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On universal time for the controllability of time-depending linear control systems

by

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Abstract. Given a family of continuous linear operators: C_t: $X \to Y$, $0 \leqslant t < +\infty$ (X, Y — Banach spaces) with $C_t X \subset C_{t'} X$ for $t < t'$ and such that for each $y \in Y$ there is a pair (t, x_t) for which $C_t x_t = y$. Then there exists a t_u such that $C_{t_u} X = Y$.

This result is applied to the proof of the existence of the universal time for controllability of systems described by a differential equation in a Banach space:

$$dx/dt = Ax + Bu,$$

where A does not depend on time t. A modification of the main result allows us to prove a similar fact concerning the so-called zero controllability.

Let a control system be described by a differential equation in a Banach space

$$\frac{dx}{dt} = A(t)x + B(t)u, \qquad 0 \leqslant t < +\infty, \tag{1}$$

where x belongs to a Banach space E, u belongs to a Banach space F, $A(t)$ is a linear (not necessarily continuous) operator acting in E, $B(t)$ is a continuous linear operator mapping F into E.

The problem of the existence of a solution of equation (1) with an initial condition

$$x(0) = x_0 \tag{2}$$

and the form of the solution is discussed in detail in [1].

The standard tool for representing the solution is the so-called evolution operators.

We shall make the standard assumptions warranting the existence of the so-called evolution operators of equation (1) (see for example [1], Chapter II).

We shall not recall all the properties of evolution operators. For our consideration it will only be important that evolution operators, i.e., a family of continuous linear operators $U(t, s)$ mapping Y into itself, depending on two real parameters t, s and such that

1° $U(t,t) = I$,

2° $U(t,s) = U(t,\tau)U(\tau,s)$ for $0 \leqslant s \leqslant \tau \leqslant t$,

3° the so-called generalized solution of equation (1) with the initial condition

$$x_s(s) = x_1, \tag{3}$$

can be written in the form

$$x_s(t) = U(t,s)x_1 + \int_s^t U(t,\tau)B(\tau)u(\tau)\,d\tau \tag{4}$$

(see for example [1], Chapter II).

We say that systems (1) is *controllable* if for each pair of elements x_0, x_1 of E there is a time t_0 and a control $u(\cdot)$ such that the generalized solution corresponding to the control $u(\cdot)$ and the initial condition (2) satisfies the condition $x(t_0) = x_1$.

In the case where x_1 is fixed and equal to zero we speak of zero-controllability.

We say that there is a universal time t_u for controllability if for an arbitrary pair x_0, x_1 there is a control $u(\cdot)$ such that the generalized solution of equation (1) corresponding to initial condition (2) and the control u satisfies the condition $x(t_u) = x_1$.

In a similar way we can determine the existence of a universal time for zero-controllability.

In [5] Zabczyk has proved the existence of a universal time when $E = F$ are Hilbert spaces and the coefficient operators A, B do not depend on time.

In this paper the result of Zabczyk is extended onto arbitrary Banach spaces E, F. Moreover, it is shown that there is a universal time for zero-controllability for non-constant operators $A(t)$ and $B(t)$.

The method of the solution of the problem is based on an abstract approach developed in papers [2], [3], [4].

By a *time-depending linear control system* we shall understand a system of two Banach spaces over reals X, Y and a family of linear continuous operators C_t depending on a real parameter t, $0 \leqslant t < \infty$, called time,

$$(X \xrightarrow{C_t} Y). \tag{5}$$

We say that system (5) is *controllable* if for all $y \in Y$ there is a pair (t, x_t) such that

$$C_t x_t = y. \tag{6}$$

We shall say that time t_0 is *universal for the controllability of system* (5) if $C_{t_0}X = Y$.

Let X be the Cartesian product of two Banach spaces $X = X_0 \times X_1$. We say that a system

$$(X_0 \times X_1 \xrightarrow{\tilde{C}_t} Y) \tag{7}$$

is *zero-controllable* if for each $x_0 \in X_0$ there is a t and a $u \in X_1$ such that

$$\tilde{C}_t(x_0, u) = 0. \tag{8}$$

Write

$$X_t = \{x \in X_0\colon \text{there is a } u \in X_1 \text{ such that } \tilde{C}_t(x, u) = 0\}. \tag{9}$$

THEOREM 1. *Suppose that*

$$X_t \subset X_{t_1} \quad \textit{for} \quad t \leqslant t_1. \tag{10}$$

If system (7) *is zero-controllable, then there is a universal time t_u such that for every $x_0 \in X_0$ there is a $u_0 \in X$ such that*

$$\tilde{C}_{t_u}(x_0, u_0) = 0. \tag{11}$$

Proof. The set $W_t = \{(x, u) \in X_0 \times X_1\colon \tilde{C}_t(x, u) = 0\}$ is closed in the space $X_0 \times X_1$ as an inverse image of a continuous operator. Let P be a projection operator mapping $X_0 \times X_1$ onto X_0, $P(x, u) = x$. Observe that $X_t = PW_t$. Hence, by the Banach theorem on open maps, either $X_t = X_0$ or X_t is of the first category.

By (10) and the zero-controllability of system (7)

$$X_0 = \bigcup_{n=1}^{\infty} X_n. \tag{12}$$

Hence there is an n_0 such that $X_0 = X_{n_0}$, which by the definition of X_t implies the theorem.

Let $X_0 \overset{\text{df}}{=} E$. As X_1 we shall take a space $L^p([0, \infty)\colon F)$, $1 \leqslant p < +\infty$, $(C[0, \infty); F)$ of functions (bounded continuous functions) with values in U such that

$$\|u(\cdot)\| = \Big(\int \|u(t)\|_F^p dt\Big)^{1/p} < +\infty, \quad 1 \leqslant p < +\infty, \tag{13}$$

$$\|u(\cdot)\| = \operatorname*{ess\,sup}_{0 \leqslant t \leqslant \infty} \|u(\cdot)\|_F \quad \text{for} \quad p = +\infty \tag{13'}$$

and for $C([0, \infty)\colon F)$.

The norm in X is defined by formulae (13) and (13′) in the way as before. Let Y be the second Banach space.

Let

$$\tilde{C}_t(x_0, u) = U(t, 0)x_0 + \int_0^t U(t,\tau)B(\tau)u(\tau)\,d\tau. \tag{14}$$

Under conditions warranting the existence of evolution operators and a condition which warrants the local integrability of $B(t)u(t)$ for all $u(\cdot)\in X_1$ operator (14) maps $X_0\times X_1$ into $Y = X_0$.

Observe now that in this case inclusion (10) holds. In fact, let $t_0\leqslant t_1$. Suppose that $x_0\in X_{t_0}$. This means that there is a control $u_0\in X_1$ such that

$$U(t_0, 0)x_0+\int_0^{t_0} U(t_0, \tau)B(\tau)u_0(\tau)\,d\tau = 0. \tag{15}$$

Let $u_1(\tau) = u_0(\tau)\chi[0, t_0]$. Because of the special character of the space X_1, we also have $u_1(\tau)\in X_1$.

By the definition of $u_1(\cdot)$,

$$U(t_1, 0)x_0+\int_0^{t} U(t, \tau)B(\tau)u_1(\tau)\,d\tau = U(t_1, 0)x_0+\int_0^{t_0} U(t, \tau)B(\tau)u_0(\tau)\,d\tau \tag{16}$$

and by property 2° of evolution operators the right-hand side of equality (16) is equal to

$$U(t_1, t_0)\Big[U(t_0, 0)x_0+\int_0^{t_0} U(t_0, \tau)B(\tau)u_0(\tau)\,d\tau\Big] = 0. \tag{17}$$

Hence $x_0\in X_{t_1}$, which implies (10). Finally we get:

COROLLARY 1. *If system (1) is zero-controllable, then there is a universal time t_u such that for every $x_0\in Y$ there is a control $u(\cdot)\in X$ such that the corresponding solution $x(t)$ of equation (1) with the initial condition (2) satisfies the final condition $x(t_u) = 0$.*

In general, zero-controllability does not imply controllability. However, if we assume that $U(t, s)$ are invertible, then zero-controllability is equivalent to controllability and the existence of a universal time for zero-controllability is equivalent to the existence of a universal time for controllability.

It follows from the fact that under the assumption that $U(t, s)$ is invertible the existence of a control from x_0 to y_0 in the time interval $[s, t]$ is equivalent to the existence of a control from $x_0-[U(t, s)]^{-1}Y_0$ to 0.

PROBLEM. Suppose that the system described by differential equation (1) is controllable.

Does a universal time for the controllability of the system exist without the hypothesis that $U(t, s)$ is invertible?

As a consequence of Theorem 1 we get

THEOREM 2. *If system (5) is controllable and*

$$C_t(X)\subset C_{t'}(X) \quad \text{for} \quad t\leqslant t', \tag{18}$$

then there exists a universal time for the controllability of system (5).

Proof. Putting $X_0 = X$, $X_1 = Y$, $Y = Y$ and

$$\tilde{C}_t(y, x) = y-C_t x,$$

we trivially get Theorem 2.

Let $Y = X_0 = E$ and $X_1 = X$ be as before. Let B be a linear continuous operator mapping F into E. Let C_t, $0\leqslant t< +\infty$, be defined by the formula

$$C_t\big(u(\cdot)\big) = \int_0^t S(t-\tau)Bu(\tau)\,d\tau, \tag{19}$$

where $S(t)$ is a strongly continuous semigroup of linear operators of the class c_0.

It is easy to verify that C_t satisfies condition (18). Therefore by Theorem 2 we infer that if the system described above is controllable, then there exists a universal time for the controllability of the system.

Observe that $x(t) = C_t\big(u(\cdot)\big)$ can be interpreted as a generalized solution of a non-homogenous differential equation in a Banach space with constant coefficients

$$\frac{dx}{dt} = Ax(t)+Bu(t) \tag{20}$$

with the initial condition

$$x(0) = 0. \tag{21}$$

A denotes here the infinitesimal generator of the semigroup $S(s)$.

Therefore we obtain

COROLLARY 2. *If for all $y_0\in Y$ there is a control $u(\cdot)X$ such that the corresponding solution of equation (20) with the initial condition (21) satisfies equality $y(t_{y_0}) = y_0$ for certain t_{y_0} depending on y_0, then there is a universal time t_u such that for every $y_0\in Y$ there is a control $u(\cdot)\in X$ such that for the corresponding solution of equation (20) with the initial condition (21) we have $y(t_u) = y_0$.*

COROLLARY 3. *If for all $x_1, y_1\in Y$, there is a control $u(\cdot)\in X$ such that the corresponding solution of equation (20) with the initial condition*

$$x(0) = x_1 \tag{22}$$

satisfies equality $y(t) = y_1$ for certain t depending on x_1, y_1, then there is a universal time t_u such that for all $x_1, y_1\in Y$ there is a control $u(\cdot)\in X$, such that the corresponding solution of equation (20) with the initial condition (22) satisfies the equality $y(t_u) = y_1$.

Proof. By Corollary 2 there is a universal time t_u such that, for every x^0, there is a control u such that the solution of equation (20) with the initial condition (21) corresponding to the control $u(\cdot)$ satisfies (23) $x(t_u) = x^0$.

Putting $x^0 = y_1 - S(t_u)x_1$, we obtain the required control.

Of course the existence of a universal time t_u in system (5) implies that

(24) $$\bigcup_{0\leqslant t\leqslant t_u} C_t X = Y.$$

If C_t has only a countable number of values and (5) is controllable, then (25) holds. An assumption of type (18) (or about a countable number of values of C_t) cannot be replaced by continuity, as follows from the two examples given below, even in finite-dimensional spaces.

EXAMPLE 1. Let $X = Y = C$ be a complex plane considered as a two-dimensional real Banach space.

Let $C_t z = e^{\pi\left(\frac{t}{t+1}\right)i}\operatorname{Re} z$, where $\operatorname{Re} z$ denotes the real part of z. It is easy to verify that

(25) $$Y = \bigcup_{0\leqslant t\leqslant +\infty} C_t X$$

and that for every $t_0 < +\infty$

(26) $$Y \neq \bigcup_{0\leqslant t\leqslant t_0} C_t X.$$

On the other hand, C_t is continuous in the norm topology.

The next example shows that we can replace (26) by a stronger condition,

(27) $$Y = \bigcap_{t\leqslant 0} \bigcup_{t\leqslant \tau<+\infty} C_\tau X,$$

and still inequality (26) holds.

EXAMPLE 2. Let X, Y be as before. Let $C_t z = e^{\pi\left(\frac{t}{t+1}\right)\sin^2 ti}\operatorname{Re} z$. It is easy to verify that C_t is norm-continuous. Of course, C_t satisfies (27) and (26).

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Saturation for Favard operators in weighted function spaces

Dedicated to Jean Favard on the occasion of the tenth anniversary of his death on January 21, 1965

by

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Abstract. This note continues the investigation of the operators

$$F_n^\gamma f(x) := \frac{1}{\sqrt{\pi\gamma n}} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \exp\left(-\frac{n}{\gamma}\left(\frac{k}{n}-x\right)^2\right) \quad (\gamma > 0,\ n \in \mathbf{N})$$

introduced by J. Favard in 1944 for $\gamma = 1$ as a discrete analog of the familiar Gauss–Weierstrass convolution integral. These Favard operators give approximation on the whole real axis $\mathbf{R}$ and are of special interest with regard to approximation in locally convex spaces. The saturation problem for $F_n^\gamma f$ on the Banach space

$$X_N := \{f \in C(\mathbf{R});\ (1+x^{2N})^{-1} f(x) = o(1),\ |x| \to \infty\} \quad (N \in \mathbf{N})$$

is solved by employing a theorem of H. F. Trotter (1958/59) on the convergence of semigroups of operators. Thus the family of noncommutative operators $\{F_n^\gamma;\ n \in \mathbf{N}\}$ is associated with a family of commutative operators having the same saturation class, in this case just the Gauss–Weierstrass integral. For this purpose asymptotic estimates are derived which are needed for verifying the hypotheses of the Trotter theorem. Finally, instead of the weight functions $(1+x^{2N})^{-1}$, also the functions $\exp(-\beta x^2)$, $\beta > 0$, are considered.

1. Introduction. In this note we would like to study the Favard operators

(1.1) $$F_n^\gamma f(x) := \frac{1}{\sqrt{\gamma\pi n}} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \exp\left(-\frac{n}{\gamma}\left(\frac{k}{n}-x\right)^2\right)$$

with $\gamma > 0$, $n \in \mathbf{N}$, the set of positive integers. These operators were introduced by Favard [8], pp. 229, 239, in 1944 for $\gamma = 1$ as discrete analogs of the familiar Weierstrass operators

(1.2) $$W_n^\gamma f(x) := \sqrt{\frac{n}{\gamma\pi}} \int_{-\infty}^{\infty} f(u) \exp\left(-\frac{n}{\gamma}(u-x)^2\right) du.$$