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UNIVERSITY OF SOUTH CAROLINA
DEPARTMENT OF MATHEMATICS
and
COMPUTER SCIENCE
COLUMBIA, SOUTH CAROLINA

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On power series in the operators s^a *

by

M. SKENDŽIĆ and B. STANKOVIĆ (Novi Sad, Yugoslavia)

Abstract. Necessary and sufficient conditions for convergence, in the field of Mikusiński operators, of the series $S = \sum_{n=0}^{\infty} \gamma_n s^{an}$ and the uniqueness of this representation are given. Here a is real positive, γ_n are complex and s is the differentiation operator.

This extends a result of T. K. Boehme when $a = 1$. It is also shown that J. Wloka's sufficient condition for convergence is also a necessary one.

1. Introduction. In the field of Mikusiński operators M the convergence class is defined. But nobody has investigated the conditions for convergence or divergence of series in operators in general case.

The special class of power series in the operator s^a

$$\sum_{n=0}^{\infty} \gamma_n s^{an} \lambda^n, \tag{1.1}$$

where γ_n and λ are complex numbers, a real and positive, s the differentiation operator, has an important role in the operational calculus and its applications.

In the case $a = 1$ we know one sufficient condition for the convergence of the series (1.1) and one for its divergence [4]. We know also generalization of these results to the case $a > 0$ [6]. J. Wloka [8] found a sufficient condition for the convergence of the series (1.1), in case $a = 1$, too. In the mentioned paper he asked the question: "Is this condition also a necessary condition?". Recently, T. K. Boehme [1] gave a necessary and sufficient condition for the convergence of the series (1.1) in the case $a = 1$. Our aim is to enlarge the result of Boehme to the case $a > 0$. We prove two propositions, both containing sufficient and necessary conditions for the convergence of the series (1.1). We give also the answer to the question of J. Wloka and a proposition about the uniqueness of the development of an element of M in a series of the form (1.1) for a fixed $a > 0$.

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2. Notations and the known results which we use. The field M is the quotient field for the ring C of continuous functions with the addition and composition as inner operations. The operator which corresponds to the function $f(t) \in C$ will be denoted by f or $\{f(t)\}$. We shall also use the notation $|f|$ for $\{|f(t)|\}$, l for integration operator and s for differentiation operator. In this paper the convergence is in the sense of the first definition of convergence classes [3].

The convergence of the series (1.1) in M can be related to the not quasi-analytic class of functions. For this reason we shall give the definition and some properties of this class which we shall use.

Let $\{(M_n)|\ n = 0, 1, 2 \ldots\}$ be a sequence of positive real numbers. By $C_I\{M_n\}$ we mean the class of all infinitely differentiable functions $f(t)$ such that there are constants $\beta_f > 0$ and B_f depending on $f(t)$ and

$$\max_{t\in I} |f^{(n)}(t)| \leqslant \beta_f B_f^n M_n, \quad \text{for each } n = 0, 1, 2, \ldots, \tag{2.2}$$

$$I = [a, b] \subset [0, \infty).$$

We shall allow M_n to be infinite so long as infinitely many M_n are finite. We always suppose:

$$M_0 = 1, \quad 0 < M_{n+1} \leqslant \infty \quad \text{for each } n = 0, 1, 2, \ldots; \tag{2.3}$$
$$M_n < \infty \quad \text{for infinitely many } n.$$

DEFINITION 2.1. A sequence (M_n) is said to be *logarithmically convex* if

$$M_n^2 \leqslant M_{n-1}M_{n+1} \quad \text{for each } n = 1, 2, \ldots$$

DEFINITION 2.2. $C_I\{M_n\}$ is said to be *quasi-analytic* if $f(t) \in C_I\{M_n\}$, $t_0 \in I$ and

$$f^{(n)}(t_0) = 0 \quad \text{for each } n = 0, 1, 2, \ldots$$

implies $f(t) \equiv 0$ on I.

We are particularly interested in those $C_I\{M_n\}$ which are not quasi-analytic.

PROPOSITION A ([1], p. 312). *Suppose $C_I\{M_n\}$ is not quasi-analytic. Then there exists a logarithmically convex sequence $(\bar{M}_n)$ such that $C_I\{\bar{M}_n\} \subset C_I\{M_n\}$, $C_I\{\bar{M}_n\}$ is not quasi-analytic, and for every $B > 0$*

$$\sum_{n=0}^{\infty} \frac{B^n \bar{M}_n}{M_n} < \infty. \tag{2.4}$$

PROPOSITION B ([1], p. 314). *Suppose $C_I\{M_n\}$ is not quasi-analytic. If $I' \subset I = [a, b]$, there is a nontrivial function $f(t) \in C_I\{M_n\}$ with support in I'.*

PROPOSITION C ([2], Carleman). *Let*

$$\mu_n = (M_n)^{1/n} \quad \text{for each } n = 1, 2, \ldots$$

and

$$\mu_n^* = \min_{k\geqslant 0} \mu_{n+k} \quad \text{for each } n = 1, 2, \ldots$$

Then $C_I\{M_n\}$ is not quasi-analytic if and only if $\sum_{n=1}^{\infty} \frac{1}{\mu_n^*} < \infty$.

PROPOSITION D ([5], p. 376). *Suppose $M_0 = 1$, $M_n^2 < M_{n-1}M_{n+1}$ for $n = 1, 2, \ldots$; then $C_I\{M_n\}$ is not quasi-analytic if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{1}{M_n}\right)^{1/n} < \infty \tag{2.5}$$

or

$$\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} < \infty. \tag{2.6}$$

PROPOSITION E ([7], p 21). *Suppose that for each $i_0 \in N$ there exists $i \geqslant i_0$ such that $\gamma_i \neq 0$. Then*

$$P_n = \sum_{i=0}^{n} \gamma_i \lambda^i s^{ia} f \in C \quad \text{for each } n = 0, 1, 2, \ldots$$

if and only if

$$f^{(i)}(0) = 0 \quad \text{for each } i = 0, 1, 2, \ldots$$

PROPOSITION F ([7], p. 21). *If the series (1.1) is convergent in M for one $\lambda = \lambda_0 \neq 0$, it is convergent for every complex number λ.*

3. The convergence of the series (1.1). Proposition F allows us to analyse the series

$$\sum_{n=0}^{\infty} \gamma_n s^{an} \tag{3.1}$$

instead of the series (1.1) without loss of generality.

First we shall construct a sequence of complex numbers $(a_n(a))$ in order to prove propositions on the convergence of the series (3.1).

For $0 < a \leqslant 1$

$$a_n(a) = \max_{[pa]=n} |\gamma_p|, \tag{3.2}$$

for $a \geqslant 1$

$$a_n(a) = \begin{cases} |\gamma_p| & \text{if there exists } p \in N \text{ such that } n = [ap], \\ 0 & \text{if such } p \text{ does not exist} \end{cases}$$

($[y]$ denotes the biggest integer $\leqslant y$).

PROPOSITION 1. *The series* (3.1) *is convergent in* M *if and only if* $C_I\left\{\frac{1}{a_n(a)}\right\}$ *is not quasi-analytic class.*

Proof. The condition is necessary. Let the series (3.1) be convergent. This means that a function $f(t) \in C$ exists and $f(t) \not\equiv 0$. By Proposition E, $f(t)$ has all derivatives and $f^{(i)}(0) = 0$, $i = 0, 1, \ldots$ Let Ω_f be the support of the function $f(t)$. We can suppose that $\Omega_f \cap I \neq \emptyset$. If it is not so, one may use a translation which does not change the property of differentiability. The same properties has the function $g(t)$, $g = lf$, and

$$s^{[ap]}lf = l^{pa-[pa]+1}s^{pa}f = \left\{\frac{t^{pa-[pa]}}{\Gamma(pa-[pa]+1)}\right\}s^{pa}f. \tag{3.3}$$

From the convergence of the series (3.1) follows the existence of a constant K such that:

$$\sup_{t\in[0,T]} |\gamma_p s^{pa} f| < K, \quad p = 0, 1, 2, \ldots \tag{3.4}$$

Using (3.3) and (3.4), we have

$$\sup_{t\in I} |s^{[pa]}lf| \leqslant \frac{Kb^{pa-[pa]+1}}{|\gamma_p|\,\Gamma(pa-[pa]+2)} < \frac{K^1}{|\gamma_p|}; \quad p = 0, 1, 2, \ldots; \tag{3.5}$$

K^1 is a constant.

For such n for which there exists an integer p so that $n = [pa]$ we have

$$\sup_{t\in I} |g^{(n)}(t)| \leqslant \frac{K^1}{|\gamma_p|} = \frac{K^1}{a_n(a)} \quad \text{for} \quad a \geqslant 1, \tag{3.6}$$

and

$$\sup_{t\in I} |g^{(n)}(t)| \leqslant \frac{K^1}{\operatorname*{Max}_{[pa]=n} |\gamma_p|} = \frac{K^1}{a_n(a)} \quad \text{for} \quad 0 < a < 1. \tag{3.7}$$

From (3.6) and (3.7) it follows that $g(t) \in C_I\left\{\frac{1}{a_n(a)}\right\}$ and that $C_I\left\{\frac{1}{a_n(a)}\right\}$ is not quasi-analytic class.

The condition is sufficient. Let the class $C_I\left\{\frac{1}{a_n(a)}\right\}$ be not quasi-analytic class. By Proposition B there exists $f(t) \not\equiv 0$ from $C_I\left\{\frac{1}{a_n(a)}\right\}$ with a nonempty support in $I' = [a', b'] \subset I$ for which is valid the inequality

$$\operatorname*{Max}_{t\in I'} |f^{(n)}(t)| \leqslant \beta_f B_f^n \frac{1}{a_n(a)}, \quad n = 0, 1, 2, \ldots \tag{3.8}$$

Let us construct the function $F(t) = f(t/2B_f)$. The support of this function is in the interval $I'' = [2B_f a', 2B_f b']$.

From the inequality

$$\sup_{t\in I''} |F^{(n)}(t)| = \sup_{t\in I'} \left|f^{(n)}\left(\frac{t}{2B_f}\right)\left(\frac{1}{2B_f}\right)^n\right| < \beta_f \frac{1}{2^n a_n(a)} \tag{3.9}$$

for $n = 0, 1, 2 \ldots$ follows $F(t) \in C_{I''}\left\{\frac{1}{a_n(a)}\right\}$, and from (3.9)

$$\begin{aligned} \sup_{t\in[0,T]} |\gamma_n s^{na} l^2 F| &= \sup_{t\in[0,T]} |\gamma_n l^{[na]-na+2} F^{([na])}| \\ &\leqslant \frac{C|\gamma_n|\beta_f}{2^{[na]} a_{[na]}(a)} \leqslant 2C\left(\frac{1}{2^a}\right)^n \beta_f. \end{aligned} \tag{3.10}$$

This inequality shows that the series (3.1) converges in M.

COROLLARY 1. *If*

$$|C_n| \leqslant |\gamma_n h^n|, \quad n = 0, 1, 2, \ldots, \tag{3.11}$$

where h is an arbitrary complex number, and if the series (3.1) *converges in* M, *the series* $\sum_{n=0}^{\infty} C_n s^{an}$ *converges in* M, *too.*

Proof. Let $(C_n(a))$ and $(d_n(a))$ be sequences constructed by using the sequences (C_n) and $(\gamma_n h^n)$, respectively, as it is done in (3.2). From the convergence of the series (3.1) follows that the class $C_I\left\{\frac{1}{d_n(a)}\right\}$ is not quasi-analytic and from (3.11) it follows that the class $C_I\left\{\frac{1}{C_n(a)}\right\}$ is also not quasi-analytic. Proposition 1 says that the series $\sum_{n=0}^{\infty} C_n s^{an}$ is convergent.

COROLLARY 2. *If*

$$|C_n| \geqslant |\gamma_n h^n|, \quad n = 0, 1, 2, \ldots, \tag{3.12}$$

where h is an arbitrary complex number and if the series (3.1) *is divergent in* M, *the series* $\sum_{n=0}^{\infty} C_n s^{an}$ *is divergent in* M, *too.*

PROPOSITION 2. *The series* (3.1) *is for* $a > 0$ *convergent in* M *if and only if there exists a sequence* (b_n) *of positive real numbers such that*:

1. $|\gamma_n| \leqslant b_n, \quad n \geqslant n_0,$
2. $(b_n)^{\frac{1}{[an]}} \geqslant (b_{n+1})^{\frac{1}{[a(n+1)]}}$ *for* $[an] \geqslant 1$, $n = 0, 1, 2, \ldots,$
3. $\sum_{n=n_1}^{\infty} (b_n)^{\frac{1}{[an]}} < \infty \quad ([an_1] \neq 0).$

Proof. The proof will be devided into two parts, for $0 < a < 1$ and for $a \geqslant 1$.

The condition is sufficient. Without loss of generality we can take that the supposition 1 is valid for $n = 0, 1, \ldots$ Let us suppose first that $0 < a < 1$. We shall use the following sequences

$$b_n^* = \operatorname*{Max}_{[ap]=n} b_p, \quad p = 0, 1, 2, \ldots, \quad n = 0, 1, 2, \ldots;$$

$$\mu_n = (b_n^*)^{-1/n}, \; n = 1, 2, 3, \ldots; \quad \mu_n^* = \operatorname*{Min}_{k\geqslant 0} \mu_{n+k}, \; n = 1, 2, 3, \ldots$$

From the condition (2) it follows that (μ_n) is a non-decreasing sequence and $\mu_n^* = \mu_n$, $n = 1, 2, \ldots$

Now we have

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^*} = \sum_{n=1}^{\infty} \frac{1}{\mu_n} = \sum_{n=1}^{\infty} (b_n^*)^{1/n} < \infty, \tag{3.13}$$

because $\sum_{n=1}^{k} (b_n^*)^{1/n} < \sum_{p=p_0}^{k_1} (b_p)^{\frac{1}{[ap]}} < A$ (A is a constant which exists by supposition 3; k_1 is the biggest integer for which $[k_1 a] = k$ and p_0 is the smallest p such that $[p_0 a] = 1$.

From (3.13) and Proposition C it follows that the class $C_I\left\{\frac{1}{b_n^*}\right\}$ is not quasi-analytic. Let $(a_n(a))$ be the sequence defined in (3.2). By supposition 1 we have

$$C_I\left\{\frac{1}{b_n^*}\right\} \subset C_I\left\{\frac{1}{a_n(a)}\right\}$$

and the class $C_I\left\{\frac{1}{a_n(a)}\right\}$ is not quasi-analytic, too. Proposition 1 says that the series (3.1) is convergent in M for $0 < a < 1$.

The case $a \geqslant 1$. We shall construct the following sequences:

$$B_n = \begin{cases} b_p & \text{for} \quad n = [ap], \; p = 0, 1, 2, \ldots, \\ 0 & \text{for} \quad n \neq [ap], \; p = 0, 1, 2, \ldots, \end{cases}$$

$n = 0, 1, 2, \ldots;$

$$G_n = (B_n)^{-1/n}, \quad G_n^* = \operatorname*{Min}_{k\geqslant 0} G_{n+k}; \quad n = 1, 2, \ldots$$

If we separate from the sequence (G_n) those elements which are not limited, we obtain a non-decreasing sequence and

$$G_n^* = \begin{cases} (b_p)^{-1/[ap]} & \text{for} \quad n = [ap], \; p = 1, 2, 3, \ldots, \\ (b_{p+1})^{-1/[a(p+1)]} & \text{for} \quad [ap] < n \leqslant [a(p+1)], \; p = 0, 1, 2, \ldots \end{cases}$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{G_n^*} < \infty \tag{3.14}$$

because

$$\sum_{n=1}^{k} \frac{1}{G_n^*} < ([a]+1) \sum_{p=1}^{k_1} (b_p)^{1/[ap]} < A,$$

(A is a constant which exists by supposition 3; k_1 satisfies the condition $[k_1 a] \geqslant k$). Proposition C says that the class $C_I\left\{\frac{1}{B_n}\right\}$ is not quasi-analytic. From supposition 1 it follows that

$$C_I\left\{\frac{1}{B_n}\right\} \subset C_I\left\{\frac{1}{a_n(a)}\right\}$$

and $C_I\left\{\frac{1}{a_n(a)}\right\}$ is also not quasi-analytic. By Proposition 1 the series (3.1) is convergent in M for $a \geqslant 1$.

The condition is necessary. Let the series (3.1) be convergent in M. Then the class $C_I\left\{\frac{1}{a_n(a)}\right\}$ is not quasi-analytic. Now, there exists a logarithmically convex sequence $(\bar{M}_n)$ (Proposition A) such that $C_I\{\bar{M}_n\} \subset C_I\left\{\frac{1}{a_n(a)}\right\}$ and $C_I\{\bar{M}_n\}$ is not a quasi-analytic class. Without loss of generality we can take $\bar{M}_0 = 1$. From $\bar{M}_0 = 1$ and the fact that $(\bar{M}_n)$ is a logarithmically convex sequence it follows that

$$(\bar{M}_n)^{1/n} \leqslant (\bar{M}_{n+1})^{1/(n+1)} \quad \text{for} \quad n > 0. \tag{3.15}$$

From the fact that $C_I\{\bar{M}_n\}$ is not quasi-analytic class and from Proposition D we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{\bar{M}_n}\right)^{1/n} < \infty \tag{3.16}$$

and the relation (2.4) implies that for every $B > 0$ there exists one constant k such that

$$B^n \bar{M}_n a_n(a) \leqslant k,$$

that is,

$$a_n(\alpha) \leqslant \frac{1}{\bar{M}_n} \quad \text{for } n \geqslant n_0. \tag{3.17}$$

Let $0 < \alpha < 1$. We can construct the sequence $b_p = 1/\bar{M}_n$ for $[p\alpha] = n$, $p = 0, 1, 2, \ldots$; then from (3.15) it follows that

$$(b_p)^{1/[\alpha p]} \geqslant (b_{p+1})^{1/[\alpha(p+1)]} \quad \text{for} \quad [p\alpha] > 0 \tag{3.18}$$

and from (3.16)

$$\sum_{p=p_1}^{\infty} (b_p)^{1/[p\alpha]} < \infty \tag{3.19}$$

p_1 is the smallest p such that $[p_1\alpha] = 1$; because

$$\sum_{p=1}^{k} (b_p)^{1/[\alpha p]} \leqslant \left(\left[\frac{1}{\alpha}\right]+1\right) \sum_{n=1}^{k_1} \left(\frac{1}{\bar{M}_n}\right)^{1/n}, \qquad k_1 = [\alpha k].$$

Since $a_n(\alpha) = \operatorname*{Max}_{[p\alpha]=n} |\gamma_n|$, we have $|\gamma_p| < 1/\bar{M}_n$ (see (3.17)) for $[p\alpha] = n$ and $n \geqslant n_0$, that is,

$$|\gamma_p| \leqslant b_p \quad \text{for} \quad p \geqslant p_1, \quad \text{where} \quad [p_1\alpha] = n_0. \tag{3.20}$$

The constructed sequence (b_p), on account of (3.18), (3.19) and (3.20) satisfies suppositions 1, 2, and 3 of Proposition 2 for $0 < \alpha < 1$.

We shall suppose now that $\alpha \geqslant 1$. Let us construct the sequence (b_p) in such a manner that $b_p = 1/\bar{M}_n$ for $[p\alpha] = n$, $p = 0, 1, 2, \ldots$

On account of (3.15), the sequence (b_p) satisfies the condition

$$(b_p)^{1/[\alpha p]} \geqslant (b_{p+1})^{1/[\alpha(p+1)]} \tag{3.21}$$

and by (3.16)

$$\sum_{p=1}^{\infty} (b_p)^{1/[\alpha p]} < \infty, \tag{3.22}$$

because

$$\sum_{p=1}^{k} (b_p)^{1/[\alpha p]} < \sum_{n=1}^{k_1} (1/\bar{M}_n)^{1/n}, \qquad k_1 = [\alpha k].$$

Since $|\gamma_p| = a_n(\alpha)$ for $n = [p\alpha]$, $p = 0, 1, 2, \ldots$, we have from (3.17)

$$|\gamma_p| \leqslant \frac{1}{\bar{M}_n}, \quad \frac{1}{\bar{M}_n} = b_p \quad \text{for} \quad [p\alpha] \geqslant n_0. \tag{3.23}$$

On account of (3.21), (3.22) and (3.23), the sequence (b_p) satisfies the suppositions of Proposition 2 for $\alpha \geqslant 1$.

Remark. When the sequence (b_n) from Proposition 2 is selected in a special manner we have sufficient condition of Ryll-Nardzewski ($b_n = n^{-cp}$, $c > 1$) [4]; of B. Stanković [6] ($b_n = n^{-\delta/\alpha[\alpha n]}$, $\delta > \alpha$); of J. Wloka [8] ((b_n) is a logarithmically convex sequence which defines not quasi-analytic class).

A new sufficient condition could be obtained by using the sequence $\left(b_n = \{n \ln^c(n+e)\}^{-[\alpha n]}\right)$, $c > 1$, which allows us to use a bigger class of convergent series (3.1) because

$$n^{-\delta/\alpha[\alpha n]} < \{n \ln^c(n+e)\}^{-[\alpha n]}, \quad \delta > \alpha, \ c > 1.$$

Sufficient conditions for divergence of the series (3.1) given by Ryll-Nardzewski [4] and B. Stanković [6] show that in this case $|\gamma_n|$ are of such a form that no sequence (b_n) from Proposition 2 can have property 3, because the minorant sequences used by the mentioned authors do not satisfy property 3.

The answer to the question of J. Wloka is given by the following proposition.

PROPOSITION 3. *The sufficient and necessary condition that the series (3.1) be convergent in $\mathfrak{M}$ for $\alpha = 1$ is*

$$|\gamma_n| = O\left(\frac{1}{h^n M_n}\right), \quad n \to \infty, \tag{3.24}$$

where $h > 0$ is arbitrary, $M_0 = 1$, and M_n is logarithmically convex sequence so that $\sum_{n=1}^{\infty} (1/M_n)^{1/n} < \infty$.

Proof. The condition is necessary. From the convergence of the series (3.1) it follows that the class $C_I\{1/|\gamma_n|\}$ is not quasi-analytic (Proposition 1). Proposition A asserts that there exists a logarithmically convex sequence (M_n) such that $C_I\{M_n\}$ is not a quasi-analytic class. By Proposition D we have: $\sum_{n=1}^{\infty} (1/M_n)^{1/n} < \infty$. From condition (2.4) and Proposition A follows: $h^n M_n |\gamma_n| < k$, $h > 0$ but arbitrary; this inequality shows that (3.24) is valid.

The condition is sufficient. From $M_0 = 1$, $M_n^2 \leqslant M_{n-1} M_{n+1}$ and $\sum_{n=1}^{\infty} (1/M_n)^{1/n}$ follows that $C_I\{M_n\}$ is not a quasi-analytic class (Proposition D). Relation (3.24) gives $k/|\gamma_n| h^n > M_n$ and the class $C_I\{k/|\gamma_n| h^n\}$ is not quasi-analytic too. Thus the series (3.1) is convergent in $\mathfrak{M}$ (Proposition 1 and F).

Consequently, the condition of J. Wloka [8] is sufficient and necessary for the convergence of the series (3.1) if one omits the condition for the sequence $(M_{n+1})^{1/n} = O\left((M_n)^{1/n}\right)$ for which J. Wloka himself says that it is introduced only to make easier the proof.

4. The uniqueness of the representation of one operator by the series of the form (1.1). The uniqueness follows by

PROPOSITION 4. *An element of the field M can be represented by only one series of the form* (1.1) *for a fixed $a > 0$.*

To prove Proposition 4 we shall use the following lemma.

LEMMA. *If f is a convergence factor of the series* (3.1), *then f is also a convergence factor of the series*

$$S(\lambda) = \sum_{n\geqslant 0} \gamma_n \lambda^n s^{an} \tag{4.1}$$

for every $|\lambda| < 1$. If $S(1) = 0$, then $S(\lambda) = 0$ for every $0 < \lambda \leqslant 1$.

Proof of the lemma. Let $M_T = \operatorname*{Max}_{0\leqslant t\leqslant T} |\gamma_n s^{n\alpha} f|$; M_T is bounded for every $T < \infty$, because f is the convergence factor of (3.1). We have

$$\operatorname*{Max}_{0\leqslant t\leqslant T}\Big|\sum_{n=m+1}^{m+p} \gamma_n \lambda^n s^{n\alpha} f\Big| < M_T \sum_{n=m+1}^{m+p} |\lambda|^n < M_T |\lambda|^{m+1} \to 0, \quad \text{as} \quad m\to\infty,$$

and the series $S(\lambda)$ is convergent for every $|\lambda| < 1$.

Now let p be an integer greater than α; then

$$\sum_{n\geqslant 0} \gamma_n \lambda^n s^{n\alpha} f = \sum_{n\geqslant 0} \gamma_n \lambda^n \Big\{\int_0^t \frac{(t-u)^{n(p-\alpha)}}{\Gamma(np-n\alpha+1)} f^{(np+1)}(u)\,du\Big\}; \tag{4.2}$$

after the substitution $t = xw$, $w = \lambda^{1/\alpha}$, $0 < \lambda < 1$ in (4.2) we have

$$\begin{aligned}\sum_{n\geqslant 0} \gamma_n \lambda^n s^{n\alpha} f &= \sum_{n\geqslant 0} \gamma_n \lambda^n \Big\{\int_0^{xw} \frac{(xw-u)^{n(p-\alpha)}}{\Gamma(np-n\alpha+1)} f^{(np+1)}(u)\,du\Big\} \\ &= \sum_{n>0} \gamma_n s^{n\alpha} F = 0,\end{aligned} \tag{4.3}$$

where $F(t) = f(t\lambda^{1/\alpha})$.

Proof of Proposition 4. We shall prove that $S(1) = 0$ implies $\gamma_n = 0$, $n \in N$. From our lemma we have that $S(\lambda)f = \{g(\lambda, t)\}$, where $g(\lambda, t)$ is a regular function in λ in the disc $|\lambda| < 1$ for every $t \geqslant 0$. But this function equals zero for $0 < \lambda \leqslant 1$, $t \geqslant 0$. This is possible only if $\gamma_n = 0$, $n \in N$.

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