x \operatorname{Im}(\lambda\omega) + \ldots + |1+\lambda\omega^{n-1}|^x \operatorname{Im}(\lambda\omega^{n-1}) = 0$$

holds for each $x \in R$. But the LHS of (1) defines an entire function of x, so (1) can hold for only a finite number of x in any finite interval. Since $\frac{d}{d\theta} F_p(\theta)\big|_{\theta=0} = 0$ implies that (1) holds for $x = p-2$, this proves the lemma.

Now fix $n = 3, 4, \ldots$ and regard $Z(n) = \{0, 1, \ldots, n-1\}$ as the character group of the subgroup G_n of T consisting of the nth roots of unity. Let $E_1 \subseteq Z(n)$ be the set $\{0, 1\}$. A computation shows that if the Haar measure on G_n is normalized, then for any E_1-polynomial on G_n we have

$$\|f\|_{L^{2n-2}(G_n)}^{2n-2} = |\hat{f}(0)^{n-1}|^2 + |n\hat{f}(0)^{n-2}\hat{f}(1)|^2 + \left|\frac{n(n-1)}{2}\hat{f}(0)^{n-3}\hat{f}(1)^2\right|^2 + \ldots$$
$$\ldots + |\hat{f}(1)^{n-1}|^2.$$

Thus $\|f\|_{L^{2n-2}}$ depends only on $|\hat{f}(0)|$, $|\hat{f}(1)|$, and so each function φ_θ defined on E_1 by $\varphi_\theta(0) = 1$, $\varphi_\theta(1) = e^{i\theta}$ $(\theta \in R)$ has $\|\varphi_\theta\|_{M_{E_1}^{2n-2}} = 1$.

If, on the other hand, $\|\varphi_\theta\|_{M_{E_1}^p} \leqslant 1$ for some value of p, then it must be the case that for any number $\hat{f}(0), \hat{f}(1)$, we have the inequality

$$(2) \qquad \left(\frac{1}{n}\sum_{j=0}^{n-1} |\hat{f}(0)+\hat{f}(1)e^{i\theta}\omega^j|^p\right)^{1/p} \leqslant \left(\frac{1}{n}\sum_{j=0}^{n-1} |\hat{f}(0)+\hat{f}(1)\omega^j|^p\right)^{1/p},$$

where ω is a primitive nth root of unity. This is so because the RHS of (2) is the norm in $L^p(G_m)$ of the E_1-polynomial having Fourier coefficients $\hat{f}(0)$ and $\hat{f}(1)$. Let $\{\theta_j'\}_{j=1}^\infty$ $(\{\theta_j''\}_{j=1}^\infty)$ be a sequence of positive (negative) numbers such that $\theta_j' \to 0$ $(\theta_j'' \to 0)$. Taking $\hat{f}(0) = 1$, $\hat{f}(1) = \lambda$ in (2), it follows from Lemma 5.2 that if $p \in (2, 2n-2) \setminus \{p_i\}_{i=1}^l$, then there exists some $\theta = \theta_j'$ or $\theta = \theta_j''$ such that $\|\varphi_\theta\|_{M_{E_1}^p} > 1$.

Now let $E \subseteq \overset{\infty}{\underset{i=1}{P^*}} Z(n)$ be the set $\overset{\infty}{\underset{i=1}{P^*}} E_1$. That is, E is the set of all $x = (x_1, x_2, \ldots) \in \overset{\infty}{\underset{i=1}{P^*}} Z(n)$ such that $x_i \in E$, $i = 1, 2, \ldots$ Let $\{\theta_j\}_{j=1}^\infty$ be a sequence in which each θ_j' and each θ_j'' occur infinitely often, and define $\varphi \in l^\infty(E)$ by the formula

$$\varphi(x_1, x_2, \ldots) = \overset{\infty}{\underset{i=1}{P}} \varphi_{\theta_i}(x_i).$$

(This makes sense because $x_i = 0$ for all large i and $\varphi_\theta(0) = 1$.) Then it follows easily from Lemma 2.4 and the equalities $\|\varphi_{\theta_i}\|_{M_{E_1}^{2n-2}} = 1$, $i = 1, 2, \ldots$, that $\|\varphi\|_{M_E^{2n-2}} = 1$. But if $p \in (2, 2n-2) \setminus \{p_i\}_{i=1}^l$, then $\|\varphi_{\theta_i}\|_{M_{E_1}^p} \geqslant 1+\varepsilon$ for some $\varepsilon > 0$ and an infinite number of indices i together with Lemma 2.4 imply that $\varphi \notin M_E^p$. This completes the proof of Theorem 5.1.

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Gaussian measures on L_p spaces $0 \leqslant p < \infty$

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Abstract. Using the correspondence between measures on L_p spaces, $0 \leqslant p < \infty$, and measurable processes with paths in L_p, given in [2], we prove that the independence of random elements with values in L_p as well as their distributions are determined by finite-dimensional distributions of the corresponding measurable processes. This result is applied to the investigation of Gaussian random elements with values in L_p, $0 \leqslant p < \infty$.

It is well known that if X_1, X_2 are independent real random variables such that X_1+X_2 and X_1-X_2 are independent, then X_1, X_2 are Gaussian. This property of real Gaussian random variables, proved already by Bernstein, has been used by Fréchet as one of two (equivalent) definitions of Gaussian random elements with values in a Banach space [5]. This definition allows us to consider Gaussian random elements in metric linear spaces which admit no nontrivial continuous linear functionals. The best known examples of such spaces are $L_p \equiv L_p(m)$ spaces, where m is the Lebesgue measure on [0, 1] and $0 \leqslant p < 1$. Of course, in such spaces the classical definition of Gaussian elements cannot be used. In this paper we investigate Gaussian random elements on L_p spaces, $0 \leqslant p < \infty$. For $p \geqslant 1$ these results were proved by Rajput [9].

Section 1 is preliminary. In Section 2 we prove two results. In Theorem 2.1 we prove that the support of a symmetric Gaussian measure defined on a linear metric space is a closed linear subspace. In Theorem 2.2 we give a short proof of the 0–1 law for Gaussian measures defined on complete separable metric linear spaces.

In Section 3 we consider measurable processes with paths in L_p. Theorem 1.1 proved in [2] gives the correspondence between measures on L_p and measurable processes with paths in L_p. We prove that the independence of random elements with values in L_p as well as their distributions are determined by finite-dimensional distributions of the corresponding measurable processes.

In Section 4 we apply the results of the preceding sections to prove results analogous to those obtained by Rajput [9] for $p \geqslant 1$. Theorem