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Complementably universal Banach spaces

by

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Abstract. There is no separable Banach space which is complementably universal for the class of separable Banach spaces.

I. Introduction. A Banach space X is said to be *complementably universal* for a class $\mathscr{A}$ of Banach spaces provided every space in $\mathscr{A}$ is isomorphic to a complemented subspace of X. A. Pełczyński proved in [11] that there exists a separable Banach space which is complementably univeral for the class of all Banach spaces with Schauder basis. This result was extended by Kadec [8], who constructed a separable space which is complementably universal for the class of all separable Banach spaces which possess the bounded approximation property (b.a.p.). Actually, the spaces constructed by Kadec and Pełczyński are isomorphic (cf. [7], [12]).

In Section II we prove

A. Basic result. *There is no separable Banach space which is complementably universal for the class of all separable Banach spaces.*

This result is a simple consequence of Enflo's counterexamples to the approximation problem [3].

Section III and Section IV contain extensions of our basic result. In Section III it is shown that, in contrast to Kadec's theorem,

B. *There is no separable Banach space which is complementably universal for the class of separable Banach spaces which possess the approximation property.*

In Section IV it is shown that Davie's construction in [2] yields

C. *For each $2 < p < \infty$, there is no separable Banach space which is complementably universal for the class of all subspaces of l_p.*

Of course, results B and C both contain the basic result A. We have included a separate proof of A because it is short and the proof is accessible to anyone who is willing to accept the "axiom" that there are subspaces of some special spaces which fail a certain approximation condition mentioned at the beginning of Section II. The proof in Section III is also not very difficult, but uses many more "axioms", while the proof in Section IV requires the reader's detailed knowledge of Davie's construction [2].

We use standard Banach space theory notation as may be found, for example, in [10].

II. The basic result. Let us say that a Banach space Y has the *bounded compact approximation property* (b.c.a.p.) provided there is a uniformly bounded net of compact operators on Y which converges strongly to the identity. It was pointed out to us by T. Figiel that the criterion used by Enflo [3] (and then Davie [2] and Figiel [4]) to guarantee that a space fails the approximation property actually guarantees that it fails the b.c.a.p. (We include in an appendix at the end of the paper Figiel's proof of this assertion.) Thus by Enflo's result [3] (see [2] or [4] for more readable expositions) there is for each $p, 1 < p < \infty$, a subspace E_p of $(l_\infty^1 \oplus l_\infty^2 \oplus l_\infty^3 \oplus \ldots)_{l_p}$ which fails the b.c.a.p.

Suppose now that X is a separable Banach space and for each p there is a complemented subspace Y_p of X which is isomorphic to E_p. Letting Q_p be a projection of X onto Y_p, we have that there is $\lambda < \infty$ and an uncountable set $A \subset (1, \infty)$ so that $\|Q_p\| \leqslant \lambda$ for each $p \in A$.

Since each Y_p fails the b.c.a.p., there are finite sets $(y_i^p)_{i=1}^{n(p)}$ of unit vectors in Y_p and $\varepsilon_p > 0$ so that if T is a compact operator on Y_p for which $\|y_i^p - Ty_i^p\| < \varepsilon_p$ for $1 \leqslant i \leqslant n(p)$, then $\|T\| > \lambda^2$. Choose an uncountable subset B of A so that $n(p) = n$ $(p \in B)$ and $\inf\limits_{p \in B} \varepsilon_p = \varepsilon > 0$.

Since B is uncountable and X is separable, there exist $p < r$ in B so that $\|y_i^p - y_i^r\| < (\lambda + \lambda^2)^{-1}\varepsilon$ for $1 \leqslant i \leqslant n$. Also, note that every operator from Y_r to Y_p is compact. Indeed, it is essentially contained in Banach's book [1] (cf. also the appendix to [13]) that every operator from l_r to l_p is compact when $p < r$; the proof goes over to show that every operator from a subspace of $(l_\infty^1 \oplus l_\infty^2 \oplus l_\infty^3 \oplus \ldots)_{l_r}$ into $(l_\infty^1 \oplus l_\infty^2 \oplus l_\infty^3 \oplus \ldots)_{l_p}$ is compact.

Let $T\colon Y_r \to Y_r$ be the restriction to Y_r of $Q_r Q_p$. In view of the above discussion, T is compact, and obviously $\|T\| \leqslant \lambda^2$. But note that for $1 \leqslant i \leqslant n$,

$$\begin{aligned}
\|y_i^r - Ty_i^r\| &= \|y_i^r - Q_r Q_p y_i^r\| \\
&\leqslant \|y_i^r - Q_r y_i^p\| + \|Q_r y_i^p - Q_r Q_p y_i^r\| \\
&= \|Q_r(y_i^r - y_i^p)\| + \|Q_r Q_p (y_i^p - y_i^r)\| \\
&\leqslant (\lambda + \lambda^2)\,\|y_i^p - y_i^r\| < \varepsilon.
\end{aligned}$$

This contradicts the choice of $(y_i^r)_{i=1}^n$ and completes the proof of the basic result.

III. Spaces with the approximation property. A basis (x_n) is said to be *block Hilbertian* (resp., *block Besselian*) provided that there is a constant K so that for each normalized block basic sequence (y_n) of (x_n) and

scalars (a_n), $\|\sum a_n y_n\| \leqslant K(\sum |a_n|^2)^{1/2}$ (resp., $\|\sum a_n y_n\| \geqslant K(\sum |a_n|^2)^{1/2}$). It is clear that a basis is block Hilbertian if and only if the biorthogonal functionals to (x_n) are block Besselian.

In [9], Lindenstrauss extended the technique of James [6] to show that for each separable Banach space Y there is a space Z so that $Z^{**} = Z \oplus Y^*$ isometrically with the projection of Z^{**} onto Y^* perpendicular to Z having norm at most 2. (Here we have Z embedded in Z^{**} in the canonical way.) Further, Z was constructed to have a shrinking basis (x_n) so that the biorthogonal functionals to (x_n) are block Besselian, and hence (x_n) is block Hilbertian. We will need the fact that if X has a block Hilbertian basis then every operator from X into $(l_\infty^1 \oplus l_\infty^2 \oplus l_\infty^3 \oplus \ldots)_{l_p}$ is compact for $1 < p < 2$. (The proof of this assertion is identical to the proof in the special case $X = l_2$.)

For $1 < p < 2$, let E_p be a subspace of $(l_\infty^1 \oplus l_\infty^2 \oplus l_\infty^3 \oplus \ldots)_{l_p}$ which fails the b.c.a.p. and let Z_p be a James–Lindenstrauss space which has a shrinking block Hilbertian basis and which satisfies $Z_p^{**} = Z_p \oplus E_p$. Note that if S is an operator on E_r and S factors through Z_p^{**} for some $r \neq p$, then S is compact. Using this and a lemma from [5], we will show:

PROPOSITION 1. *There does not exist a separable Banach space X and a $\lambda < \infty$ so that for each $1 < p < 2$ and each equivalent norm $|\cdot|$ on Z_p^*, $(Z_p^*, |\cdot|)$ is λ-isomorphic to a λ-complemented subspace of X.*

Note that since Z_p has a shrinking basis, Z_p^* has a basis and hence the approximation property.

The result B is an immediate consequence of Proposition 1, Proposition 2 below, and the obvious fact that $(Y_1 \oplus Y_2 \oplus \ldots)_{l_p}$ $(1 \leqslant p < \infty)$ has the approximation property if each Y_n does.

PROPOSITION 2. *Suppose that X is complementably universal for a class $\mathscr{A}$ and there is $1 \leqslant p < \infty$ so that for every sequence (Y_n) in $\mathscr{A}$, $(Y_1 \oplus Y_2 \oplus \ldots)_{l_p}$ is in $\mathscr{A}$. Then there is $\lambda < \infty$ so that every Y in $\mathscr{A}$ is λ-isomorphic to a λ-complemented subspace of X.*

Proof. If not, then there are Y_n in $\mathscr{A}$ so that $\|P_n\| d(X_n, Y_n) > n$ for any projection P_n from X onto a subspace X_n. (Here $d(Y, Z)$ is the Banach–Mazur distance coefficient $\inf\{\|T\| \cdot \|T^{-1}\|\colon T$ is an isomorphism from Y onto $Z\}$.) It is clear that $(Y_1 \oplus Y_2 \oplus \ldots)_{l_p}$ is not isomorphic to a complemented subspace of X.

Before proving Proposition 1, we restate Proposition 1 of [5] in a form suitable for our needs.

LEMMA 1. *Let $(Y, \|\cdot\|)$ be a Banach space, $\beta < \infty$, $\varepsilon > 0$, and U a finite subset of Y^*. There is an equivalent norm $|\cdot|$ on Y, a $\delta > 0$, and a finite set F of $\|\cdot\|$-unit vectors in Y so that if T is an operator on Y which satisfies $|Tx - x| < \delta$ for $x \in F$ and $|T| \leqslant \beta$, then $\|T\| \leqslant [1 + 2\varepsilon^{-1}\beta]\beta$ and $\|T^*u - u\| < \varepsilon$ for each $u \in U$.*

We turn now to the proof of Proposition 1. Assume, for contradiction, that there is such an X and λ. Fix $0 < \varepsilon < 2^{-1}$ and $1 < p < 2$. Since $(E_p, \|\cdot\|_p)$ fails the b.c.a.p., the proof of Proposition 2 in [5] shows that there is a finite set U_p of unit vectors in E_p so that if S is a compact operator on E_p and $\|Su-u\|_p < 2\varepsilon$ for $u \in U_p$, then $\|S\|_p > 2[1+2\varepsilon^{-1}\lambda^3]\lambda^3$. By Lemma 1, there is an equivalent norm $|\cdot|_p$ on Z_p^*, a $\delta_p > 0$, and a finite set $(x_i^p)_{i=1}^{n(p)}$ of unit vectors in Z_p^* so that if T is an operator on Z_p^* which satisfies $|Tx_i^p - x_i^p|_p < \delta_p$ and $|T|_p \leqslant \lambda^3$, then $\|T\|_p < [1+2\varepsilon^{-1}\lambda^3]\lambda^3$ and $\|T^*u-u\|_p < \varepsilon$ for all $u \in U_p$. (Here we regard Z_p^{**} as being isometric to $Z_p \oplus E_p$, where E_p has norm $\|\cdot\|_p$, and denote the norm on Z_p^* and Z_p^{**} also by $\|\cdot\|_p$.) By the hypothesis on X, we can assume that Z_p^* is contained in $(X, \|\cdot\|)$, that $\lambda^{-1}\|x\| \leqslant |x|_p \leqslant \|x\|$ for $x \in Z_p^*$, and that there is a projection Q_p from X onto Z_p^* with $\|Q_p\| \leqslant \lambda$.

As in Section II, we have that there exists $1 < p < r < 2$ so that $n(p) = n(r) = n$ and for $1 \leqslant i \leqslant n$, $\|x_i^p - x_i^r\| < (\lambda+\lambda^2)^{-1}\delta_r$. Letting T be the restriction of Q_rQ_p to Z_r^*, we have just as in Section II, that $\|T\| \leqslant \lambda^2$ and $\|Tx_i^r - x_i^r\| < \delta_r$ so that also $|T|_r \leqslant \lambda^3$ and $|Tx_i^r - x_i^r|_r < \delta_r$. Therefore, we have that $\|T\|_r \leqslant (1+2\varepsilon^{-1}\lambda^3)\lambda^3$ and $\|T^*u-u\|_r < \varepsilon$ for all $u \in U_r$. Finally, let P_r be the projection of Z_r^{**} onto E_r perpendicular to Z_r and let S be the restriction of P_rT^* to E_r. Since $\|P_r\|_r \leqslant 2$, it follows that $\|S\|_r \leqslant 2(1+2\varepsilon^{-1}\lambda^3)\lambda^3$ and $\|Su-u\|_r < 2\varepsilon$ for all $u \in U_r$. However, S factors through the space $Z_p^{**} = Z_p \oplus E_r$ and therefore is compact. This contradiction completes the proof.

IV. Subspaces of l_p, $2 < p < \infty$. Throughout this section p is a fixed number with $2 < p < \infty$ and $\mathscr{A}$ denotes the class of all subspaces of l_p. The result C is an immediate consequence of Lemma 2 and the construction of the X_t's given below.

LEMMA 2. *There is no separable Banach space which is complementably universal for $\mathscr{A}$, provided*

(*) *for every $m = 1, 2, \ldots$ there exist an uncountable family $(X_t)_{t \in \Gamma_m} \subset \mathscr{A}$ and points $e_1^t, \ldots, e_m^t \in X_t$, $t \in \Gamma_m$ such that whenever $T\colon X_t \to X_s$, $t \neq s \in \Gamma_m$ satisfies*

$$\sum_{i=1}^{m} \|Te_i^t - e_i^s\| \leqslant 1,$$

then $\|T\| > a_m$ with $\lim\limits_{m\to\infty} a_m = \infty$.

Proof. Suppose X is complementably universal for $\mathscr{A}$. The class $\mathscr{A}$ obviously satisfies the assumptions of Proposition 2 so let $\lambda < \infty$ be such that every Y in $\mathscr{A}$ is λ-isomorphic to a λ-complemented subspace of X.

Let m be so big that $a_m > \lambda^2$.

For each $t \in \Gamma_m$ we fix an embedding $T_t\colon X_t \to X$ and a projection $P_t\colon X \xrightarrow{\text{onto}} T_tX_t$ so that

$$\|P_t\| \leqslant \lambda \quad \text{and} \quad \|x\| \leqslant \|T_tx\| \leqslant \lambda\|x\| \quad \text{for every} \quad x \in X_t.$$

Fix an $\varepsilon > 0$. Since X is separable and Γ_m is uncountable, we can find a pair $t \neq s$ in Γ_m such that

$$\|T_te_i^t - T_se_i^s\| < \varepsilon \quad \text{for} \quad i = 1, \ldots, m.$$

Define now $T\colon X_t \to X_s$ by $T = T_s^{-1}P_sT_t$.

We have $T_s^{-1}P_s(T_te_i^t - T_se_i^s) = Te_i^t - e_i^t$ and therefore

$$\|Te_i^t - e_i^s\| \leqslant \|T_t^{-1}\|\,\|P_s\|\,\varepsilon \leqslant 1\cdot\lambda\cdot\varepsilon,$$

on the other hand, $\|T\| \leqslant 1\cdot\lambda\cdot\lambda$.

By taking $\varepsilon = \lambda^{-1}m^{-1}$ we get a contradiction.

Construction of the X_t's. We exploit Davie's construction to such an extent, that it seems more reasonable to emphasize the changes made rather than to rewrite the major part of [2].

The main point is that we strengthen condition (1) in [2] by requiring the following four conditions to hold simultaneously for $\varepsilon = 0, 1$ and $\eta = 0, 1$

$$\Big|2\sum_{j=\varepsilon 2^{k-1}+1}^{(\varepsilon+1)2^{k-1}} \sigma_j^k(g) - \sum_{j=\eta\cdot 2^k+1}^{(\eta+1)2^k} \tau_j^k(g)\Big| \leqslant Ak^{1/2}2^{k/2} \quad \text{for all} \quad g \in G_k,\ k = 1, 2, \ldots$$

(here A is an absolute constant).

For $\varepsilon = 0, 1$ we put $J(k, \varepsilon) = \{\varepsilon\cdot 2^{k-1}+1, \varepsilon\cdot 2^{k-1}+2, \ldots, (\varepsilon+1)2^{k-1}\}$.

Let $t = \big(t(n)\big)_{n=1}^{\infty} \in \{0, 1\}^{\aleph_0}$. We define

$$X_t = \operatorname{span}\{e_j^k\colon j \in J\big(k, t(k)\big),\ k = 1, 2, \ldots\}$$

(here the e_j^k's are defined as in (2) of [2]), and we set $X = \operatorname{span}\{X_t\colon t \in \{0, 1\}^{\aleph_0}\}$.

For any operator $T\colon X_t \to X$ we define

$$\beta_t^k(T) = 2^{-k+1}\sum_{j \in J(k, t(k))} \alpha_j^k(Te_j^k)$$

(the α_j^k's are defined as in (3) or (4) of [2]).

The further argument of [2] yields also in our case

$$|\beta_t^{k+1}(T) - \beta_t^k(T)| \leqslant \sup\{\|T\Phi_g^{k,t}\|\colon g \in G_k\}$$

for some $\Phi_g^{k,t}$, where $\|\Phi_g^{k,t}\| = O(k^{-2})$, uniformly on t and g.

This gives us that for every $\delta > 0$ there exists a $k = k(\delta)$ such that for every $m > k$,

$$|\beta_t^m(T) - \beta_t^k(T)| \leqslant \delta\,\|T\|$$

for every $t \in \{0, 1\}^{\aleph_0}$ and for every operator $T\colon X_t \to X$.

Suppose now that $t, s \in \{0,1\}^{\aleph_0}$ coincide on the first k places but are different, i.e. $t(n) \neq s(n)$ for some $n > k$. Suppose also that $T\colon X_t \to X_s$ is such that

$$\sum_{j \in J(k,t(k))} \|Te_j^k - e_j^k\| \leqslant 1.$$

This gives

$$\beta_t^k(T) \geqslant 1 - 2^{-k+1}.$$

On the other hand, since $t(n) \neq s(n)$ and (e_j^m, a_j^m) is a biorthogonal system we have

$$\beta_t^n(T) = 0$$

and therefore $\|T\| \geqslant \frac{1}{2}\delta^{-1}$.

Now, for $m \geqslant 2^{k-1}$ we take $\Gamma_m = \{t \in \{0,1\}^{\aleph_0}\colon t_i = 0 \text{ for } i = 1, \ldots, k\}$ and $e_j^t = e_j^k$ for $j = 1, \ldots, 2^{k-1}$ and all $t \in \Gamma_m$; e_j^t are arbitrary for $j = 2^{k-1} + 1, \ldots, m$.

Appendix. We wish to thank T. Figiel for permission to include the following lemma, which was mentioned already in Section II.

LEMMA. *Let E be a Banach space, (e_n) a bounded sequence in x, and (e_n^*) a bounded sequence in X^*. Suppose that (t_n) is a sequence of positive reals and (Δ_n) is a pairwise disjoint sequence of sets of positive integers such that*

$$(*) \qquad \sup_n \sum_{j \in \Delta_n} t_j < \infty.$$

For $T \in L(E)$, set $\varphi_n(T) = \sum_{j \in \Delta_n} t_j e_j^*(Te_j)$. If either $e_n \xrightarrow{w} 0$ or $e_n^* \xrightarrow{w^*} 0$, then $\varphi_n(T) \to 0$ for each compact operator T.

Proof. Assume, e.g., that $e_n^* \xrightarrow{w^*} 0$. Since the set $K = \text{closure}(Te_n)$ is compact, it follows that $\lim_n \sup_{x \in K} |e_j^*(x)| = 0$. This and $(*)$ yield the conclusion.

The space E constructed in [3], [2], or [4] is endowed with a sequence (φ_n) as a above and a compact set K_0 such that for any $A \in L(E)$, $|\lim_n \varphi(A)| \leqslant \sup_{x \in K_0} \|Ax\|$ and $\lim_n \varphi_n(I) = 1$, where I is the identity on X. Consequently, for any compact operator T on E, one has $\sup_{x \in K_0} \|(I-T)x\| \geqslant |\lim_n \varphi_n(I-T)| = 1$, which implies that E fails the b.c.a.p.

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