5. The product $\delta^{(r)}(x)\,\delta^{(p)}(x)$. When $r=0$ in equation (5.1) we have

$$H(x)\,\delta^{(p)}(x) = \tfrac{1}{2}\,\delta^{(p)}(x)$$

for $p = 0, 1, 2, \ldots$ On using the theorem it follows that

$$\delta(x)\,\delta^{(p)}(x) = \tfrac{1}{2}\,\delta^{(p+1)}(x) - H(x)\,\delta^{(p+1)}(x) = 0$$

for $p = 0, 1, 2, \ldots$

We will now assume that

$$\delta^{(r)}(x)\,\delta^{(p)}(x) = 0$$

for some r and $p = 0, 1, 2, \ldots$ Then using the theorem we have

$$\delta^{(r+1)}(x)\,\delta^{(p)}(x) = 0 - \delta^{(r)}(x)\,\delta^{(p+1)}(x) = 0$$

for $p = 0, 1, 2, \ldots$ It follows by induction that

$$\delta^{(r)}(x)\,\delta^{(p)}(x) = 0$$

for $r, p = 0, 1, 2, \ldots$

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An inequality for integrals

by

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Abstract. An inequality for n-fold integrals of products of functions of less than n variables is obtained and applied to obtain a Sobolev type inequaliy.

Consider the following identity

$$\int_{R^n}\Big[\prod_{j=1}^{n} f_j(x_j)\Big]dx = \prod_{j=1}^{n}\Big[\int_{-\infty}^{+\infty} f_j(x_j)\,dx_j\Big], \tag{1}$$

where R^n is the n-dimensional Euclidean space and $dx = dx_1 dx_2 \ldots dx_n$. This identity can be generalized to an inequality for integrals of products of functions of less than n variables. For example, if $f_{ij}(x_i, x_j) \geqslant 0$ then

$$\int_{R^3} f_{12}(x_1, x_2) f_{13}(x_1, x_3) f_{23}(x_2, x_3)\,dx_1 dx_2 dx_3$$
$$\leqslant \Big[\int_{R^2} f_{12}^2(x_1, x_2)\,dx_1 dx_2\Big]^{1/2}\Big[\int_{R^2} f_{13}^2(x_1, x_3)\,dx_1 dx_3\Big]^{1/2}\Big[\int_{R^2} f_{23}^2(x_2, x_3)\,dx_2 dx_3\Big]^{1/2}.$$

In order to describe the general result of which this is a special case, consider subsets ω of the set of indices $\{1, 2, \ldots, n\}$ and denote by $|\omega|$ the number of their elements. Let x_ω denote the set of variables $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$, where $\{i_1, i_2, \ldots, i_k\} = \omega$, and let f_ω denote a function depending only on x_ω. Then the inequality

$$\int_{R^n}\Big[\prod_{|\omega|=k} f_\omega(x_\omega)\Big]dx \leqslant \prod_{|\omega|=k}\Big[\int_{R^k} f_\omega^r(x_\omega)\,dx_\omega\Big]^{1/r} \tag{2}$$

holds, where r is the binomial coefficient $\binom{n-1}{k-1}$ and the products extend over all subsets ω of $\{1, 2, \ldots, n\}$ with $|\omega| = k$.

For $k = 1$, (2) is actually an equality, namely (1), and for $k = n$ the two sides of (2) become the same. Thus in order to prove our assertion we may assume that $2 \leqslant k < n$, and argue by induction on n.

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Suppose that (2) holds for $n-1$ and $k<n-1$. The sets ω can be grouped into two classes S_1 and S_2, S_1 consisting of those ω which do not contain the index n and S_2 of those which do. For $\omega\in S_2$ let $\varpi=\omega-\{n\}$, that is, ϖ is obtained from ω by removing the index n from it. Then we have

$$(3)\quad \int_{R^n}\Big[\prod_{|\omega|=k} f_\omega(x_\omega)\Big]dx = \int_{R^n}\Big[\prod_{\omega\in S_1} f_\omega(x_\omega)\Big]\Big[\prod_{\omega\in S_2} f_\omega(x_\omega)\Big]dx$$

$$= \int_{-\infty}^{+\infty}\Big[\int_{R^{n-1}}\Big[\prod_{\omega\in S_1} f_\omega(x_\omega)\Big]\Big[\prod_{\omega\in S_2} f_\omega(x_\omega)\Big]dx_1\ldots dx_{n-1}\Big]dx_n$$

$$\leqslant \Big[\int_{R^{n-1}}\Big[\prod_{\omega\in S_1} f_\omega(x_\omega)\Big]^p dx_1\ldots dx_{n-1}\Big]^{1/p}\int_{-\infty}^{+\infty}\Big[\prod_{\omega\in S_2}\int_{R^{n-1}} f_\omega(x_\omega)\Big]^q dx_1\ldots dx_{n-1}\Big]^{1/q}dx_n,$$

where $p=\dfrac{n-1}{n-k}$ and $q=\dfrac{n-1}{k-1}$. Now, by our inductive hypothesis we have

$$\Big[\int_{R^{n-1}}\Big[\prod_{\omega\in S_1} f_\omega(x_\omega)\Big]^p dx_1\ldots dx_{n-1}\Big]^{1/p}\leqslant\prod_{\omega\in S_1}\Big[\int_{R^k} f_\omega^{pr_1}(x_\omega)\,dx_\omega\Big]^{1/pr_1},$$

where $r_1=\binom{n-2}{k-1}$. Since $p=\dfrac{n-1}{n-k}$, we have $pr_1=\binom{n-1}{k-1}=r$ and the inequality above becomes

$$(4)\quad \Big[\int_{R^{n-1}}\Big[\prod_{\omega\in S_1} f_\omega(x_\omega)\Big]^p dx_1\ldots dx_{n-1}\Big]^{1/p}\leqslant\prod_{\omega\in S_1}\Big[\int_{R^k} f_\omega^{r}(x_\omega)\,dx_\omega\Big]^{1/r}.$$

On the other hand,

$$\Big[\int_{R^{n-1}}\Big[\prod_{\omega\in S_2} f_\omega(x_\omega)\Big]^q dx_1\ldots dx_{n-1}\Big]^{1/q}\leqslant\prod_{\omega\in S_2}\Big[\int_{R^{k-1}} f_\omega^{qr_2}(x_\omega)\,dx_\omega\Big]^{1/qr_2},$$

where now $r_2=\binom{n-2}{k-2}$, since for $\omega\in S_2$ and x_n fixed, $f_\omega(x_\omega)$ depends on $k-1$ variables. But as is readily verified, $qr_2=r$, and since there are precisely $r=\binom{n-1}{k-1}$ factors on the right above, integrating with respect to x_n, and using the inequality

$$\int_{-\infty}^{+\infty} g_1g_2\ldots g_r\,dx_n\leqslant\prod_1^r\Big[\int_{-\infty}^{+\infty} g_j^r\,dx_n\Big]^{1/r},$$

we obtain

$$(5)\quad \int_{-\infty}^{+\infty}\Big[\int_{R^{n-1}}\Big[\prod_{\omega\in S_2} f_\omega(x_\omega)\Big]^q dx_1\ldots dx_{n-1}\Big]^{1/q}dx_n\leqslant\prod_{\omega\in S_2}\Big[\int_{R^k} f_\omega^{r}(x_\omega)\,dx_{\varpi}\,dx_n\Big]^{1/r}.$$

But for $\omega\in S_2$ we have $dx_{\varpi}dx_n=dx_\omega$, so that substituting (4) and (5) in the last expression in (3), inequality (2) follows.

As an application of (2) we shall give a simple proof of a result of L. Nirenberg [1].

Let $F(x)$, $x\in R^n$, be a function with continuous integrable derivatives and such that $F(x)\to 0$ as $|x|\to\infty$. Then $F\in L^{n/n-1}$ and

$$(6)\quad \|F\|_{n/n-1}\leqslant\prod_1^n\Big\|\frac{\partial F}{\partial x_j}\Big\|_1^{1/n}.$$

To show this let

$$f_j(x)=\int_{-\infty}^{+\infty}\Big|\frac{\partial F}{\partial x_j}(x_1,x_2,\ldots,x_j+t,\ldots,x_n)\Big|dt.$$

Then, clearly, $f_j(x)$ does not depend on x_j and $|F(x)|\leqslant f_j(x)$ and therefore

$$|F(x)|^{n/n-1}\leqslant\prod_1^n f_j(x)^{1/n-1};$$

thus, integrating using (2) with $|\omega|=n-1$ and observing that

$$\int_{R^{n-1}} f_j(x)\,dx_1\ldots dx_{j-1}dx_{j+1}\ldots dx_n=\int_{R^n}\Big|\frac{\partial F}{\partial x_j}\Big|dx=\Big\|\frac{\partial F}{\partial x_j}\Big\|_1,$$

we obtain (6).

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