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On an infinite linear combination of partial sums of Fourier series

by

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Abstract. Given a Fourier series, we consider an infinite linear combination of its partial sums which converges almost everywhere to an integrable function.

Introduction. For $f \in L$, let $\sum_{n=0}^{\infty} A_n(f, x)$ be its Fourier series, where $A_n(f, x) = a_n(f)\cos nx + b_n(f)\sin nx$. We write

$$S_n(f, x) = \sum_{j=0}^{n} A_j(f, x).$$

In [3] it was shown that for $f \in L \log L$ and φ of bounded variation,

$$\lim_{n\to\infty} \sum_{j=0}^{n} a_{2j+1}(\varphi) S_j(f, x)$$

exists for a.e. x. The limit function is integrable and its Fourier series can be obtained via certain multipliers. The purpose of this paper is to extend the results in [3] and answer the questions raised therein.

In the first section we show that if $f, \tilde{f} \in L$, then for a.e. x,

$$\sum_{n=1}^{\infty} [S_n(f, x) - f(x)] A_n(\varphi, t)$$

converges uniformly in t, for all φ of bounded variation. We also show that $f_\varphi(x) \equiv \sum_{j=1}^{\infty} a_j(\varphi) S_j(f, x)$ is in L^p whenever $f \in L^p$ $(1 < p \leqslant \infty)$.

In the second section we show that for $f \in L$ and φ of bounded variation $\sum_{n=1}^{\infty} a_n(\varphi) S_n(f, x) \equiv f_\varphi(x)$ exists for a.e. x but now the convergence set may depend upon φ. Further we show that $f_\varphi(x)$ is still in L. In the last section we give some examples showing the necessity of the hypotheses in the above theorems.

Notation. All functions considered in this paper are, defined on $[-\pi, \pi]$, real integrable and periodic with period 2π. Without loss of generality it is also assumed that the constant term in the Fourier expansion of each function is zero. If a notation is not explained, it coincides with the notation of [5]. We define $S_n f(x) \equiv S_n(f, x)$ and $\sigma_n f(x) \equiv \sigma_n(f, x)$.

By BV we mean the space of all functions of bounded variation with $\|\varphi\|_{BV} = \|\varphi\|_\infty + \mathrm{Var}(\varphi)$. By AC we mean the space of all absolutely continuous function with $\|\varphi\|_{AC} = \|\varphi\|_1 + \|\varphi'\|_1$. By H we mean the space of all functions whose conjugates are also integrable.

If $\sum_{n=1}^{\infty} A_n(f, x) \sim f \in E$, then we also write $\sum_{n=1}^{\infty} A_n(f, x) \in E$. By $g(x, t) \in E(dt)$ we mean $g(x, \cdot) \in E$.

1. THEOREM 1. *Let $f \in H$. Then $\exists E \subset [-\pi, \pi]$ with $|E| = 2\pi$ such that for all $x \in E$, $\sum_{n=1}^{\infty} \big(S_n(f, x) - f(x)\big) A_n(\varphi, t)$ converges uniformly in t for all $\varphi \in$ BV.*

Proof. From [5], I, p. 50, 5.5,

$$S_n^*(f, x) = \frac{1}{\pi} \int_t^\pi (1 - \cos nu) \frac{\tilde{f}(x+u) - \tilde{f}(x-u)}{2\tan\frac{1}{2}u} du. \tag{1.1}$$

For $\pi \geqslant t > 0$, define

$$h(x, t) \equiv \int_t^\pi \frac{\tilde{f}(x+u) - \tilde{f}(x-u)}{2\tan\frac{1}{2}u} du. \tag{1.2}$$

Then

$$S_n^*(f, x) = -\lim_{t\to 0+} \frac{1}{\pi} \int_t^\pi (1 - \cos nu) \frac{\partial}{\partial u} h(x, u)\, du.$$

Integrating by parts and using that the limit of $h(x, t)$ as $t \to 0+$ exists for a.e. x ([5], I, p. 131), we get

$$S_n^*(f, x) = \frac{n}{\pi} \int_0^\pi (\sin nu) h(x, u)\, du \quad \text{for a.e. } x. \tag{1.3}$$

Now defining

$$H^*(x, t) = \begin{cases} \frac{1}{2}[h(x, t) - (\pi - t) f(x)] & \text{for} \quad \pi \geqslant t > 0, \\ 0 & \text{for} \quad t = 0, \end{cases}$$

and extending it as an odd periodic function, we see that for a.e. x,

$$\sum_{n=1}^{\infty} \frac{S_n^*(f, x) - f(x)}{n} \sin nt \sim H^*(x, t) \in C(dt). \tag{1.4}$$

Since $\sum_{n=1}^{\infty} \frac{A_n(f, x)}{n} \sin nt \in AC(dt)$, we have

$$\sum_{n=1}^{\infty} \frac{S_n(f, x) - f(x)}{n} \sin nt \sim H(x, t) \in C(dt) \quad \text{for a.e. } x. \tag{1.5}$$

Thus,

$$\begin{aligned} &\|S_N H(x, \cdot) - H(x, \cdot)\|_\infty \\ &\leqslant \|S_N H(x, \cdot) - \sigma_N H(x, \cdot) + \sigma_N H(x, \cdot) - H(x, \cdot)\|_\infty \\ &\leqslant \frac{1}{N+1} \sum_{n=1}^{N} |S_n(f, x) - f(x)| + \|\sigma_N H(x, \cdot) - H(x, \cdot)\|_\infty = o(1) \text{ for a.e. } x \end{aligned} \tag{1.6}$$

by strong convergence of partial sums ([5], II, p. 184) and uniform convergence of Cesàro sums.

Let $E \subset [-\pi, \pi]$ be the set of all those x for which (1.6) is true. Let $\varphi \in$ BV. Then

$$\begin{aligned} x_0 \in E \ &\Rightarrow \lim_{N\to\infty} \|S_N H(x_0, \cdot) - H(x_0, \cdot)\|_\infty = 0 \\ &\Rightarrow \lim_{N\to\infty} \|S_N H(x_0, \cdot) * d\varphi - H(x_0, \cdot) * d\varphi\|_\infty = 0 \\ &\Rightarrow \sum_{n=1}^{\infty} [S_n(f, x_0) - f(x_0)] A_n(\varphi, t) \text{ converges uniformly in } t. \end{aligned}$$

Remark 1. The set E in Theorem 1 is independent of $\varphi \in$ BV, and this, as we shall see, will no longer be true if we only assume $f \in L$.

The following result shows how Theorem 3 of [3] becomes a particular case of the above theorem.

COROLLARY 1.1. *Let $f \in H$. Then for a.e. x, $\sum_{n=1}^{\infty} A_{kn}(\varphi) S_n(f, x)$ and $\sum_{n=0}^{\infty} A_{(2n+1)k}(\varphi) S_n(f, x)$ converge for all $k \geqslant 1$ and $\varphi \in$ BV.*

Proof. We have

$$\sum_{n=1}^{\infty} A_n(\varphi) \cos nt \sim \tfrac{1}{2}[\varphi(t) + \varphi(-t)] \in \mathrm{BV}.$$

By [1], I, p. 257, Ex. 16, $\sum_{n=1}^{\infty} A_{kn}(\varphi) \cos nt \sim \varphi_k \in$ BV. Using the theorem, we get the first part of the corollary.

Now

$$\varphi_2 \in \mathrm{BV} \Rightarrow \sum_{n=1}^{\infty} A_{2n}(\varphi) \cos 2nt \in \mathrm{BV} \Rightarrow \sum_{n=0}^{\infty} A_{2n+1}(\varphi) \cos(2n+1)t \in \mathrm{BV}. \tag{1.7}$$

Let

$$f(x) \sim \sum_{n=1}^{\infty} (c_n \cos nx + d_n \sin nx), \quad \text{and}$$

$$f^*(x) \equiv f(2x) \sim \sum_{n=1}^{\infty} (c_n \cos 2nx + d_n \sin 2nx).$$

Then $S_{2n+1}(f^*, x) = S_n(f, 2x)$.

By (1.7) and the theorem, for a.e. x, $\sum_{n=0}^{\infty} a_{2n+1}(\varphi) S_{2n+1}(f^*, x)$ converges, i.e. for a.e. x, $\sum_{n=1}^{\infty} a_{2n+1}(\varphi) S_n(f, x)$ converges.

Replacing φ by φ_k, we get the second part of the corollary.

THEOREM 2. *Let* $f \in H$, $\sum_{j=1}^{\infty} a_j(\varphi) \cos jt \sim \varphi \in \mathrm{BV}$ *and* $\sum_{j=1}^{\infty} a_j(\varphi) S_j(f, x) \equiv f_\varphi(x)$ (which exists a.e. from Theorem 1); *then*

(i) $\|f_\varphi\|_p \leqslant A \|f\|_p \|\varphi\|_{\mathrm{BV}}$ *for* $1 \leqslant p \leqslant \infty$ *where* A *is an absolute constant. Moreover,*

(ii) $f_\varphi(x) \sim \sum_{n=1}^{\infty} \lambda_n(\varphi) A_n(f, x)$ *where* $\lambda_n(\varphi) = \sum_{j=n}^{\infty} a_j(\varphi)$.

Proof. (i) It is sufficient to prove the result for $f_\varphi^*(x)$ where

$$f_\varphi^*(x) \equiv \sum_{j=1}^{\infty} a_j(\varphi) S_j^*(f, x) \quad \text{for a.e. } x.$$

Define

$$S_\varphi(x) \equiv \sum_{n=1}^{\infty} a_n(\varphi) [S_n^*(f, x) - f(x)]. \tag{1.8}$$

From Theorem 1,

$$\sum_{n=1}^{\infty} \frac{S_n^*(f, x) - f(x)}{n} \sin nt \sim H^*(x, t) \in C(dt) \quad \text{for a.e. } x.$$

Now

$$\sum_{n=1}^{\infty} [S_n^*(f, x) - f(x)] a_n(\varphi) \cos nt \sim H^*(x, \cdot) * d\varphi(t) \in C(dt).$$

Since the series on the left-hand side converges for $t = 0$, by Theorem 1,

$$S_\varphi(x) = H^*(x, \cdot) * d\varphi(0) = -\frac{1}{\pi} \int_0^{\pi} H^*(x, t)\, d\varphi(t). \tag{1.9}$$

Since ([5], I, p. 295, Ex. 3(b))

$$\|H^*(\cdot, t)\|_p \leqslant 2\pi \|f\|_p \quad \text{for} \quad 0 < t \leqslant \pi \text{ and } 1 \leqslant p \leqslant \infty,$$

and clearly $\|H^*(\cdot, 0)\|_p = 0$, we see that

$$\|S_\varphi\|_p \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \|H^*(\cdot, t)\|_p \, d|\varphi| \leqslant \|f\|_p \|\varphi\|_{\mathrm{BV}}. \tag{1.10}$$

Hence from (1.8)

$$\|f_\varphi^*\|_p \leqslant 2 \|f\|_p \|\varphi\|_{\mathrm{BV}} + \Big| \sum_{n=1}^{\infty} a_n(\varphi) \Big| \|f\|_p \leqslant 3 \|f\|_p \|\varphi\|_{\mathrm{BV}}.$$

(ii) The argument is standard. First we take $g \in L^2$ and find the Fourier expansion of $g_\varphi(x)$.

By partial summation,

$$\sum_{n=1}^{N} a_n(\varphi) S_n(g, x) = \sum_{n=1}^{N-1} \lambda_n(\varphi) A_n(g, x) + \lambda_N(\varphi) S_N(g, x).$$

Since $\|\lambda_N(\varphi) S_N(g, \cdot)\|_2 \to 0$ as $N \to \infty$,

$$g_\varphi(x) \sim \sum_{n=1}^{\infty} \lambda_n(\varphi) A_n(g, x).$$

Now find $\{g_k\} \subset L^2$ such that $\|g_k - f\|_1 \to 0$ as $k \to \infty$.

Since the operator T mapping f to f_φ is continuous on H, for all n and x,

$$A_n(Tf, x) = \lim_{k\to\infty} A_n(Tg_k, x) = \lim_{k\to\infty} \lambda_n(\varphi) A_n(g_k, x) = \lambda_n(\varphi) A_n(f, x). \ ■$$

COROLLARY 1.2. *Let* $\varphi \sim \sum_{n=1}^{\infty} a_n \cos nt \in \mathrm{BV}$ *and let* $\lambda_n = \sum_{j=n}^{\infty} a_j$. *Then* $\sum_{n=1}^{\infty} \lambda_n \cos nt$ *is a Fourier–Stieltjes series.*

Proof. From Theorem 2, for any $f \in L^\infty$,

$$f_\varphi \sim \sum_{n=1}^{\infty} \lambda_n A_n(f, x) \quad \text{and} \quad \|f_\varphi\|_\infty \leqslant A \|f\|_\infty.$$

Hence by [5], I, p. 176, Th. 11.4, $\sum_{n=1}^{\infty} \lambda_n \cos nt$ is a Fourier–Stieltjes series.

Remark 2. An unpublished result of G. Goes asserts that with the hypotheses of Corollary 1.2, $\sum_{n=1}^{\infty} \lambda_n \cos nt$ is a Fourier series.

2. THEOREM 3. *Let* $f \in L$ *and* $\sum_{n=1}^{\infty} a_n \cos nt \sim \varphi \in \mathrm{BV}$. *Then there exists* $E_\varphi \subset [-\pi, \pi]$ *with* $|E_\varphi| = 2\pi$ *such that* $\sum_{n=1}^{\infty} a_n S_n(f, x) \equiv f_\varphi(x)$ *exists for all* $x \in E_\varphi$. *Moreover,* $f_\varphi(x)$ *is integrable and its Fourier series is* $\sum_{n=1}^{\infty} \lambda_n A_n(f, x)$ *where* $\lambda_n = \sum_{j=n}^{\infty} a_j$.

Proof. Let $\lambda_n = \sum_{j=n}^{\infty} a_j$. Then from Corollary 1.2, $\sum_{n=1}^{\infty} \lambda_n A_n(f, x)$ is a Fourier series of a function $f_\lambda \in L$. We will show that $f_\lambda = f_\varphi$. Let

$$\sigma_N f_\lambda(x) \equiv \sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)\lambda_n A_n(f, x)$$

and

$$\tau_N f_\varphi(x) \equiv \sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right) a_n S_n(f, x).$$

Using summation by parts,

$$\frac{N+1}{N}\sigma_N f_\lambda(x) = \tau_{N-1} f_\varphi(x) + \frac{\lambda_1 S_1(f, x)+\ldots+\lambda_N S_N(f, x)}{N}.$$

Hence

$$|\tau_{N-1} f_\varphi(x) - f_\lambda(x)| \leqslant \left|\frac{N+1}{N}\sigma_N f_\lambda(x) - f_\lambda(x)\right| + \frac{|\lambda_1 S_1(f, x)+\ldots+\lambda_N S_N(f, x)|}{N}.$$

Since $\lambda_n \to 0$ and $S_n(f, x)\to f(x)$ strongly for a.e. x,

$$\tau_{N-1} f(x) \to f_\lambda(x) \quad \text{for a.e. } x.$$

Therefore,

$$\sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right) a_n [S_n(f, x) - f(x)] \to f_\lambda(x) - f(x)\sum_{n=1}^{\infty} a_n \quad \text{for a.e. } x.$$

But

$$\left|\sum_{n=1}^{N}(S_n(f, x)-f(x))a_n - \sum_{n=1}^{N}\left(1-\frac{n}{N+1}\right)a_n(S_n(f, x)-f(x))\right|$$

$$\leqslant \frac{A}{N+1}\sum_{n=1}^{N}|S_n(f, x)-f(x)| \text{ (where } A \text{ is a constant)} = o(1) \quad \text{for a.e. } x.$$

Hence,

$$\sum_{n=1}^{N}(S_n(f, x)-f(x))a_n \to f_\lambda(x) - f(x)\sum_{n=1}^{N} a_n \quad \text{for a.e. } x$$

or

$$\sum_{n=1}^{N} S_n(f, x) a_n \to f_\lambda(x) \quad \text{for a.e. } x. \;\blacksquare$$

Remark 3. If $f \in H$, then from Theorem 1, E_φ can be chosen independent of φ. In the next theorem we show that there exists $f\in L$ for which this is not possible.

Similar to Corollary 1.1 we have

COROLLARY 2.1. *If $f\in L$ and $\varphi \in \mathrm{BV}$ then for all $k \geqslant 1$, $\sum_{n=1}^{\infty} a_{kn}(\varphi)S_n(f, x)$ and $\sum_{n=0}^{\infty} a_{(2n+1)k}(\varphi)S_n(f, x)$ converge for a.e. x.*

For the remaining cases we have following counterexample.

EXAMPLE. Let $k>2$, $1\leqslant d\leqslant k-1$ and $d\neq k/2$. Then there exists $\varphi\in \mathrm{BV}$ and $f\in C$ such that $\sum_{n=0}^{\infty} a_{nk+d}(\varphi)S_n(f, x)$ diverges for all x.

Construction. Take $f(x)\equiv 1$ and $\psi(t)\sim \sum_{n=1}^{\infty}\frac{1}{n}\sin nt$. Take

$$\varphi(t) = \psi\left(\frac{2\pi}{k}+t\right)+\psi\left(\frac{2\pi}{k}-t\right) \sim \sum_{n=1}^{\infty}\frac{2}{n}\sin\frac{2\pi n}{k}\cos nt.$$

Then $\varphi\in\mathrm{BV}$. Now

$$\sum_{n=0}^{\infty} a_{nk+d}(\varphi)S_n(f, x) \equiv \sin\frac{2\pi d}{k}\sum_{n=0}^{\infty}\frac{2}{nk+d}.$$

But $\sum_{n=0}^{\infty}\frac{1}{nk+d} = \infty$ and $\sin\frac{2\pi d}{k}\neq 0$. Hence the given series diverges for all x.

3. LEMMA 3.1. *Let $f\in L$ such that for a.e. x, $\sum_{n=1}^{\infty} S_n(f, x)a_n(\varphi)$ converges for all $\varphi\in\mathrm{AC}$. Then for a.e. x, there exists M_x (independent of φ) such that*

$$\left|\sum_{n=1}^{\infty} S_n(f, x)a_n(\varphi)\right| \leqslant M_x\|\varphi\|_{\mathrm{AC}} \quad \textit{for all } \varphi\in\mathrm{AC}.$$

Proof. Fix an x such that $\sum_{n=1}^{\infty} S_n(f, x)a_n(\varphi)$ converges for all $\varphi\in\mathrm{AC}$. Define $T_N\colon \mathrm{AC}\to\boldsymbol{R}$ as $T_N(\varphi) = \sum_{n=1}^{N} S_n(f, x)a_n(\varphi)$. Then $|T_N\varphi|\leqslant M(N, x)\|\varphi\|_{\mathrm{AC}}$, where $M(N, x)$ is a constant depending upon N and x only. Since the limit of $T_N\varphi$, as $N\to\infty$, exists in $\boldsymbol{R}$, by the uniform boundedness principle we get the result.

Remark 4. Using Corollary 1.1 and the above lemma we get a generalization of [3], p. 139, Th. 2.

THEOREM 4. *There exists a function $f\in L$ for which there is no E satisfying the following properties:*

(i) *$E\subset[-\pi, \pi]$ and $|E| = 2\pi$;*

(ii) *$\lim_{N\to\infty}\sum_{n=1}^{N} S_n(f, x)a_n(\varphi)$ exists for all $x\in E$ and $\varphi\in\mathrm{AC}$.*

Proof. From Lusin's Theorem ([1], II, p. 95) there exists an absolutely continuous function $F(x)$ such that

(3.1) *the Fourier series of its derivative* $f(x)$ *and its conjugate converge almost everywhere*,

(3.2) *the function* $\tilde{F}(x)$, *conjugate to* $F(x)$, *is essentially unbounded in any interval* $[a, b] \subset [0, 2\pi]$.

So from (3.1) we have $f \in L$ and $\tilde{S}_n(f, x) = O(1)$ for a.e. x.

Suppose for this f that there exists an E satisfying (i) and (ii) in the theorem, i.e., for a.e. x, $\sum_{n=1}^{\infty} S_n(f, x) a_n(\varphi)$ converges for all $\varphi \in AC$.

Then similar to Corollary 1.1 we would get that for a.e. x,

$$\Big|\sum_{n=0}^{\infty} a_{2n+1}(\varphi) S_n(f, x)\Big| < \infty \quad \text{for all } \varphi \in AC. \tag{3.3}$$

Hence from Lemma 3.1, for a.e. x, $\exists M_x$ such that

$$\Big|\sum_{n=0}^{\infty} a_{2n+1}(\varphi) S_n(f, x)\Big| \leqslant M_x \|\varphi\|_{AC} \quad \text{for all } \varphi \in AC. \tag{3.4}$$

Hence from [4], p. 133, Th. 3, $\sum_{n=1}^{\infty}{}' \frac{1}{n} A_n(f, x) \sim -\tilde{F}(x)$ is equivalent to a function differentiable a.e. which contradicts (3.2). ■

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Best order conditions in linear spaces, with applications to limitation, inclusion, and high indices theorems for ordinary and absolute Riesz means

by

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Abstract. It is the first purpose of this paper to obtain simple order conditions which hold for certain sequences of continuous linear functionals on a Fréchet space with a Schauder basis, and to investigate best-possible order conditions. We then specialize the results to Banach spaces and to summability fields of matrices. By using results on summability fields and absolute summability fields of Riesz typical means and generalized Cesàro means, some of them new, we are able to obtain some best-possible order conditions in these fields; in particular, we can specify the best-possible limitation theorems for a sequence (or series) which is limitable, or absolutely limitable, by the Riesz method. We then apply our limitation theorems to obtain two equivalence theorems, of 'high-indices' type, for ordinary and absolute Riesz summability. Finally, we can obtain improved forms of two inclusion theorems which specify necessary and sufficient conditions for an arbitrary matrix method to include the ordinary or absolute Riesz methods.

1. Introduction. A locally convex linear topological (Hausdorff) space (over the complex number field) which is complete and metrizable has a topology generated by a countable set of seminorms $p = \{p_j\}$, and such a space (X, p) is a Fréchet space (F-*space*). We may assume without loss of generality that no seminorm p_j is identically zero. An F-space with a norm topology, $(X, \|\cdot\|_X)$, is a Banach space (B-*space*). A *sequence space* is a vector subspace of ω, the space of all complex-valued sequences. An FK-*space* (X, p) is a (locally convex) Fréchet sequence space for which the coordinate functionals (i.e., the maps $P_n(x) = x_n$, $n = 0, 1, \ldots$) are continuous; an FK-space has a unique FK-topology. A BK-*space* is a Banach sequence space with continuous coordinates. Examples of BK-spaces are the spaces m, c, c^0 of bounded, convergent, null sequences, respectively, all with

$$\|x\| = \sup_k |x_k|; \qquad l = \Big\{x\colon \|x\| = \sum_{k=0}^{\infty} |x_k| < \infty\Big\};$$

$$v = \Big\{x\colon \|x\| = \lim_k |x_k| + \sum_{k=0}^{\infty} |x_k - x_{k+1}| < \infty\Big\}; \qquad v^0 = c^0 \cap v.$$

A countable collection of points, $\{a^k\}$, of an F-space (X, p), is a (*Schauder*) *basis* for (X, p) if there are unique functionals f_k $(k = 0, 1, \ldots)$