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Independent sets and measure algebras

by

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Abstract. Let G be a non-discrete locally compact abelian group. Let $M(G)$ be the measure algebra on G. In this paper, at first, we shall consider the relation between independent sets and prime L-subalgebras of $M(G)$. Finally, we shall show the existence of measures with some properties in case of G being compact and the dual group of G having an infinite independent set.

0. Introduction. Throughout this paper $G(\tau_0)$ denotes a non-discrete locally compact abelian group with an underlying group G and a topology τ_0. We shall denote by $\mathscr{T}(G(\tau_0))$ the set of all those locally compact group topologies on G which are stronger than or equal to the original topology τ_0. For any $\tau \in \mathscr{T}(G(\tau_0))$, let $M(G(\tau))$ be the algebra of all bounded regular Borel measures on $G(\tau)$ under convolution multiplication, and $L^1(G(\tau))$ the ideal consisting of all those measures which are absolutely continuous with respect to the Haar measure m_τ on $G(\tau)$.

A closed subalgebra N of $M(G(\tau_0))$ is called an *L-subalgebra* if $\mu \in N$ and $\nu \ll \mu$ implies $\nu \in N$. An L-subalgebra N is said to be *prime* if $N^\perp$ is an ideal, where $N^\perp$ is the set of measures ν such that $\nu \perp \mu$ for all $\mu \in N$.

A collection $\mathscr{F}$ of σ-compact subsets of $G(\tau_0)$ is called a *Raikov system* if the following conditions are satisfied:

(R1) If $A_1 \in \mathscr{F}$ and A_2 is a σ-compact subset of A_1, then $A_2 \in \mathscr{F}$;

(R2) The union of each countable subcollection of $\mathscr{F}$ is also in $\mathscr{F}$;

(R3) If $A \in \mathscr{F}$ and $x \in G$, then $A + x \in \mathscr{F}$;

(R4) If $A \in \mathscr{F}$, then $A + A \in \mathscr{F}$. A Raikov system $\mathscr{F}$ is said to be *symmetric* if the following additional condition is satisfied:

(R5) If $A \in \mathscr{F}$, then $-A \in \mathscr{F}$.

To each topology τ in $\mathscr{T}(G(\tau_0))$ there corresponds the Raikov system $\mathscr{F}_\tau$ of all those subsets which are σ-compact with respect to τ.

Let Φ be the homomorphism of $M(G(\tau))$ to $M(G(\tau_0))$ which is induced by the canonical injective mapping of $G(\tau)$ to $G(\tau_0)$, then $\Phi(M(G(\tau))) = M(\mathscr{F}_\tau)$ ([4]). Thus we may identify $M(\mathscr{F}_\tau)$ with $M(G(\tau))$. Each Raikov system $\mathscr{F}$ gives rise to an L-subalgebra $M(\mathscr{F})$ of all those measures which are concentrated on suitable sets in $\mathscr{F}$. This subalgebra is prime ([2], Section 33).

In Section 2, by constructing a Raikov system of a special type, we shall show that there exists a non-trivial prime L-subalgebra which is different from $M(G(\tau))$ for all $\tau \in \mathscr{T}(G(\tau_0))$. In this connection, we shall prove further that, if a symmetric Raikov system $\mathscr{F}_1$ with a single generator is contained properly in a Raikov system $\mathscr{F}_2$, then there is a Raikov system $\mathscr{F}$ which is contained properly in $\mathscr{F}_2$ and contains properly $\mathscr{F}_1$. In these constructions the notion of independent sets and that of semi-independent ones play a decisive role. The rest of the paper concerns construction of measures of special types in connection with independent sets. We prove the existence of a measure with independent powers. Finally, when the dual group admits an infinite independent set, then we can construct a measure which answers affirmatively to a problem raised by J. L. Taylor in [10].

1. Independent sets and prime L-subalgebras. Now we shall show that for a suitable symmetric Raikov system $\mathscr{F}$, $M(\mathscr{F})$ is a non-trivial prime L-subalgebra which is different from $M(G(\tau))$ for all $\tau \in \mathscr{T}(G(\tau_0))$.

A Raikov system $\mathscr{F}$ such that $m_{\tau_0}(A) = 0$ for every set $A \in \mathscr{F}$, where m_{τ_0} is the Haar measure on $G(\tau_0)$, will be called a *proper Raikov system*. For a subset E of G and a positive integer n we write that

$$nE = \{x_1 + \ldots + x_n \colon x_1, \ldots, x_n \in E\},$$

and, in particular, we set $0E = \{0\}$. If a Borel subset E of $G(\tau_0)$ is locally negligible with respect to m_τ, then we call E *locally τ-negligible*.

LEMMA 1.1. *For any non-discrete topology $\tau \in \mathscr{T}(G(\tau_0))$ and a positive number n, suppose that E is a Borel subset such that kE is a Borel subset of $G(\tau)$ for $k = 1, 2, \ldots, n$ and nE is non-locally τ-negligible. If E' is a subset of E such that $E \setminus E'$ is finite, then nE' has non-locally τ-negligible Borel subsets.*

Proof. Put $F = E \setminus E'$, from $kE + (n-k)F \subset kE + (n-k)E = nE$, we have that $nE = \bigcup_{k=0}^{n} (kE + (n-k)F)$. Let k_0 be the smallest positive integer such that $k_0 E + (n-k_0)F$ is non-locally τ-negligible. Since $(n-k_0)F$ is finite, $k_0 E$ is non-locally τ-negligible. From

$$k_0 E = k_0 E' \cup \bigcup_{k=1}^{k_0} \big((k_0 - k)E' + kF\big),$$

we have that

$$k_0 E \setminus \bigcup_{k=1}^{k_0} \big((k_0 - k)E + kF\big) \subset k_0 E'.$$

Since $(k_0 - k)E + kF$ is locally τ-negligible $(k = 1, 2, \ldots, k_0)$ and $k_0 E' \subset nE'$, nE' has a non-locally τ-negligible Borel subset. ■

A subset E of G is said to be *independent* if E has the following property: for every choice of distinct elements $x_1, \ldots, x_k$ of E and integers $n_1, \ldots, n_k$, either

$$n_1 x_1 + \ldots + n_k x_k \neq 0$$

or

$$n_1 x_1 = \ldots = n_k x_k = 0.$$

LEMMA 1.2. *Let P be an independent subset in G, and $Q = P \cup (-P)$. If $\tau \in \mathscr{T}(G(\tau_0))$ is non-discrete and kQ is a Borel subset in $G(\tau)$ for any positive integer k, then the group $G(P)$ generated by P is locally τ-negligible.*

Proof. Since $G(P) = \bigcup_{k=1}^{\infty} kQ$, it is enough to show that kQ is locally τ-negligible for any positive integer k. Assume that for some fixed positive integer there is a τ-compact subset E of kQ such that $E = -E$ and $m_\tau(E) > 0$. If χ_E is the characteristic function of E and $f = \chi_E * \chi_E$, then f is continuous on $G(\tau)$ and $f(0) = m_\tau(E) > 0$, hence $f(x) > 0$ for all x in some τ-neighborhood V of 0 which is contained in $2E$ ([7], p. 108).

Let U be a τ-neighborhood of 0 such that $(2k+1)U \subset V$. Since $G(\tau)$ is non-discrete, there is a non-empty finite subset $\{x_1^{(1)}, \ldots, x_{q_1}^{(1)}\}$ of P such that $n_1^{(1)} x_1^{(1)} + \ldots + n_{q_1}^{(1)} x_{q_1}^{(1)}$ is a non-zero element of U for some choice of non-zero integers $n_1^{(1)}, \ldots, n_{q_1}^{(1)}$.

Put $P_1 = P \setminus \{x_1^{(1)}, \ldots, x_{q_1}^{(1)}\}$ and $Q_1 = P_1 \cup (-P_1)$. In view of Lemma 1.1, kQ_1 is non-locally τ-negligible, thus $2kQ_1$ contains a neighborhood of 0 in $G(\tau)$ ([7]). By induction, we can obtain distinct elements

$$x_1^{(1)}, \ldots, x_{q_1}^{(1)}, \ldots, x_1^{(2k+1)}, \ldots, x_{q_{2k+1}}^{(2k+1)}$$

of P and non-zero integers

$$n_1^{(1)}, \ldots, n_{q_1}^{(1)}, \ldots, n_1^{(2k+1)}, \ldots, n_{q_{2k+1}}^{(2k+1)}$$

such that

$$n_1^{(1)} x_1^{(1)} + \ldots + n_{q_1}^{(1)} x_{q_1}^{(1)} + \ldots + n_1^{(2k+1)} x_1^{(2k+1)} + \ldots + n_{q_{2k+1}}^{(2k+1)} x_{q_{2k+1}}^{(2k+1)}$$

is an element of $(2k+1)U$. On the other hand, from $(2k+1)U \subset V \subset 2kQ$, there are distinct elements $y_1, \ldots, y_r \in P$ $(1 \leqslant r \leqslant 2k)$ and non-zero integers $m_1, \ldots, m_r$ such that

$$n_1^{(1)} x_1^{(1)} + \ldots + n_{q_1}^{(1)} x_{q_1}^{(1)} + \ldots + n_1^{(2k+1)} x_1^{(2k+1)} + \ldots + n_{q_{2k+1}}^{(2k+1)} x_{q_{2k+1}}^{(2k+1)} = m_1 y_1 + \ldots + m_r y_r.$$

This contradicts the independence of P. ■

THEOREM 1.3. *Let $\mathscr{F}$ be a symmetric Raikov system generated by a compact independent set P. Then $M(\mathscr{F})$ is a symmetric prime L-subalgebra such that $M(\mathscr{F}) \perp L^1(G(\tau))$ for any non-discrete topology $\tau \in \mathscr{T}(G(\tau_0))$, therefore*

$M(\mathscr{F})$ is a non-trivial prime L-subalgebra which is different from $M(G(\tau))$ for all $\tau \in \mathscr{T}(G(\tau_0))$.

Proof. Let H be the σ-compact group generated by P. Put $Q = P \cup \cup(-P)$, then kQ is τ_0-compact, so that kQ is τ-closed for any $\tau \in \mathscr{T}(G(\tau_0))$. Thus from Lemma 1.2, H is locally τ-negligible for any non-discrete topology $\tau \in \mathscr{T}(G(\tau_0))$. Since any $A \in \mathscr{F}$ is contained in the union of some countable cosets of H, A is locally τ-negligible, therefore $M(\mathscr{F}) \perp L^1(G(\tau))$ for any non-discrete topology $\tau \in \mathscr{T}(G(\tau_0))$. ■

Our next purpose is to show the following theorem.

THEOREM 1.4. *If $\mathscr{F}_1$ is a symmetric Raikov system generated by a σ-compact group H, and if $\mathscr{F}_2$ is a strictly larger symmetric Raikov system, then there exists a symmetric Raikov system $\mathscr{F}$ such that*

(i) $M(\mathscr{F}_1) \subsetneqq M(\mathscr{F}) \subsetneqq M(\mathscr{F}_2)$

and

(ii) $M(\mathscr{F}) \neq M(G(\tau))$ *for all* $\tau \in \mathscr{T}(G(\tau_0))$.

Given a subset E of G, we say that a subset P of G is *semi-E-independent*, if P has the property that for every choice of distinct points $x_1, \ldots, x_{N+1} \in P$ and integers $n_1, \ldots, n_N$ the relation

$$\sum_{r=1}^{N} n_r x_r + x_{N+1} \notin E$$

holds. In particular, if $E = \{0\}$, then we call briefly P *semi-independent.*

LEMMA 1.5 (cf. [7], p. 108). *Let H be a σ-compact subgroup of $G(\tau_0)$ and P a compact semi-H-independent subset of $G(\tau_0)$. If P has a cluster point with respect to a topology $\tau \in \mathscr{T}(G(\tau_0))$, then the group $H+G(P)$ is locally τ-negligible.*

Proof. If $Q = P \cup(-P)$, then $H+G(P) = \bigcup_{k=1}^{\infty}(H+kQ)$. Suppose that $H+nQ$ is non-locally τ-negligible for some integer n. Then there exists a τ-compact subset A of $H+nQ$ such that $m_\tau(A) > 0$ and $A = -A$. We define the mapping ψ of $G(\tau)^{2n+2}$, the direct sum of $2n+2$ copies of $G(\tau)$, to $G(\tau)$ as follows:

$$\psi\big((x_1, x_2, \ldots, x_{2n+2})\big) = \sum_{k=1}^{2n+2} (-1)^k x_k.$$

Let x be a τ-cluster point of P. Put $x_k = x$ for $k = 1, 2, \ldots, 2n+2$; then

$$\psi\big((x_1, x_2, \ldots, x_{2n+2})\big) = 0.$$

Since $m_\tau(A) > 0$, $A+A$ contains a neighborhood V of 0 with respect to τ ([7], p. 108). Since x is a τ-cluster point and ψ is continuous, there is a element $(y_1, y_2, \ldots, y_{2n+2}) \in P^{2n+2}$ such that $y_i \neq y_j$ if $i \neq j$, and $y_1 - y_2 + \ldots + y_{2n+1} - y_{2n+2} \in V$. From $V \subset A+A \subset H+2nQ$, we can choose $w_1, \ldots, w_{2n} \in Q$ and $h \in H$ such that

$$y_1 - y_2 + \ldots + y_{2n+1} - y_{2n+2} = h + w_1 + \ldots + w_{2n}.$$

It follows that

$$r_1 z_1 + \ldots + r_{m-1} z_{m-1} + z_m = h,$$

where $r_1, \ldots, r_{m-1}$ are integers, $z_1, \ldots, z_m$ are distinct elements of P. This contradicts the semi-H-independence of P. ■

EXAMPLE. For some locally compact abelian group $G(\tau_0)$, there is an example of a semi-independent compact perfect set P of $G(\tau_0)$ such that $G(P)$ is open with respect to some non-discrete locally compact group topology on G. Let $G(\tau_0) = T \oplus D_2$, where T is the circle group and D_2 is the complete direct sum of countable many copies of the cyclic group of order 2 ([7], p. 254). Let P_0 be an independent Cantor set of D_2 ([7], p. 103). Let φ_1 be a homeomorphic mapping of P_0 onto D_2. Put $D_2 = \prod_{n=1}^{\infty}\{0, 1\}_n$. We define the mapping φ_2 of D_2 to T as follows:

$$\varphi_2\big((x_1, x_2, \ldots)\big) = \sum_{n=1}^{\infty} x_n/2^n$$

for $(x_1, x_2, \ldots) \in D_2$. If we put

$$\varphi(x) = \big(x, \varphi_2 \circ \varphi_1(x)\big) \quad (x \in P_0),$$

then φ is a continuous mapping of P_0 to $D_2 \oplus T$. Since $(\varphi_2 \circ \varphi_1)^{-1}(t)$ is a finite set for any $t \in T$, $\varphi(P_0)$ has no τ_T-cluster point, where τ_T is the weakest locally compact group topology on $D_2 \oplus T$ such that T is open. Clearly, $\varphi(P_0)$ is a compact semi-independent subset of $D_2 \oplus T$. Now, we have $\{2x: x \in \varphi(P_0)\} = T$, thus the group generated by $\varphi(P_0)$ is τ_T-open.

LEMMA 1.6. *Let K be a compact subgroup of $G(\tau_0)$ and let α be the canonical homomorphism of $G(\tau_0)$ to $G(\tau_0)/K$. If $\mathscr{F}$ is a Raikov system of $G(\tau_0)$, then $\alpha(\mathscr{F}) = \{\alpha(A)\colon A \in \mathscr{F}\}$ is a Raikov system of $G(\tau_0)/K$.*

Proof. It is clear that $\alpha(\mathscr{F})$ satisfies (R2), (R3) and (R4) of Section 0. Given a set $A \in \mathscr{F}$, for any σ-compact set $B \subset \alpha(A)$, there is a σ-compact subset B' of A such that $\alpha(B') = B$ ([6]), and so $B \in \alpha(\mathscr{F})$. ■

LEMMA 1.7. *Let $\mathscr{F}$ be a Raikov system generated by a σ-compact group H, and A a compact perfect set such that $(H-x) \cap A$ is of the first category in A for each $x \in G(\tau_0)$. Then there is a compact group K such that $G(\tau_0)/K$ is metrizable and $\alpha(A) \notin \alpha(\mathscr{F})$, where α is the canonical homomorphism of $G(\tau_0)$ to $G(\tau_0)/K$.*

Proof. Suppose that $H = H_1 \cup H_2 \cup \ldots$, where each H_n is compact and symmetric. We may assume $H_1 \subset H_2 \ldots$ We can choose compact

τ_0-neighborhoods V_n and finite subsets $\{x_1^{(n)}, \ldots, x_{2^n}^{(n)}\}$ of A $(n = 1, 2, \ldots)$ such that

(i) $V_n = -V_n$,

(ii) $V_{n+1} + V_{n+1} \subset V_n$,

(iii) $x_i^{(n+1)} + V_{n+1} \subset x_j^{(n)} + V_n$ with $i \leqslant 2j \leqslant i+1$,

(iv) $\big((x_i^{(n)} + V_n) - (x_j^{(n)} + V_n)\big) \cap H_n = \emptyset$ if $i \neq j$ (cf. [8], [12]).

Put $\Lambda = \{\lambda = (\lambda_1, \lambda_2, \ldots)\colon \lambda_i = 0$ or 1 $(i = 1, 2, \ldots)\}$. For $\lambda \in \Lambda$ choose a set $\{x_{\lambda(1)}^{(1)}, x_{\lambda(2)}^{(2)}, \ldots\}$ with $\lambda(1) = 2 - \lambda_1$ and $\lambda(k) = 2\lambda(k-1) - \lambda_k$ $(k = 2, 3, \ldots)$. For $\lambda, \lambda' \in \Lambda$, if $\lambda_k \neq \lambda_k'$, then $\lambda(m) \neq \lambda'(m)$ for every $m \geqslant k$. Let $K = \bigcap_{n=1}^{\infty} V_n$, then, from (i) and (ii), K is a compact group such that $G(\tau_0)/K$ is metrizable. Let x_λ be a cluster point of $\{x_{\lambda(1)}^{(1)}, x_{\lambda(2)}^{(2)}, \ldots\}$, then, from (iii), $x_\lambda \in \bigcap_{n=1}^{\infty}(x_{\lambda(n)}^{(n)} + V_n)$. Suppose $\lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$, then for some integer k it follows that $\lambda(n) \neq \lambda'(n)$ if $n \geqslant k$, so that

$$\begin{aligned}\big((x_\lambda + K) - (x_\lambda' + K)\big) \cap H_n &\subset \big((x_{\lambda(n)}^{(n)} + V_{n+1} + K) - (x_{\lambda'(n)}^{(n)} + V_{n+1} + K)\big) \cap H_n \\ &\subset \big((x_{\lambda(n)}^{(n)} + V_{n+1} + V_{n+1}) - (x_{\lambda'(n)}^{(n)} + V_{n+1} + V_{n+1})\big) \cap H_n \\ &\subset \big((x_{\lambda(n)}^{(n)} + V_n) - (x_{\lambda'(n)}^{(n)} + V_n)\big) \cap H_n = \emptyset\end{aligned}$$

for every $n \geqslant k$. Thus $\lambda \neq \lambda'$ implies

$$\big((x_\lambda + K) - (x_{\lambda'} + K)\big) \cap H = \emptyset,$$

that is,

$$x_\lambda - x_{\lambda'} \notin H + K.$$

Since Λ is uncountable,

$$\alpha(\{x_\lambda\}_{\lambda \in \Lambda}) \not\subset \bigcup_{n=1}^{\infty} \big(\alpha(H) + y_n\big)$$

for every countable set $\{y_n\}_{n=1}^{\infty}$ of $G(\tau_0)/K$. Thus $\alpha(A) \notin \alpha(\mathscr{F})$. ∎

Proof of Theorem 1.4. Since $\mathscr{F}_2$ contains properly $\mathscr{F}_1$, there is a compact perfect set A in $\mathscr{F}_2$ such that $(H - x) \cap A$ is of the first category in A for each $x \in G$ (cf. [12]). From Lemma 1.7, there exists a compact group K such that $G(\tau_0)/K$ is a metrizable group and $\alpha(A) \notin \alpha(\mathscr{F}_1)$, where α is the canonical homomorphism of $G(\tau_0)$ to $G(\tau_0)/K$. From Lemma 1.6, $\alpha(\mathscr{F}_2)$ is a Raikov system which contains properly a Raikov system $\alpha(\mathscr{F}_1)$ generated by σ-compact group $\alpha(H)$. Since $G(\tau_0)/K$ is metrizable, there is a compact perfect semi-$\alpha(H)$-independent set $P \in \alpha(\mathscr{F}_2) \setminus \alpha(\mathscr{F}_1)$. Let P_0 be a compact perfect subset such that $P \setminus P_0$ contains perfect subset. If $\mathscr{F}'$ is a symmetric Raikov system which is generated by $\alpha(H)$ and P_0, then we have $\alpha(\mathscr{F}_1) \subsetneqq \mathscr{F}' \subsetneqq \alpha(\mathscr{F}_2)$, because $(P \setminus P_0) \cap \big(G(P_0) + H + z\big)$ is a set consisting of at most one point for each $z \in G$. There is a compact set $P_1 \in \mathscr{F}_2$ such that $\alpha(P_1) = P_0$ ([6]). Let $\mathscr{F}$ be the symmetric Raikov system generated by P_1 and H, then $\alpha(\mathscr{F}) = \mathscr{F}'$ and $\mathscr{F}_1 \subsetneqq \mathscr{F} \subsetneqq \mathscr{F}_2$.

The rest of the proof is to show that $M(\mathscr{F}) \neq M\big(G(\tau)\big)$ for any $\tau \in \mathscr{T}\big(G(\tau_0)\big)$. It is enough to show that $M(\mathscr{F}') \neq M\big((G/K)(\tau)\big)$ for any $\tau \in \mathscr{T}\big(G(\tau_0)/K\big)$ ([7], p. 54). If P_0 is a τ-discrete set, then $M_c(P_0) \cap M\big((G/K)(\tau)\big) = \{0\}$, where $M_c(P_0)$ is the subspace of $M\big(G(\tau_0)/K\big)$ consisting of all continuous measures whose supports are contained in P_0, so that in this case we have $M(\mathscr{F}') \neq M\big((G/K)(\tau)\big)$. If P_0 has a τ-cluster point, by Lemma 1.5, we obtain that $\alpha(H) + G(P_0)$ is locally τ-negligible. Thus, it follows that $M(\mathscr{F}') \neq M\big((G/K)(\tau)\big)$. ∎

2. Independent power measures with respect to $M(\mathscr{F})$. If f is a polynomial in elements of $M\big(G(\tau_0)\big)$, with coefficients $a_{r_1 r_2 \ldots} \in M\big(G(\tau_0)\big)$, we write $|f|$ for the polynomial with scalar coefficients $\|a_{r_1 r_2 \ldots}\|$. Let $M(\mathscr{F})$ be a given Raikov system. A set X of non-zero measures $\{\mu_i\}$ has *independent powers* with respect to $M(\mathscr{F})$ if for each polynomial f with coefficients in $M(\mathscr{F})$, we have

$$\|f(\mu_1, \ldots, \mu_n)\| = |f|(\|\mu_1\|, \ldots, \|\mu_n\|)$$

for all $\mu_1, \ldots, \mu_n \in X$ (cf. [11]).

Throughout this section we shall assume that $G(\tau_0)$ is metrizable.

LEMMA 2.1 ([8], [12]). *Let $\mathscr{F}$ be a proper symmetric Raikov system which is generated by a σ-compact group H. Let $\{P_i\}_{n=1}^{\infty}$ be a disjoint collection of subsets of $G(\tau_0)$, with $P = \bigcup_{n=1}^{\infty} P_i$ semi-H-independent, and for each i let μ_i be a continuous measure concentrated on $Q_i = P_i \cup (-P_i)$. If $a, b \in M(\mathscr{F})$ and $(r_1, \ldots, r_N) \neq (s_1, \ldots, s_N)$, where $r_1, \ldots, r_N, s_1, \ldots, s_N$ are non-negative integers, then*

$$a * \mu_1^{r_1} * \ldots * \mu_N^{r_N} \perp b * \mu_1^{s_1} * \ldots * \mu_N^{s_N}.$$

Given a subset E of G, we say (as in [12]) that a subset X of G is *$(E, 2)$-independent* if the relation

$$n_1 x_1 + \ldots + n_r x_r \in E,$$

where $n_1, \ldots, n_r$ are integers satisfying

$$|n_i| \leqslant 2 \quad (1 \leqslant i \leqslant r)$$

and $x_1, \ldots, x_r$ are distinct elements of X, is possible only if $n_1 x_1 = \ldots = n_r x_r = 0$. For a given subset E of G, we put

$$2 \times E = \{2x\colon x \in E\}.$$

The proof of the following proposition is essentially the same as that of Proposition 2 in [12].

PROPOSITION 2.2. *Let $\mathscr{F}$ be a proper symmetric Raikov system generated by a σ-compact group H. Let μ_i $(i = 1, \ldots, r)$ be mutually singular continuous measures which are concentrated on $P \cup (-P)$. If P is $(H, 2)$-independent, then the set of measures $\{\mu_i\}_{i=1}^r$ has independent powers with respect to $M(\mathscr{F})$.*

THEOREM 2.3. *Let $\mathscr{F}_1$ be a Raikov system generated by a σ-compact group H, and $\mathscr{F}_2$ a strictly larger symmetric Raikov system. Then $2\times H = H$ implies that there is a compact perfect $(H,2)$-independent set P in $\mathscr{F}_2$, so that the set of continuous measures $\{\mu_i\}_{i=1}^n$ in $M(\mathscr{F}_2)$, which are concentrated on $P \cup (-P)$ and are mutually singular, has independent powers with respect to $M(\mathscr{F}_1)$.*

Proof. Let $\mathscr{P}$ be the family consisting of all compact perfect semi-H-independent sets which belong to $\mathscr{F}_2$, then $\mathscr{P}$ is non-empty (cf. [8], [12]). If $2\times P' \not\subset H$ for each $P' \in \mathscr{P}$, then it is easy to show the existence of compact perfect $(H, 2)$-independent sets (cf. [12]). We consider the case that $2\times P_0 \subset H$ for some $P_0 \in \mathscr{P}$. Let $G_2 = \{x \in G(\tau_0)\colon 2x = 0\}$. Since $2\times P_0 \subset 2\times H = H$, for each $p \in P_0$ there exists $h \in H$ such that $p - h \in G_2$. Then we have that $(P_0 - H)\cap G_2 \in \mathscr{F}_2 \setminus \mathscr{F}_1$. In fact, suppose $(P_0 - H)\cap G_2 \subset \bigcup_{n=1}^{\infty}(H + x_n)$ for some countable set $\{x_n\}_{n=1}^{\infty}$. For any $p \in P_0$ there is an element h of H such that $p - h \in G_2$, so that $p - h \in \bigcup_{n=1}^{\infty}(H + x_n)$. Since H is a group, $p \in \bigcup_{n=1}^{\infty}(H + x_n)$, that is, $P_0 \subset \bigcup_{n=1}^{\infty}(H + x_n)$. On the other hand, by Lemma 2.1, $P_0 \in \mathscr{F}_2 \setminus \mathscr{F}_1$, this is a contradiction. Choose a compact perfect semi-H-independent subset P of $(P_0 - H)\cap G_2$ (cf. [12]). Since $2\times G_2 = \{0\}$, every semi-H-independent subset P of G_2 is $(H, 2)$-independent. Thus P is a compact perfect $(H, 2)$-independent subset in $\mathscr{F}_2$. By Proposition 2.2, if $\{\mu_i\}_{i=1}^n$ is a set of continuous measures which are concentrated on $P \cup (-P)$, then $\{\mu_i\}_{i=1}^n$ has independent powers with respect to $M(\mathscr{F}_1)$. ∎

EXAMPLE. There is an example of a locally compact abelian group $G(\tau_0)$ such that the statement of Proposition 2.2 is not established, that is, there is a Raikov system $\mathscr{F}$ of $G(\tau_0)$ generated by a σ-compact group H such that for any compact perfect semi-H-independent set P every non-zero continuous Hermitian measure μ concentrated on $P \cup (-P)$ does not have independent powers with respect to $M(\mathscr{F})$.

Let $\{Z_4^{(n)}\}_{n=1}^{\infty}$ be the family of cyclic groups of order 4 and $Z_2^{(n)}$ the subgroup of $Z_4^{(n)}$ of order 2. We shall define the groups as follows $G(\tau_0) = \prod_{n=2}^{\infty} Z_4^{(n)}$ and $H = \prod_{n=2}^{\infty} Z_2^{(n)}$. Let P be any compact perfect semi-H-independent set of $G(\tau_0)$. Then we can assume that $P \subset \{a_1^{(1)}\} \times \prod_{n=1}^{\infty} Z_4^{(n)}$, where $a_1^{(1)}$ is an element of $Z_4^{(1)}$ of order 4. Let $a_2^{(1)}$ is a non-zero element of $Z_2^{(1)}$.

If m_0 is the normalized Haar measure on $H_1 = \{a_0^{(1)}\} \times \prod_{n=2}^{\infty} Z_2^{(n)}$, where $a_0^{(1)}$ is a unit element of $Z_2^{(1)}$, then it is clear that

$$m_0 \perp m_0 * \delta_{a_2^{(1)}},$$

where $\delta_{a_2^{(1)}}$ is the probability measure concentrated at the point $a_2^{(1)}$. Let μ be any continuous probability measure which is concentrated on P. We shall show that

$$m_0 * \mu \text{ non } \perp m_0 * \delta_{a_2^{(1)}} * \mu^*.$$

If E_0 and E_1 are any Borel sets on which $m_0 * \mu$ and $m_0 * \delta_{a_2^{(1)}} * \mu^*$ are concentrated respectively, then we have

$$m_0 * \mu(E_0) = \int m_0(E_0 - x)\, d\mu(x) = 1.$$

Write $A_0 = \{x \in P\colon m_0(E_0 - x) = 1\}$; it follows that

$$\mu(A_0) = 1$$

and

$$m_0\big(H_1 \setminus (E_0 - x)\big) = 0 \quad \text{for each } x \in A_0.$$

Similarly, we get

$$m_0 * \delta_{a_2^{(1)}}\big((H_1 + a_2^{(1)}) \setminus (E_1 + x)\big) = 0 \quad \text{for each } x \in A_1,$$

where $A_1 = \{x \in P\colon m_0 * \delta_{a_2^{(1)}}(E_1 + x) = 1\}$. From $\mu(A_0) = \mu(A_1) = 1$, it follows that $A_0 \cap A_1$ is non-empty. Given $x_0 \in A_0 \cap A_1$, then we have

$$H_1 + x_0 = H_1 + a_2^{(1)} - x_0,$$

and so

$$m_0 * \delta_{x_0} = m_0 * \delta_{a_2^{(1)}} * \delta_{-x_0}.$$

Thus, we have

$$m_0 * \delta_{x_0}\big((H_1 + x_0) \setminus E_0\big) = m_0 * \delta_{a_2^{(1)}} * \delta_{-x_0}\big((H_1 + a_2^{(1)} - x_0) \setminus E_1\big) = 0.$$

Hence, it follows that $E_0 \cap E_1 \neq \emptyset$. This shows that $m_0 * \mu$ non $\perp m_0 * \delta_{a_2^{(1)}} * \mu^*$.

Thus, it follows that

$$\|m_0 * \mu - m_0 * \delta_{a_2^{(1)}} * \mu^*\| < \|m_0 * \mu\| + \|m_0 * \delta_{a_2^{(1)}} * \mu^*\| = 2\,\|m_0\|\,\|\mu\|.$$

Define the polynomial f in elements of $M\big(G(\tau_0)\big)$ as follows

$$f(\nu) = (m_0 - m_0 * \delta_{a_2^{(1)}}) * \nu \qquad \big(\nu \in M\big(G(\tau_0)\big)\big).$$

From $m_0 \perp m_0 * \delta_{a_2^{(1)}}$ and μ being non-negative,

$$|f|(\|\mu+\mu^*\|) = 4\|m_0\|\|\mu\|.$$

On the other hand,

$$\|f(\mu+\mu^*)\| = \|m_0*\mu - m_0*\delta_{a_2^{(1)}}*\mu^*\| + \|m_0*\mu^* - m_0*\delta_{a_2^{(1)}}*\mu\|$$
$$< 2\|m_0\|\|\mu\| + \|m_0*\mu^*\| + \|m_0*\delta_{a_2^{(1)}}*\mu\| = 4\|m_0\|\|\mu\|.$$

Therefore, it follows that

$$\|f(\mu+\mu^*)\| < |f|(\|\mu+\mu^*\|),$$

that is, every non-zero continuous Hermitian measure concentrated on $P\cup(-P)$ does not have independent powers with respect to $M(\mathscr{F})$, where $\mathscr{F}$ is the Raikov system generated by $H = \prod_{n=1}^{\infty} Z_2^{(n)}$.

Next we shall show the following theorem.

THEOREM 2.4. *Let $\mathscr{F}_1$ be a symmetric Raikov system generated by a compact independent set P_1, and F_2 a strictly larger symmetric Raikov system. If there exists a compact set $A \in \mathscr{F}_2 \setminus \mathscr{F}_1$ such that $2\times A \subset P_1$, then there exists a compact perfect set P in $\mathscr{F}_2$ such that any non-zero continuous measure μ concentrated on $P\cup(-P)$ has independent powers with respect to $M(\mathscr{F}_1)$.*

We shall prove the next lemma to show this theorem.

LEMMA 2.5. *For a σ-compact subgroup H of $G(\tau_0)$, let P be a compact semi-H-independent set such that $2\times P\subset H$. Suppose that a_1 and a_2 are concentrated on $H-z$ and H, respectively; then $(H-z)\cap H = \emptyset$ implies that*

$$a_1*\mu^n \perp a_2*\mu^n \quad (n = 1, 2, \ldots)$$

for any continuous measure μ which is concentrated on $P\cup(-P)$.

Proof. The measures $a_1*\mu^n$ and $a_2*\mu^n$ are concentrated on $H-z+ +n(P\cup(-P))$ and $H+n(P\cup(-P))$, respectively. Evidently, if the sets are not disjoint, we have

$$h_1+x_1+\ldots+x_n-z = h_2+y_1+\ldots+y_n$$

for some $h_1, h_2\in H$ and $x_1, \ldots, x_n, y_1, \ldots, y_n \in P\cup(-P)$.

Denote by S the set of points $(x_1, \ldots, x_n)\in P\cup(-P)\times\ldots\times P\cup(-P)$ such that for some $h\in H$ we have

$$h+x_1+\ldots+x_n \in H+n(P\cup(-P))+z.$$

Since H is a group, S is the set of points $(x_1, \ldots, x_n)\in P\cup(-P)\times\ldots \ldots\times P\cup(-P)$ such that

$$x_1+\ldots+x_n \in H+n(P\cup(-P))+z.$$

If we can show that $\mu\times\ldots\times\mu(S) = 0$, then it will follows that $a_1\times\mu\times\ldots \ldots\times\mu((H-z)\times S) = 0$, which implies that $a_1*\mu^n(H+n(P\cup(-P))) = 0$. Let $z = h'+y_1'+\ldots+y_n'-x_1'-\ldots-x_n'$ with $h'\in H$ and $y_1', \ldots, y_n', x_1', \ldots \ldots, x_n'\in P\cup(-P)$. If $(x_1, \ldots, x_n)\in S$, then

$$z = h+x_1+\ldots+x_n-y_1-\ldots-y_n$$

for some $h\in H$ and $y_1, \ldots, y_n\in P\cup(-P)$. Thus we have that

$$x_1+\ldots+x_n-y_1-\ldots-y_n+x_1'+\ldots+x_n'-y_1'-\ldots-y_n'\in H.$$

Since the set P is semi-H-independent and $2\times P\subset H$ and $z\notin H$, the set S is contained in a finite union of sets of the form

$$A_1 = \bigcup_{1\leqslant i\neq j\leqslant n} \{(x_1, \ldots, x_n)\colon x_i = x_j \text{ or } -x_j\},$$

$$A_2 = \bigcup_{1\leqslant i,j\leqslant n} \{(x_1, \ldots, x_n)\colon x_i = x_j' \text{ or } -x_j'\},$$

$$A_2 = \bigcup_{1\leqslant i,j\leqslant n} \{(x_1, \ldots, x_n)\colon x_i = y_j' \text{ or } -y_j'\}.$$

Since μ is continuous, these sets are of $(\mu\times\ldots\times\mu)$-measure zero. It follows that $a_1*\mu^n(H+n(P\cup(-P))) = 0$, and so $a_1*\mu^n$ and $a_2*\mu^n$ are mutually singular. ∎

For a given compact subset Q of $G(\tau_0)$ and non-negative integer n, we call a measure μ *of order nQ* provided that μ is concentrated on nQ and $\mu(kQ) = 0$ if $0\leqslant k\leqslant n-1$.

Proof of Theorem 2.4. Let $Q_1 = P_1\cup(-P_1)$ and $H = \bigcup_{n=1}^{\infty} nQ_1$. We can choose a compact perfect semi-H-independent subset P of A. Let μ be a non-zero continuous measure concentrated on $P\cup(-P)$. Take mutually singular measures ω_1 and ω_2 which are concentrated on H. We have to show that

$$\omega_1*\mu^n \perp \omega_2*\mu^n \quad (n = 1, 2, \ldots).$$

Without loss of generality we may assume that ω_i have order r_iQ_1 $(i = 1, 2)$, with $r_1\geqslant r_2$. Let $A^{(1)}$ and $A^{(2)}$ be mutually disjoint Borel sets on which ω_1 and ω_2 are concentrated, respectively. For any $x\in A^{(1)}$, denote by $S^{(x)}$ the set of points $(q_1, \ldots, q_n)\in Q\times\ldots\times Q$ such that for some $y\in A^{(2)}$ and $q_1', \ldots, q_n'\in Q$, we have

$$x+q_1+\ldots+q_n = y+q_1'+\ldots+q_n'.$$

We may assume that $q_1, \ldots, q_n$ are all different, and so are $q_1', \ldots, q_n'$, since μ is continuous. We can write the elements x and y as follows:

$$x = m_1^{(x)}p_1^{(x)}+\ldots+m_{k(x)}^{(x)}p_{k(x)}^{(x)}$$

and

$$y = m_1^{(y)}p_1^{(y)}+\ldots+m_{k(y)}^{(y)}p_{k(y)}^{(y)},$$

where

$$p_1^{(x)}, \ldots, p_{k(x)}^{(x)}, p_1^{(y)}, \ldots, p_{k(y)}^{(y)} \in P_1,$$

$$|m_1^{(x)}| + \ldots + |m_{k(x)}^{(x)}| = r_1 \quad \text{and} \quad |m_1^{(y)}| + \ldots + |m_{k(y)}^{(y)}| = r_2.$$

Then we get

$$q_1 + \ldots + q_n - q_1' - \ldots - q_n'$$
$$= m_1^{(y)} p_1^{(y)} + \ldots + m_{k(y)}^{(y)} p_{k(y)}^{(y)} - m_1^{(x)} p_1^{(x)} - \ldots - m_{k(x)}^{(x)} p_{k(x)}^{(x)} \neq 0.$$

From P being semi-H-independent, it follows that

$$2q_1'' + \ldots + 2q_r'' - m_1^{(y)} p_1^{(y)} - \ldots - m_{k(y)}^{(y)} p_{k(y)}^{(y)} +$$
$$+ m_1^{(x)} p_1^{(x)} + \ldots + m_{k(x)}^{(x)} p_{k(x)}^{(x)} = 0,$$

with $\{q_1'', \ldots, q_r''\} \subset \{q_1, \ldots, q_n\}$. Then, since $2 \times P \subset P_1$, $r_1 \geqslant r_2$ and P_1 is independent, $S^{(x)}$ is contained in a finite union of sets of the form

$$S_{(j,r)}^{(x)} = \{(q_1, \ldots, q_n) \in S^{(x)}: 2q_j = \pm p_r^{(x)}\} \quad (j = 1, \ldots, n, r = 1, \ldots, k(x)).$$

If $\mu(\{q \in Q: 2q = \pm p_r^{(x)}\}) \neq 0$, then there is a compact perfect $(H, 2)$-independent subset in $\{q \in Q: 2q = p_r^{(x)}\} - q_0$, with $2q_0 = \pm p_r^{(x)}$. Thus, in this case, the statement of this proposition is established. If $\mu\{(q \in Q^{(1)}: 2q = p_r^{(x)})\} = 0$ for all $x \in A^{(1)}$, then

$$\int \chi_{(A^{(1)}+nQ)^{(x)} \cap (A^{(2)}+nQ)} d\omega_1 * \mu^n(x)$$
$$= \int \chi_{A^{(1)(x)}} \Big\{\int \ldots \int \chi_{S^{(x)}}(q_1, \ldots, q_n) d\mu(q_1) \ldots d\mu(q_n)\Big\} d\omega_1(x) = 0,$$

and so $\omega_1 * \mu^n \perp \omega_2 * \mu^n$. Thus, on the basis of Lemma 2.1 and Lemma 2.3, the rest of the proof is quite similar to that of the analogous part of Proposition 2 in [12]. ∎

3. Independent sets and certain measures. In [9] J. L. Taylor showed that there exists a compact commutative topological semigroup S with identity and an order preserving isometric isomorphism θ of $M(G(\tau_0))$ into $M(S)$, where $M(S)$ is the Banach algebra consisting of all bounded regular Borel measures on S, such that

(1) the image of $M(G(\tau_0))$ in $M(S)$ is weak*-dense;

(2) each non-zero multiplicative linear functional h on $M(G(\tau_0))$ has the form $h(\mu) = \int_S f d\theta\mu$ for some non-zero continuous semicharacter f on S;

(3) there are enough non-zero continuous semicharacters on S to separate points; and

(4) if $\mu \in M(G)$, $\nu \in M(S)$ and $\nu \ll \theta\mu$, then there is a measure $\omega \in M(G)$ such that $\omega \ll \mu$ and $\theta\omega = \nu$.

We call S the *structure semigroup of* $M(G(\tau_0))$. The space of all non-zero semicharacters on S is denoted by $\hat{S}$. We may consider $\hat{S}$ to be the maximal ideal space of $M(G(\tau_0))$.

Given an idempotent p of S, let K_p denote the maximal group of S with p as unit, and N_p the set of those measures μ in $M(G(\tau_0))$ for which $\theta\mu$ are concentrated on K_p. In [10] Taylor showed that if N_p is non-trivial then there is a topology $\tau \in \mathscr{T}(G(\tau_0))$ such that N_p coincides with the radical $L^{1/2}(G(\tau))$ of $L^1(G(\tau))$ in $M(G(\tau))$, the intersection of all maximal ideals containing the ideal $L^1(G(\tau))$. Let K stand for the union of all K_p, where p runs over the set of idempotents of S. We shall denote by $M_K(G(\tau_0))$ the set of all those measures μ in $M(G(\tau_0))$ for which $\theta\mu$ are concentrated on K but vanish on K_p for every idempotent p.

The purpose of this section is to show that under suitable restriction $M_K(G(\tau_0))$ is not trivial. This will give an affirmative answer to the problem raised by Taylor in [10]. It should be remarked that K. Izuchi ([5]) also proved independently the non-triviality of $M_K(G(\tau_0))$ for the case of the Bohr compactification of the real line group.

Let us introduce several notations. Let

$$\Lambda_n = \{(a_0, a_1, a_2, \ldots): a_0 = 1, a_j = 1 \text{ or } 2 \text{ for } 1 \leqslant j \leqslant n \text{ and } a_j = 0 \text{ for } n+1 \leqslant j\}$$

and $\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n$. For $\alpha \in \Lambda$ we write $|\alpha| = n$ if α belongs to Λ_n. Let $\alpha = (a_0, a_1, \ldots)$ and $\beta = (\beta_0, \beta_1, \ldots)$ be elements of Λ. If $a_j = \beta_j$ $(0 \leqslant j \leqslant n)$ and $a_{n+1} \neq \beta_{n+1}$, then we denote by $\alpha \wedge \beta$ the element $(a_0, a_1, \ldots, a_n, 0, 0, \ldots)$. The notation $\alpha \geqslant \beta$ will mean the relation $\alpha \wedge \beta = \beta$. If $\alpha \neq \beta$, then $\alpha' \wedge \beta' = \alpha \wedge \beta$ whenever $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$.

Let $G(\tau_0)$ be the complete direct sum of a family $\{H_\alpha\}_{\alpha \in \Lambda}$ of infinite compact groups, in particular, H_α is uncountable. Each $x \in G(\tau_0)$ may be thought of as a string $x = (\ldots, x_\alpha, \ldots)$, the group operating being componentwise addition. For each $\alpha_0 \in \Lambda$ let

$$G_{\alpha_0} = \{(\ldots, x_\beta, \ldots) \in G(\tau_0): x_\beta = 0 \text{ if } \beta \leqslant \alpha_0\}.$$

Let m_α be the normalized Haar measure on G_α. We define the measure μ_n $(n = 1, 2, \ldots)$ as follows,

$$\mu_n = \frac{1}{2^n} \sum_{\alpha \in \Lambda_n} m_\alpha.$$

LEMMA 3.1. *The countable set* $\{\mu_n\}_{n=1}^{\infty}$ *has the unique weak-* cluster point.*

Proof. Since $G(\tau_0)$ is compact, by the uniqueness of Fourier–Stieltjes transform, if $\{\hat{\mu}_n(\gamma)\}_{n=1}^{\infty}$, where $\hat{\mu}_n$ is the Fourier–Stieltjes transform of

μ_n, is convergent for each continuous character γ of $G(\tau_0)$, then $\{\mu_n\}_{n=1}^{\infty}$ has the unique weak-* cluster point. If $\alpha, \beta \in \Lambda$ and $\beta \leqslant \alpha$, then $G_\alpha \subset G_\beta$. Since $\hat{m}_\alpha(\gamma) = 1$ if $\gamma = 1$ on G_α and $\hat{m}_\alpha(\gamma) = 0$ if $\gamma \neq 1$ on G_α ([7], p. 10), $\beta \leqslant \alpha$ implies

$$\hat{m}_\beta(\gamma) \leqslant \hat{m}_\alpha(\gamma).$$

Thus, if $1 \leqslant n \leqslant m$, then

$$\hat{\mu}_m(\gamma) = \frac{1}{2^m}\sum_{\alpha \in \Lambda_m} \hat{m}_\alpha(\gamma) = \frac{1}{2^m}\sum_{\beta \in \Lambda_n}\sum_{\beta \leqslant \alpha \in \Lambda_m} \hat{m}_\alpha(\gamma)$$
$$\geqslant \frac{1}{2^m}\sum_{\beta \in \Lambda_n} 2^{m-n}\hat{m}_\beta(\gamma) = \frac{1}{2^n}\sum_{\beta \in \Lambda_n}\hat{m}_\beta(\gamma) = \hat{\mu}_n(\gamma),$$

so that $\{\hat{\mu}_n(\gamma)\}_{n=1}^{\infty}$ is a non-decreasing sequence. Thus $\{\hat{\mu}_n(\gamma)\}_{n=1}^{\infty}$ is convergent for each continuous character γ of $G(\tau_0)$. This completes the proof. ∎

Let μ be the weak-* limit of $\{\mu_n\}_{n=1}^{\infty}$, then clearly μ is a probability measure. Given $\alpha \in \Lambda$ and an integer $n \geqslant |\alpha|$, we put

$$\mu_n^\alpha = \frac{1}{2^n}\sum_{\beta \in \Lambda_n^\alpha} m_\beta,$$

where $\Lambda_n^\alpha = \{\beta \in \Lambda_n:\ \alpha \leqslant \beta\}$. Then $\{\mu_n^\alpha\}_{n=|\alpha|}^{\infty}$ has the unique weak-* cluster point μ^α with the norm $\frac{1}{2^{|\alpha|}}$ whose support is contained in G_α. Furthermore, we obtain

$$\mu = \sum_{\alpha \in \Lambda_n}\mu^\alpha \quad (n = 1, 2, \ldots).$$

LEMMA 3.2. *If $\alpha \neq \beta$ and $|\alpha| = |\beta|$, then*

$$\mu^\alpha * \mu^\beta = \frac{1}{2^{2|\alpha|}} m_{\alpha\wedge\beta}.$$

Proof. Since $\alpha' \geqslant \alpha$ and $\beta' \geqslant \beta$ implies

$$\alpha'\wedge\beta' = \alpha\wedge\beta \quad \text{and} \quad m_{\alpha'} * m_{\beta'} = m_{\alpha'\wedge\beta'} = m_{\alpha\wedge\beta},$$

it follows that

$$\mu_n^\alpha * \mu_n^\beta = \left\{\frac{1}{2^n}\sum_{\alpha' \in \Lambda_n^\alpha} m_{\alpha'}\right\} * \left\{\frac{1}{2^n}\sum_{\beta' \in \Lambda_n^\beta} m_{\beta'}\right\}$$
$$= \frac{1}{2^{2n}}\sum_{\alpha' \in \Lambda_n^\alpha}\sum_{\beta' \in \Lambda_n^\beta} m_{\alpha'} * m_{\beta'} = \frac{1}{2^{2|\alpha|}} m_{\alpha\wedge\beta}. \ ∎$$

LEMMA 3.3. *The measure $\theta\mu$ is concentrated on K.*

Proof. It is enough to prove

$$\theta\mu(A_f) = 0 \quad \text{for each } f \in \hat{S},$$

where A_f is the set of those points s of S for which $0 < |f(s)| < 1$ ([10]). If $\alpha, \beta \in \Lambda_n$ and $\alpha \neq \beta$, then the inequality, which is a consequence of multiplicativity of f,

$$\chi_{A_f}(x)\chi_{A_f}(y) \leqslant \chi_{A_f}(xy)$$

holds and Lemma 3.2 implies

$$0 \leqslant \theta\mu^\alpha(A_f)\,\theta\mu^\beta(A_f)$$
$$= \iint \chi_{A_f}(x)\chi_{A_f}(y)\,d\theta\mu^\alpha(x)\,d\theta\mu^\beta(y)$$
$$\leqslant \iint \chi_{A_f}(xy)\,d\theta\mu^\alpha(x)\,d\theta\mu^\beta(y)$$
$$= \theta\mu^\alpha * \theta\mu^\beta(A_f) = \theta(\mu^\alpha * \mu^\beta)(A_f) = \frac{1}{2^{2n}}\theta m_{\alpha\wedge\beta}(A_f).$$

Here the multiplicativity of the map θ is important. Since the measure $m_{\alpha\wedge\beta}$ is concentrated on K ([10]), it follows that

$$\theta\mu^\alpha(A_f)\,\theta\mu^\beta(A_f) = 0.$$

hence $\theta\mu^\alpha(A_f) = 0$ or $\theta\mu^\beta(A_f) = 0$. Then, since $\mu = \sum_{\alpha \in \Lambda_n}\mu^\alpha$, for each n, there is $\alpha \in \Lambda_n$ such that

$$\theta\mu(A_f) = \theta\mu^\alpha(A_f) \leqslant \frac{1}{2^n}.$$

This leads to $\theta\mu(A_f) = 0$. ∎

LEMMA 3.4. *For each idempotent $p \in S$, $\theta\mu(K_p) = 0$.*

Proof. If $\theta\mu(K_p) = \delta > 0$ for some idempotent $p \in S$, then there is topology $\tau \in \mathcal{T}(G(\tau_0))$ such that $N_p = L^{1/2}(G(\tau))$ ([10]). Let n be an integer with $\frac{1}{2^{n-1}} < \delta$. Since $\|\mu^\alpha\| = \frac{1}{2^n}$ for all $\alpha \in \Lambda_n$ and $\mu = \sum_{\alpha \in \Lambda_n}\mu^\alpha$, there exists at least three distinct elements $\alpha_1, \alpha_2, \alpha_3$ of Λ_n such that $\mu^{\alpha_i} (i = 1, 2, 3)$ are not singular to $L^{1/2}(G(\tau))$. Let ω_i be non-zero positive measure of $L^{1/2}(G(\tau))$, with $\omega_i \ll \mu^{\alpha_i} (i = 1, 2, 3)$. Then, for some integer k, $(\omega_1 * \omega_2)^k$ is not singular to $L^1(G(\tau))$ ([10], p. 112). On the other hand, from $(\omega_1 * \omega_2)^k \ll (\mu^{\alpha_1} * \mu^{\alpha_2})^k \ll (m_{\alpha_1\wedge\alpha_2})^k = m_{\alpha_1\wedge\alpha_2}$ it follows that $(\omega_1 * \omega_2)^k \in L^1(G(\tau_{\alpha_1\wedge\alpha_2}))$, where $\tau_{\alpha_1\wedge\alpha_2}$ is the weakest locally compact group topology on G such that $G_{\alpha_1\wedge\alpha_2}$ is open. Since $L^1(G(\tau)) \cap L^1(G(\tau_{\alpha_1\wedge\alpha_2})) \neq \{0\}$, $\tau = \tau_{\alpha_1\wedge\alpha_2}$.

On the other hand, since $(\omega_1 * \omega_2 * \omega_3)^k$ is not singular to $L^1(G(\tau))$ and $(\omega_1 * \omega_2 * \omega_3)^k \ll m_{\alpha_1\wedge\alpha_2\wedge\alpha_3}$, we have $\tau = \tau_{\alpha_1\wedge\alpha_2\wedge\alpha_3}$. But, from H_{α_3}

being an uncountable compact group, it follows that $\tau_{a_1\wedge a_2} \neq \tau_{a_1\wedge a_2\wedge a_3}$. Thus we have reached a contradiction. ■

Since Λ is a countable set, we can get the following theorem.

THEOREM 3.5. *If $G(\tau_0)$ is the complete direct sum of an infinite family of infinite compact abelian groups, then $M_K(G(\tau_0))$ is non-trivial.*

COROLLARY. *If $G(\tau_0)$ is a compact group such that the dual group $\hat{G}(\tau_0)$ of $G(\tau_0)$ has an infinite independent set P, then $M_K(G(\tau_0)) \neq \{0\}$.*

Proof. Let $\{P_n\}_{n=1}^\infty$ be a disjoint collection of infinite subsets of $\hat{G}(\tau_0)$, with $P = \bigcup_{n=1}^\infty P_n$. Let H be the annihilator of the group generated by P. Then $G(\tau_0)/H$ is the complete direct sum of a countable family $\{H_n\}_{n=1}^\infty$ of infinite compact groups, where H_n are dual groups of infinite discrete groups generated by P_n ([7], p. 37). Let S_1 and S_2 be the structure semigroups of $M(G(\tau_0))$ and $M(G(\tau_0)/H)$, respectively. Let θ_1 and θ_2 be the homomorphisms of $M(G(\tau_0))$ into $M(S_1)$ and of $M(G(\tau_0)/H)$ into $M(S_2)$ with properties (1)–(4) in this section, respectively. By Theorem 3.5, $M_K(G(\tau_0)/H)$ is non-trivial.

Let us denote by φ the canonical homomorphism of $G(\tau_0)$ to $G(\tau_0)/H$ and by Φ the induced Banach algebra homomorphism from $M(G(\tau_0))$ onto $M(G(\tau_0)/H)$ ([7], p. 54). There exists a non-zero positive measure $\mu \in M(G(\tau_0))$ such that $\Phi\mu$ is a measure of $M_K(G(\tau_0)/H)$. Since $\Phi(m_H)$ is the unit of $M(G(\tau_0)/H)$, we may assume that $\mu * m_H = \mu$.

At first we shall show for any $f \in \hat{S}_1$

$$|f|^2(s) = |f|(s) \quad \text{for } s \in S_1,\ \theta_1\mu \text{ a.e.},$$

if $|f|(m_H) = 0$, then

$$\int |f|\, d\theta_1\mu = |f|(\mu) = |f|(\mu * m_H) = |f|(\mu)|f|(m_H) = 0,$$

so that $|f|(s) = 0$ $\theta_1\mu$ a.e. If $|f|(m_H) \neq 0$, then $|f|(s) = 1$ for $s \in S_1$ $\theta_1 m_H$ a.e. ([10], p. 112). Thus, by Lemma 3.2 in [1], there is a positive semicharacter $g \in \hat{S}_2$ such that

$$|f|(\nu) = g(\Phi\nu) \quad \text{for each } \nu \in M(G(\tau_0)).$$

Put $A_f = \{s \in S_1\colon 0 < |f|(s) < 1\}$. If $\theta_1\mu(A_f) > 0$, then there is a non-zero positive measure $\omega \in M(G(\tau_0))$ such that $\theta_1\omega$ is the restriction measure to A_f of $\theta_1\mu$ (cf. [9]). From the inequality

$$\int g\, d\theta_2\Phi\omega = \int |f|\, d\theta_1\omega = \int_{A_f} |f|(s)\, d\theta_1\omega(s) < \|\theta_1\omega\| = \|\Phi\omega\|,$$

it follows that $\theta_2\Phi\omega(\{s \in S_2\colon g(s) < 1\}) > 0$. Since $\theta_2\Phi\omega \ll \theta_2\Phi\mu$, we have $\Phi_2\omega \in M_K(G(\tau_0)/H)$, so that $\theta_2\Phi\omega(\{s \in S_2\colon g(s) = 0\}) > 0$. Let ν be the measure on $G(\tau_0)/H$ such that $\theta_2\nu$ is the restriction to $\{s \in S_2\colon g(s) = 0\}$ of $\Phi\omega$, then ν is a non-zero positive measure and $g(\nu) = 0$. Let ν' be a non-zero positive measure on $G(\tau_0)$ such that $\nu' \ll \omega$ and $\Phi\nu' = \nu$. From $\nu' \ll \omega$, it follows that $\theta_1\nu' \ll \theta\omega$, thus

$$g(\nu) = g(\Phi\nu') = |f|(\nu') = \int_{A_f} |f|\, d\theta_1\nu' > 0.$$

This is a contradiction.

Finally, we shall show $\mu \perp L^{1/2}(G(\tau))$ for every $\tau \in \mathscr{T}(G(\tau_0))$. It is enough to show $\Phi(L^{1/2}(G(\tau))) \subset L^{1/2}(G(\tau)/H)$. Let h be any multiplicative linear functional of $M(G(\tau)/H)$ such that $h(\nu) = 0$ for every $\nu \in L^1(G(\tau)/H)$. Since $\Phi(L^1(G(\tau))) = L^1(G(\tau)/H)$ ([7], p. 55), $h \circ \Phi$ is a multiplicative linear functional of $M(G(\tau))$ such that $h \circ \Phi(L^1(G(\tau))) = 0$. Thus $h(\Phi\omega) = h \circ \Phi(\omega) = 0$ for every $\omega \in L^{1/2}(G(\tau))$. This shows that $\Phi(L^{1/2}(G(\tau))) \subset L^{1/2}(G(\tau)/H)$. ■

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