Inequalities for the maximal function relative to a metric

by

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Abstract. Weighted \( L^p \)-norm inequalities for the maximal function relative to a family of spheres defined by a pseudo-metric are obtained.

The purpose of this note is to obtain weighted \( L^p \)-norm inequalities for the maximal function defined by the spheres of a certain pseudo-metric. These inequalities generalize those known to hold in Euclidean space with the ordinary metric (see [2]), and other metrics considered by D. Kurtz [3] but they do not cover his results about maximal functions defined by certain families of rectangles.

Let \( X \) be a metric space with a measure \( \mu \) and assume that the space of continuous functions with bounded support is contained and is dense in the space of integrable functions. Further, suppose that there is given a real-valued function \( \phi(x, y) \) in \( X \times X \) (it need not be the distance function) with the following properties

(i) \( \phi(x, x) = 0 \);
(ii) \( \phi(x, y) = \phi(y, x) > 0 \) if \( x \neq y \);
(iii) there is a constant \( c \) such that \( \phi(x, z) \leq c \phi(x, y) + \phi(y, z) \) for all \( x, y, \) and \( z \);
(iv) given a neighborhood \( N \) of a point \( x \) there is an \( r, \epsilon > 0 \), such that the sphere \( B_r(x) = \{ y \mid \phi(x, y) \leq \epsilon \} \) with center at \( x \) is contained in \( N \);
(v) the spheres \( B_r(x) = \{ y \mid \phi(x, y) \leq r \} \) are measurable, the measure \( |B_r(x)| \) of \( B_r(x) \) is a continuous function of \( r \) for each \( x \), and there is a constant \( c_c \) such that

\[ c_c |B_r(x)| \leq c |B_r(x)| < \infty \]

for all \( r \) and \( x \). For convenience we shall assume that the constant here coincides with the one in (iii).

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Given a function which is integrable on all the sets $B_r(x)$, we define the maximal function $M_f$ of $f$ as

\[
M_f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, dy.
\]

On the other hand, we say that the weight function $w(x)$, $w(x) > 0$, belongs to the class $A_p$ if

\[
\left[ \int_B w(x) \, d\mu \right] \left[ \int_B w(x)^{-\frac{1}{p-1}} \, d\mu \right]^{p-1} \leq c_w |B|^p, \quad 1 < p < \infty,
\]

for all spheres $B$. Then the results of B. Muckenhoupt [2] for Euclidean space with the ordinary metric hold, namely

**Theorem 1.** If $w \in A_p$, $p > 1$, then

\[
\left[ \frac{1}{|B|} \int_B w^r \, d\mu \right]^{\frac{1}{r}} \leq c_1 \frac{1}{|B|} \int_B w \, d\mu,
\]

with $r > 1$, and $c_1$ and $r$ depend on $p$, the constant $c_w$ in (2) and the constant $c$ in (iii) and (iv).

**Theorem 2.** If $w \in A_p$, $p > 1$, then $w \in A_q$ for all $r, q > p$, where $p_0 < p$ and $p_0$ depends on $p$, $c_w$, and the constant $c$ in (iii) and (iv).

**Theorem 3.** If $w \in A_p$, $p > 1$, then for $p < q < \infty$

\[
\int_B |M_f(x)|^q \, d\mu \leq C \int_B |f(x)|^q \, d\mu,
\]

where $c_q$ depends on $c$, $c_w$, $p$ and $q$. If $p = 1$ and $w \in A_p$, the same result holds for $1 < q < \infty$, and for $p > 0$

\[
|\{x \mid M_f(x) > 2\}| \leq \frac{c_3}{\lambda} \int_B |f(x)| \, d\nu, \quad d\nu = w(x) \, d\mu,
\]

where the left-hand side represents the $\nu$-measure of the set indicated.

Except in the case of Theorem 1, the above statements can be proved by slightly modifying the arguments which have been used in the Euclidean case. Theorem 1, however, which is the key to the other two and is considerably more difficult, requires a different treatment. In spite of the close similarity of some of the following arguments to well-known ones, we give them in full detail for the sake of completeness.

**Lemma 1.** There is a constant $\gamma$ such that

\[
|B_{r_1}(x)| \leq c a^{\gamma} |B_r(x)|, \quad a \gg 1,
\]

where $c$ is the constant in (v).

**Proof.** Let $2^{a-1} \leq a < 2^b$, $k \geq 1$. Then $k \leq 1 + \log a$ and from (v) we obtain

\[
|B_{r_1}(x)| \leq |B_{r_2}(x)| \leq c \phi |B_r(x)| \leq c \phi^{\log a} |B_r(x)| = c a^{\gamma} |B_r(x)|
\]

with $\gamma = \log c$.

**Lemma 2.** For each $x$, $|B_r(x)|$ is a continuous non-decreasing function of $r$, and $|B_r(x)| > 0$ for $r > 0$, unless $\mu$ vanishes identically.

**Proof.** We only have to show that $|B_r(x)| > 0$ for $r > 0$. Suppose on the contrary that $|B_r(x)| = 0$. Then from Lemma 1 it follows that $B_{r_1}(x) = 0$ for all $s, s \geq 1$, and this clearly implies that $\mu$ vanishes identically.

**Lemma 3.** Let $\mathbb{S}$ be a family of spheres with bounded radii. Then there exists a countably family of disjoint spheres $B_{r_k}(x_k)$ such that each sphere in $\mathbb{S}$ is contained in one of the spheres $B_{r_k}(x_k)$, where $b = 3^a c$ and $c$ is the constant in (iii). Then $\mathbb{S}$ is a maximal family of spheres.

**Proof.** Let $M$ be a bound for the radii of the spheres in $\mathbb{S}$, and let $a < 1$ be such that

\[
3^a = \phi \left( 1 + \frac{1}{a} \right) + \frac{c}{a}.
\]

Since $c > 1$, such an exists. Now for each integer $k, k > 0$, we construct inductively a family of spheres with the following properties:

1. $B_{r_k}(x_k) \subset \mathbb{S}$, $\alpha M < r_k \leq \alpha^{-1} M$;
2. the $B_{r_k}(x_k)$ are disjoint for $h \neq k$;
3. for each $k$ the family is maximal with respect to properties 1 and 2.

Evidently such a family exists. Let now $B_r(x) \in \mathbb{S}$. If $\alpha M < r \leq \alpha^{-1} M$, then $B_1(x)$ intersects one of the spheres $B_{r_k}(x_k)$, $h \leq k$. But then $r_k > \sigma$ and therefore, if $x \in B_{r_k}(x)$ and $y \in B_{r_k}(x_k) \cap B_{r_k}(x)$, on account of (iii) we have

\[
\sigma(x, x_k) \leq \sigma + \sigma(x_k, x_k) \leq \sigma + \sigma(y, x_k) + \sigma(x, y) \leq \sigma + \sigma(r + r_k) \leq \sigma + \sigma(r_k) \leq \sigma + \sigma(r_k) + \sigma(x, y) \leq 3^a r_k = b r_{k+1},
\]

that is, $x \in B_{3^a r_k}(y)$,
Lemma 4. Let \( w \in A_p, 1 \leq p < \infty \), and let \( E \) be a subset of the sphere \( B \).

Then
\[
\frac{|E|}{|B|} \geq c_\lambda \left( \frac{|E|}{|B|} \right)^p, \quad dv = w \, d\mu.
\]

Proof. If \( p > 1 \), then on account of (2) we have
\[
|E| = \int_E w^{1-p} |1-w|^{-1/p} d\mu \leq \left( \frac{\int w \, d\mu}{\mu} \right)^{1-p} \left( \int w^{-1/(p-1)} |1-w|^{-1/(p-1)} d\mu \right)^{p-1}
\]
\[
= \frac{|E|}{|B|} \left( \frac{\int w^{-1/(p-1)} |1-w|^{-1/(p-1)} d\mu}{\mu} \right)^{p-1} \leq c_\lambda^p \frac{|E|}{|B|} \left( \frac{\int w \, d\mu}{\mu} \right)^{p-1}
\]
\[
= c_\lambda^p \frac{|E|}{|B|} \left( \frac{|E|}{|B|} \right)^{p-1},
\]
which is the desired inequality.

If \( p = 1 \), then since
\[
\text{essinfe}(a) \leq \frac{1}{|B|} \int_B w(a) \, d\mu = \frac{|E|}{|B|},
\]
the second inequality in (2) gives
\[
\frac{|E|}{|B|} \leq c_\lambda \left( \frac{|E|}{|B|} \right).
\]

Lemma 5. Suppose \( E \subseteq B \) and \( |E| \leq \delta |B| \). Then
\[
\frac{|E|}{|B|} \leq \left( 1 - \delta c_\lambda^{-1} \right) \frac{|B|}{|E|}.
\]

Proof. Applying the preceding lemma to \( E' = B - E \) we obtain
\[
\frac{|E'|}{|B|} = 1 - \frac{|E'|}{|B|} \leq 1 - \delta c_\lambda^{-1} \left( 1 - \delta \right)^{p}.
\]

Lemma 6. Let \( w \in A_p, 1 \leq p < \infty \), and \( w \not\equiv 0 \) \( \in L^p \), \( \lambda > 0 \); then
\[
|\{ x : (Mf)(x) > \lambda \}| \leq \frac{c_\lambda}{\lambda^p} \int_B |f|^p \, dv, \quad dv = w(x) \, d\mu,
\]
where \( c_\lambda \) depends on \( c, c_\lambda \) and \( p \).

Proof. For each \( n, n > 0 \), we define
\[
(M_n f)(x) = \sup_{r < n} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| \, d\mu,
\]
and we shall show that the inequality above holds with \( M \) replaced by \( M_n \) with \( c_n \) independent of \( n \). Once this is established, the lemma will follow by letting \( n \) tend to infinity.

Let \( B = B_r(x) \) be a sphere such that \( r \leq \delta \) and
\[
\lambda |B| < \int_B |f| \, d\mu.
\]

Then, according to (2), if \( p > 1 \) we have
\[
\lambda |B| < \int_B |f| \, d\mu \leq \int_B |f| w^{1-p} w^{-1/p} \, d\mu \leq \int_B |f|^p (1-\delta)^p \left( \frac{\int w \, d\mu}{\mu} \right)^{p-1} \left( \int w^{-1/(p-1)} |1-w|^{-1/(p-1)} d\mu \right)^{p-1}
\]
\[
\leq \int_B |f|^p \, d\mu \left( \frac{\int w \, d\mu}{\mu} \right)^{p-1} \left( \frac{\int w^{-1/(p-1)} |1-w|^{-1/(p-1)} d\mu}{\mu} \right)^{p-1}
\]
\[
= c_\lambda |B| \left( \frac{|E|}{|B|} \right)^{p-1} \int_B |f|^p \, d\mu,
\]
whence it follows that
\[
\lambda^p |B| \leq c_\lambda \int_B |f|^p \, d\mu,
\]
and for \( p = 1 \)
\[
\lambda |B| < \int_B |f| \, d\mu \leq \int_B |f| w \, d\mu \leq c_\lambda |B| \left( \frac{\int w \, d\mu}{\mu} \right)^{-1} \left( \frac{\int w^{-1/(p-1)} |1-w|^{-1/(p-1)} d\mu}{\mu} \right)^{-1}
\]
\[
= c_\lambda |B| \left( \frac{|E|}{|B|} \right)^{-1} \int_B |f|^p \, d\mu,
\]
so that (4) also holds in this case.

Let now \( \mathcal{B} \) be the family of spheres satisfying (3). As we have just shown, they also satisfy (4). Evidently, the union of all spheres in \( \mathcal{B} \) contains the set
\[
\{ x : (Mf)(x) > \lambda \}.
\]

According to Lemma 3, there exists a disjoint family of spheres \( B_{\lambda}(x) \) in \( \mathcal{B} \) such that
\[
\bigcup_{x \in \mathcal{B}} B_{\lambda}(x) = \bigcup_{x \in \mathcal{B}} \{ x : (Mf)(x) > \lambda \}.
\]

By Lemmas 1 and 4 we have
\[
\frac{|B_{\lambda}(x)|}{|B_{\lambda}(x)|} \leq c_\lambda \left( \frac{|B_{\lambda}(x)|}{|B_{\lambda}(x)|} \right)^p \leq c_\lambda \varepsilon^{bp}.
\]

so that
\[
|B_{\lambda}(x)| \leq c_\lambda \varepsilon^{bp} |B_{\lambda}(x)|.
\]

Thus since the \( B_{\lambda}(x) \) are disjoint and satisfy (4), we obtain
\[
|\{ x : (Mf)(x) > \lambda \}| \leq \sum_{x \in \mathcal{B}} |B_{\lambda}(x)| \leq c_\lambda \varepsilon^{bp} \sum_{x \in \mathcal{B}} |B_{\lambda}(x)|
\]
\[
\leq \frac{1}{\lambda^p} c_\lambda \varepsilon^{bp} \sum_{x \in \mathcal{B}} \int_{B_{\lambda}(x)} |f|^p \, dv \leq \frac{c_\lambda}{\lambda^p} \int_B |f|^p \, dv.
\]
Lemma 7. Let \( f \) be integrable on every sphere. Then
\[
\lim_{r \to 0} \frac{1}{|B_r|} \int_{|x| < r} f(x) \, dx = f(0)
\]
almost everywhere. In particular, \( f \) is \( |f| \leq M \) almost everywhere.

Proof. In view of (iv) our assertion holds for \( f \) continuous. Since continuous functions with bounded support are assumed to be dense in \( L^p \), for integrable \( f \) our lemma follows from Lemma 6 in the well-known fashion and clearly, if the lemma holds for \( f \) integrable, then it holds in general.

Proof of Theorem 1. Let \( B = B_{r_1}(x_0) \) and
\[
\lambda = \frac{1}{|B|} \int_B \omega d\mu.
\]
We shall construct a sequence \( E_1 \subseteq E_2 \subseteq \ldots \subseteq E_m \subseteq \ldots \) of subsets of \( B_{p_0+\epsilon_0}(x_0) \) such that \( \bigcap E_m = 0 \), \( \omega \leq \lambda \) almost everywhere in \( B \) and outside \( E_m \), and
\[
|E_{m+1}| \leq \delta |E_m|, \quad \delta < 1,
\]
where \( \delta \) depends only on \( c_0, \epsilon, \rho \) and \( p \). Once these sets have been constructed, taking \( \epsilon \) so that \( \epsilon^2 < 1 \) we will have
\[
\int_B \omega^{1+\epsilon} d\mu \leq \int_B \omega^{1+\epsilon} d\mu + \sum_{i=1}^m \int_{B_{p_i}(x_i)} \omega^{1+\epsilon} d\mu.
\]
\[
\leq (\epsilon \lambda)^{1+\epsilon} \int_B \omega d\mu + \sum_{i=1}^m \left( \frac{\epsilon^{p_i+\epsilon}}{\delta^i} - \frac{\epsilon^{p_i+\epsilon}}{\delta^i} \right) \int_{B_{2p_i}(x_i)} \omega d\mu.
\]
\[
\leq (\epsilon \lambda)^{1+\epsilon} |B| + \frac{\epsilon^{p+\epsilon}}{\delta} \sum_{i=1}^m \delta^{-i} \omega^{1+\epsilon}.
\]
But \( E_1 \subseteq B_{p_0+\epsilon_0}(x_0) \) and therefore, according to Lemmas 1 and 4, we have
\[
|E_1| \leq c_0 \epsilon (\rho + \epsilon)^p |B|
\]
Furthermore, since \( \lambda = |B| |B|^{-1} \), substituting above we obtain
\[
\int_B \omega^{1+\epsilon} d\mu \leq \epsilon \lambda (|B| |B|^{-1})^{1+\epsilon},
\]
which is the desired result.

To construct the sets \( E_m \) we proceed as follows. Given a point \( x \) in \( B \) consider the ratio
\[
\frac{|B_r(x)|}{|B|}
\]
This is a continuous function of \( r \) for \( r > 0 \). Furthermore, as we shall see, if \( a, a > 1 \), is sufficiently large as compared with \( c_n \), \( c_0 \) and \( c_1 \), then
\[
\frac{|B_{r_s}(x)|}{|B_{r_s}(x)|} \leq a_1,
\]
so that for each \( m, m \geq 1 \), there is a largest value of \( r, r \leq r_n \), such that
\[
|B_{r_s}(x)| \leq a_m^m \lambda,
\]
or else \( (M_a \omega)(x) \leq a_m^m \lambda \). Let us denote these spheres by \( B^{m_0} \) and by \( B^{m_0+1} \) the spheres in a subfamily as in Lemma 3. Let \( B^{m_0} \) be the spheres concentric with the \( B^{m_0+1} \) and \( b \) times their radii. Then if
\[
E_m = \bigcup B^{m_0}
\]
according to Lemma 3, \( E_m \) consists all spheres \( B^{m_0} \) and therefore \( (M_a \omega)(x) \leq a_m \lambda \) in \( B \) and outside \( E_m \) and, by Lemma 7, \( \omega \leq a_m \lambda \) almost everywhere in \( B \) and outside \( E_m \).

Let us prove (6) before proceeding to show that the sets \( E_m \) have the properties stated above. As is readily verified we have
\[
\frac{|B_{r_s}(x)|}{|B_{r_s}(x)|} \leq c_m (\rho + \epsilon)^p |B|
\]
and therefore from Lemmas 1 and 4 we obtain
\[
\frac{|B_{r_s}(x)|}{|B_{r_s}(x)|} \leq c_m (\rho + \epsilon)^p |B|
\]
that is,
\[
|B_{r_s}(x)| \leq 2^m \rho^{m+1} |B_{r_s}(x)|,
\]
and
\[
|B_{r_s}(x)| \leq 2^m \rho^{m+1} |B_{r_s}(x)|,
\]
from these inequalities it follows that
\[
\frac{|B_{r_s}(x)|}{|B_{r_s}(x)|} \leq 2^m \rho^{m+1} \left( \frac{|B_{r_s}(x)|}{|B_{r_s}(x)|} \right)^p.
\]
Returning to the sets \( E_m \), let us show next that \( E_m \supseteq E_{m+1} \). Consider one of the spheres \( B^{m+1} \). Let us denote this sphere by \( B_{r_s}(x) \). Then \( B^{m+1} \subseteq B_{r_s}(x) \) is contained in a sphere \( B^{m} \). Now \( B^{m_0+1} \) is contained in a sphere \( B^{m} \) and
\[
\frac{|B_{r_s}(x)|}{|B_{r_s}(x)|} = a_{m+1} \lambda, \quad \frac{|B_{r_s}(x)|}{|B_{r_s}(x)|} = a_m \lambda.
\]

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and therefore, according to Lemma 1,
\[ 1 = \frac{|E_{m+k}(s)|}{|E_{m+k}(s)|} = a^{-1} \left( \frac{s}{r} \right)^{d} \]

or \( s \geq r \alpha \) so that if \( a^{\alpha} \) is sufficiently large, we will have \( s \geq br \), and therefore
\[ E_{m+k} = B_{m+k}(s) \supseteq B_{m+k}(s) = B_{m+k}(s) \]

and this clearly implies that \( E_{m+k} \supseteq E_{m} \).

There remains only to prove (5), which clearly implies that \( |\bigcap_{m} E_{m}| = 0 \). To prove this we shall show that
\[ |E_{m+k} \cap B_{m+k}(s)| \leq \delta \, |B_{m+k}(s)|, \]

where \( \delta < 1 \), provided \( a \) is sufficiently large. Once this is established from Lemma 5, it will follow that
\[ |E_{m+k} \cap B_{m+k}(s)| \leq \left( 1 - c_{m}^{-1} (1 - \delta_{k})^{p} \right) |B_{m+k}(s)|, \]

and, since the \( B_{m+k}(s) \) are disjoint,
\[ |E_{m+k} \cap \left( \bigcup_{m} B_{m+k}(s) \right) | \leq \left( 1 - c_{m}^{-1} (1 - \delta_{k})^{p} \right) |\bigcup_{m} B_{m+k}(s)|. \]

On the other hand, on account of Lemmas 1 and 4 we have
\[ |B_{m+k}(s)| \leq c_{m} |B_{m+k}(s)|, \]

whence it follows that
\[ |E_{m+k} \cap B_{m+k}(s)| \leq c_{m} |B_{m+k}(s)|. \]

Thus, from (8) and (9), since \( E_{m+k} \subseteq E_{m} \), we obtain
\[ |E_{m+k} \cap \left( \bigcup_{m} B_{m+k}(s) \right) | \leq |E_{m+k} \cap B_{m+k}(s)| + |E_{m+k} \cap B_{m+k}(s)| \]

\[ \leq \left( 1 - c_{m}^{-1} (1 - \delta_{k})^{p} \right) |\bigcup_{m} B_{m+k}(s)| + |E_{m+k} \cap B_{m+k}(s)| \]

\[ = |E_{m+k} \cap B_{m+k}(s)| = \left( 1 - c_{m}^{-1} (1 - \delta_{k})^{p} \right) |\bigcup_{m} B_{m+k}(s)| \]

\[ \leq |E_{m+k} \cap B_{m+k}(s)| < (1 - c_{m}^{-1} (1 - \delta_{k})^{p} |\bigcup_{m} B_{m+k}(s)|. \]

where
\[ \delta = 1 - c_{m}^{-1} (1 - \delta_{k}) \alpha^{-p}. \]

which is the inequality (5).

To prove (7) let us consider a sphere \( B_{m+k+1}(s) \). Let us take \( B_{m+k}(s) = B_{m+k}(s) \). If \( B_{m+k+1}(s) \) intersects \( B_{m+k}(s) \) but is not entirely contained in it, and \( \delta \in B_{m+k}(s) \cap B_{m+k+1}(s) \) then we have
\[ c(y, z) \leq 2r_{1}, \quad c(x, z) \leq \sigma + 2r_{1} = r_{2} \]

and therefore \( B_{m+k+1}(s) \subseteq B_{m+k}(s) \supseteq B_{m+k+1}(s) \), and
\[ \alpha^{-p} \Delta \leq \frac{|B_{m+k}(s)|}{|B_{m+k+1}(s)|} \geq \frac{|B_{m+k}(s)|}{|B_{m+k+1}(s)|} \geq \frac{|B_{m+k}(s)|}{|B_{m+k+1}(s)|} \geq \alpha^{-p} \Delta \]

Now this and Lemma 1 yield
\[ |B_{m+k}(s)| \geq a |B_{m+k+1}(s)| \geq a |B_{m+k+1}(s)| \geq \alpha^{-p} \Delta \frac{r_{1}}{2d_{m+k+1}} |B_{m+k+1}(s)|, \]

whence it follows that
\[ r_{1} \leq 2a^{-1} \nu^{-1} \alpha^{-p} \Delta \frac{r_{1}}{2d_{m+k+1}}. \]

Consequently, since \( 4a^{-1} \nu^{-1} \alpha^{-p} \Delta \frac{r_{1}}{2d_{m+k+1}} \leq 1/2 \), we will have
\[ r_{1} \leq 4a^{-1} \nu^{-1} \alpha^{-p} \Delta \frac{r_{1}}{2d_{m+k+1}}. \]

Thus, as is readily verified, if \( a \) is sufficiently large and \( B_{m+k+1}(s) \) contains points outside \( B_{m+k}(s) = B_{m}(s) \), then \( B_{m+k+1}(s) \) does not intersect \( B_{m+k+1}(s) \).

Next, let us consider the spheres \( B_{m+k+1}(s) \) which are entirely contained in \( B_{m+k}(s) \). For such spheres, we have
\[ \sum_{\Delta} |B_{m+k+1}(s)| \leq c_{\Delta} \sum_{\Delta} |B_{m+k+1}(s)| = c_{\Delta} |B_{m+k+1}(s)| \]

\[ \leq c_{\Delta} \alpha^{-p} \Delta^{-1} |B_{m+k+1}(s)| \leq c_{\Delta} \alpha^{-p} |B_{m+k+1}(s)| \]

so that if, again, \( a \) is sufficiently large, then
\[ \sum_{\Delta} |B_{m+k+1}(s)| \leq a |B_{m+k+1}(s)|, \]

where the sum is extended over all spheres \( B_{m+k+1}(s) \) entirely contained in \( B_{m+k}(s) \). Since the other spheres \( B_{m+k+1}(s) \) do not intersect \( B_{m+k+1}(s) \), we find that
\[ |E_{m+k} \cap B_{m+k}(s)| = \left| \bigcup_{\Delta} B_{m+k+1}(s) \cap B_{m+k}(s) \right| \]

\[ \leq |B_{m+k}(s)| - |B_{m+k+1}(s)| + |B_{m+k+1}(s)| \]

\[ \leq |B_{m+k}(s)| - \frac{1}{2} |B_{m+k+1}(s)| \leq |B_{m+k}(s)| - \frac{1}{2} |B_{m+k+1}(s)| \]

whence
\[ \left[ 1 - \left( 2a^{-1} \alpha^{-p} \Delta \frac{r_{1}}{2d_{m+k+1}} \right) \right] |B_{m+k}(s)| \]

Thus (7) holds with \( \delta = 1 - (2a^{-1} \alpha^{-p} \Delta \frac{r_{1}}{2d_{m+k+1}}) \) provided \( a \) is sufficiently large as compared with \( c \). This completes the proof of the theorem.

Proof of Theorem 2. If \( \nu \gg p \gg 1 \), then, as is readily verified, \( \omega^{-1} \nu^{-1} A_{\nu} \gg (p-1)(p-1) \gg 1 \), and therefore according to Theorem 1 we have
\[ \frac{1}{|B|} \left[ \frac{1}{|B|} \int_{|B|=r_{1}} \omega^{-1} \nu^{-1} \frac{d\mu}{d\nu} \right] \leq c_{\omega} \left[ \frac{1}{|B|} \int_{|B|=r_{1}} \omega^{-1} \nu^{-1} \frac{d\mu}{d\nu} \right] \]
for some \( r < p \) and \( c_1 < \infty \), and by Hölder's inequality,

\[
\left[ \frac{1}{|B|} \int w^{-\lambda(r-1)} \, dx \right]^{-1} \leq \left[ \frac{1}{|B|} \int w^{-\lambda(p-1)} \, dx \right]^{-1} \]

for all \( r, \lambda \geq p \). Thus substituting in the first inequality in (2) we find that \( w \in A_r \) for some \( r < p \), and all \( r \geq p \). Thus the interval of values of \( r \) for which \( w \in A_r \) is open half-line containing \( p \), as we wished to show.

**Proof of Theorem 3.** If \( w \in A_p, p > 1 \), then \( w \in A_r \) for some \( r < p \), and according to Lemma 6, the maximal operator \( M : f \rightarrow (Mf) \) is of weak type \( (r, r) \) with respect to the measure \( ds = w^r \, dx \). Since \( M \) is obviously also of type \( (\infty, \infty) \), by the Marcinkiewicz interpolation theorem, it follows that \( M \) is of strong type \( (p, p) \) with respect to the measure \( r \), which is the desired result. Since the case \( p = 1 \) is covered by Lemma 6, this establishes Theorem 3.

**References**

3. N. S. Kurtz, *Weighted norm inequalities for the Hardy–Littlewood maximal function for one parameter rectangles*.

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**Methods of Hilbert Spaces**

**Krystztof Maurin**

**Monografie Matematyczne**, Vol. 45

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