Bases and basic sequences in $F$-spaces

by

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Abstract. This paper is concerned with the theory of Schauder bases in non-locally convex $F$-spaces. We first give some results on the existence problem for basic sequences, extending work of the first author (Basic sequences in $F$-spaces and their applications, Proc. Edinburgh Math. Soc. to appear). In particular it is shown that the existence of a basic sequence in every infinite-dimensional closed linear subspace of an $F$-space is equivalent to an extension property for linear functionals. Then we introduce two new classes of $F$-spaces, which we call pseudo-Fréchet and pseudo-reflexive spaces. For example, an $F$-space is pseudo-reflexive if every bounded set is relatively compact in the weak topology of its closed linear span. We give criteria for spaces with bases to be pseudo-Fréchet and pseudo-reflexive and hence are able to give non-locally convex examples. Using these examples we show the existence of non-locally convex $F$-spaces on which there exist strictly weaker vector topologies which define the same closed subspaces as the original topology.

1. Introduction. In this paper we continue the study begun by the first author in [1] of basic sequences in $F$-spaces, with the emphasis on non-locally convex spaces. In §2 we restate in a more accurate form the main result from [1] on constructing basic sequences, and derive some variations on this result. It is not known if every $F$-space contains a basic sequence. §3 contains some contributions to this existence problem. The last section of the paper treats two new classes of $F$-spaces. We call an $F$-space pseudo-Fréchet if the weak topology of each linear subspace coincides on bounded sets with the weak topology of the whole space. We call an $F$-space pseudo-reflexive if the weak topology is Hausdorff, and every bounded subset is relatively compact in the weak topology of its closed linear span. It turns out that every pseudo-reflexive $F$-space is pseudo-Fréchet, and a Fréchet space (locally convex $F$-space) is pseudo-reflexive if and only if it is reflexive. We give criteria for spaces to be pseudo-Fréchet or pseudo-reflexive which involve shrinking and boundedly complete basic sequences; and we use these results to construct

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examples of non-locally convex pseudo-Préchét and pseudo-reflexive spaces. Finally, we show that the bounded weak topology of a locally bounded, pseudo-reflexive F-space is compatible with (i.e., has the same closed subspaces as) the original topology. This provides examples of non-locally convex F-spaces which have topologies strictly weaker than, yet compatible with, the original ones; and gives non-locally convex applications of some of the results in §5 of [1].

We wish to thank Professor A. Pelczynski for suggesting to us the examples used in §4.

2. Basic results. First we recall some definitions. Let $(E, τ)$ be an F-space; then a sequence $(a_n)$ is semi-basic [1]) if for each $n$, we have $x_n \in \lim (a_{n+1}, a_{n+2}, \ldots)$. As observed in [1] we can then define continuous linear functionals $(f_n)$ on the space $E_n = \lim (x_n; n \in N)$ such that $f_n(x_n) / n \to 0$. If we further have that for $x \in E_n$, $f_n(x) = 0$ for all $n \in N$ implies that $x = 0$, then $(a_n)$ is a Markushhevich basis of $E_n$, and we shall then say that $(a_n)$ is an M-basic sequence in $E$. Finally, if for $x \in E_n$, $x = \sum_{n=1}^{\infty} f_n(x) a_n$ then $(a_n)$ is a basic sequence in $E$.

If $\varrho$ is another Hausdorff vector topology on $E$ we shall say that $\tau$ is $\varrho$-polar if $\varrho$ has a base of $\varrho$-closed neighbourhoods of 0, and $\tau$ is $\varrho$-compatible if every $\tau$-closed linear subspace of $E$ is also $\varrho$-closed. It is shown in [1] that if $\tau$ is $\varrho$-compatible then $\tau$ is $\varrho$-polar. A net $(y_\lambda; \lambda \in A)$ in $E$ is $\varrho$-regular if there is a neighbourhood $V$ of 0 such that $y_\lambda \in V$ for all $\lambda \in A$.

In Theorem 2.1 below we restate the main existence theorem for basic sequences from [1]. Part (ii) is a more accurate formulation of Corollary 3.4 of [1], for it asserts the existence of an M-basic sequence rather than simply a semi-basic sequence. It is clear that the proof of Corollary 3.4 yields this extra information, as the sequence obtained is an M-basic sequence for a topology on $E$ which is weaker than $\tau$.

**Theorem 2.1.** Let $(E, \tau)$ be an F-space and let $\varrho$ be a Hausdorff vector topology on $E$ with $\varrho \subset \tau$. Suppose $(a_\lambda; \lambda \in A)$ is a $\varrho$-regular net which converges to 0 in $\varrho$ and suppose $a_\lambda \in E$ with $a_\lambda \neq 0$.

(i) Suppose $\tau$ is $\varrho$-polar. Then there is an increasing sequence $(a(n); n \geq 2)$ such that if $a_n = a_{n+1}$, $n \geq 2$, then $(a_n; n \in N)$ is a basic sequence in $(E, \tau)$.

(ii) In general, if $\tau$ is not $\varrho$-polar, there is an increasing sequence $(a(n); n \geq 2)$ such that if $a_n = a_{n+1}, n \geq 2$ then $(a_n; n \in N)$ is an M-basic sequence in $(E, \tau)$.

In this section we modify this result by giving another condition under which basic sequences may be constructed. A sequence $(a_n)$ is of type $D^*$ if there is a continuous linear functional $\varrho$ on $\lim (a_n; n \in N)$ such that $\varrho (a_n) = 1$ for all $n$.

**Lemma 2.2.** Let $(a_n)$ be a sequence of type $D^*$ and suppose that $u \in \lim (a_n; n \in N)$. If $(a_n)$ is basic (resp. M-basic; resp. semi-basic), then $(u + a_n)$ is basic (resp. M-basic; resp. semi-basic).

**Proof.** Suppose $(a_n)$ is semi-basic, and that there are continuous linear functionals $(f_n)$ on $\lim (a_n)$ such that $f_n(a_n) = 0$. Let $\varrho$ be a continuous linear functional on $\lim (a_n)$ such that $\varrho (a_n) = 1$ for all $n$. Let $X$ be the space $\lim (a_n; u)$, and extend $f_n$ and $\varrho$ to continuous linear functionals $f_n$ and $\varrho$ defined on $X$ such that $f_n(u) = \varrho(u) = 0$. Also let $\psi$ be the linear functional defined on $X$ such that $\varrho(u) = 1$ and $\varrho(\lim (a_n)) = 0$. Then $\psi$ is a continuous functional as $\varrho^{-1}(0)$ is closed.

Now $f_n(0 + a_n) = 0 = \varrho(u + a_n)$. Thus $\varrho(u + a_n) = \varrho(u + a_n)$ for all $n$, and hence $\varrho(0 + a_n) = \varrho(0 + a_n)$.

**Theorem 2.3.** Let $(E, \tau)$ be an F-space and suppose $\varrho \subset \tau$ is a Hausdorff vector topology on $E$. Suppose $(a_\lambda; \lambda \in A)$ is a $\varrho$-Cauchy net in $E$. Suppose either that $(a_\lambda)$ converges in $\varrho$ to some $y \in \lim (a_\lambda; \lambda \in A)$ or that $(a_\lambda)$ does not converge in $\varrho$. Then

(i) if there is an increasing sequence $(a(n))$ such that $(a(n))$ is an M-basic sequence.

(ii) if $\tau$ is $\varrho$-polar, there is an increasing sequence such that $(a(n))$ is a basic sequence.

**Proof.** First suppose $(a_\lambda)$ converges to some $u \in \lim (a_\lambda; \lambda \in A)$. Then by Theorem 2.1 there is an increasing sequence $(a(n); n \geq 2)$ such that if $a_n = a_{n+1} - u$ for $n \geq 1$ then $(a_n)$ is M-basic (or basic if $\tau$ is $\varrho$-polar). There is a continuous linear functional $\varrho$ on $\lim (a_n; u)$ such that $\varrho (u) = -1$.
but \( \varphi(x_n) = 0 \) for all \( n \). Then \( \varphi(x_n) = 1 \) for \( n \geq 1 \) and so \( (x_n) \) is of type \( P^* \).

By Lemma 2.2, \( x + x_n - x_{a_n} = 0 \) is \( M \)-basic (or basic if \( \tau \) is \( \rho \)-polar).

Next suppose \( (x_n; x \in A) \) does not converge. Let \( (E, \tau) \) be the completion of \( (E, \rho) \) and \( \tau \subset \bar{E} \) be the linear span of \( E \) and \( u = \lim a_n \). For (i) we extend \( \tau \) to a topology \( \tilde{\tau} \) on \( Y \) so that \( Y = Y \cap \tilde{\tau} \), and apply the preceding proof. For (ii) suppose \( (V_n) \) is a base of balanced \( \rho \)-closed \( \tau \)-neighbourhoods of 0 satisfying \( V_{n+1} + V_{n+1} \subset V_n \). Let \( W_n \) be the closure of \( V_n \) in \( (Y, \tilde{\tau}) \). If each \( W_n \) is absorbent in \( Y \), then \( (W_n) \) defines a \( \rho \)-polar topology \( \hat{\tau} \) on \( Y \) which extends \( \tau \) (cf. Theorem 5.7 of [1]) and again we may apply the earlier proof. Otherwise \( \bigcup \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} W_{n+k} = E \), in particular \( W_{n+k} \subset E \). Thus \( W_n = V_n \) for \( n \geq m \). If we define \( U_n = W_n + (\lambda_n; |\lambda_n| < 2^{-m}) \), then \( (U_n) \) defines a topology \( \tilde{\tau} \) on \( Y \) which is \( \rho \)-polar and \( \tau = \tau \) on \( E \) (since for \( k \geq n \), \( U_k \cap \tilde{\tau} = V_k \)). Again we apply the earlier proof.

3. The existence problem. An \( F \)-space \( E \) is called minimal if there is no strictly weaker Hausdorff vector topology on \( E \). It is shown in [1] that \( \omega \), the space of all sequences, is a minimal space; however it is not known whether there are other examples. This problem is central to the problem of finding basic sequences in any \( F \)-space. In [2], Peck considers the space \( M[0, 1] \) of measurable functions on \([0, 1]\) with the \( F \)-norm

\[
|f| = \int_0^1 |f(t)| \, dt
\]

and shows that \( M[0, 1] \) is not a minimal space. His method of proof yields the following result:

Proposition 3.1. Let \( (E, \tau) \) be a minimal \( F \)-space and suppose \( (x_n) \) is an \( M \)-basic sequence in \( E \). Then \( (x_n) \) is a basic sequence equivalent to the usual basis of \( \omega \).

Two basic sequences \( (x_n) \) and \( (y_n) \) are equivalent if \( \sum a_n x_n \) converges if and only if \( \sum a_n y_n \) converges.

Proof. Let \( L_n = \lim |x_n|; k \geq n \) and define a vector topology \( \lambda \) with a base of \( \tau \)-neighbourhoods of 0 of the form \( L_n + U \), where \( U \) is a \( \tau \)-neighbourhood of 0. Then

\[
\bigcap \{L_n + U \mid \lambda \} = \bigcap \{U_n + U \mid \lambda \} = \bigcap L_n = (0)
\]

since \( (x_n) \) is \( M \)-basic. Therefore \( \lambda \) is Hausdorff and as \( \lambda \leq \tau \) we conclude \( \lambda = \tau \). For any sequence \( (t_n) \) of scalars we have

\[
\sum a_n t_n = \bigcap L_n + U
\]

for any \( \tau \)-neighbourhood \( U \). Hence \( \sum a_n t_n \) converges for any scalar sequence, and it follows that \( (x_n) \) is a basic sequence equivalent to the usual basis of \( \omega \).

Theorem 3.2. Let \( E \) be an \( F \)-space; then the following are equivalent:

(i) \( E \) is non-minimal,

(ii) \( E \) contains a regular \( M \)-basic sequence,

(iii) \( E \) contains a strongly regular \( M \)-basic sequence,

(iv) \( E \) contains a regular basic sequence.

Proof. (iv) \( \Rightarrow \) (ii). Immediate.

(ii) \( \Rightarrow \) (i). By Proposition 3.1, since the usual basis of \( \omega \) is not regular.

(iii) \( \Rightarrow \) (iii). By the proof of Theorem 2.1 (iii) \( \Rightarrow \) (ii) \( (= \) Corollary 3.4 of [1]) \( E \) contains a sequence \( (x_n) \) which is regular and basic in a weaker metrizable topology \( \mu \). If \( x \in \text{lin}(x_n) \), then \( x \) is also in the \( \mu \)-closed linear span of \( (x_n) \) and therefore

\[
\mu(x) = \frac{\sum a_n f_n(x) x_n}{\sum a_n f_n(x)}.
\]

Since \( (x_n) \) is \( \mu \)-regular, \( \lim f_n(x) = 0 \) and \( (x_n) \) is strongly regular.

(iii) \( \Rightarrow \) (iv). Let \( E_0 = \text{lin}(x_n) \), where \( (x_n) \) is a strongly regular \( M \)-basic sequence. Let \( (t_n) \) be the biorthogonal sequence of linear functionals on \( E_0 \). Since the topology induced by the functionals \( (t_n) \) is strictly weaker than the original topology on \( E_0 \), \( E_0 \) is non-minimal and contains a basic sequence \( \langle y_n \rangle \) by Theorem 4.3 of [1]. For \( x \in E_0 \), \( \sup f_n(x) < \infty \), and so by the Baire Category Theorem, the norm

\[
|\varphi| = \sup_n f_n(x)
\]

is continuous on \( E_0 \). The sequence \( \langle y_n \rangle \) is a regular basic sequence in \( E \).

Corollary 3.3. If \( E \) is an \( F \)-space, then \( E \) contains a basic sequence if and only if \( E \) contains a closed infinite-dimensional subspace \( X \) with a total family of continuous linear functionals.

Proof. One direction is trivial. Suppose \( Y \) is a closed infinite-dimensional subspace and possesses a total family of continuous linear functionals. If \( Y \) is minimal, then the weak topology on \( Y \) is the original topology and so \( \mu \). If \( Y \) is non-minimal, then \( Y \) contains a basic sequence.

Remark. Corollary 3.3 shows that the existence question for basic sequences is equivalent to Problem IV.2.4, p. 114, of Rolewicz [6].
In our last two results of this section we attempt to classify $F$-spaces in which every closed subspace contains a basic sequence. An $F$-space $E$ (of infinite dimension) will be said to have the Restricted Hahn–Banach Extension Property (RHEEP) if whenever $L \subset E$ is an infinite-dimensional closed subspace and $0 \neq x \in L$, then there is an infinite-dimensional closed subspace $M$ of $L$ with $x \in M$.

**Proposition 3A.** Let $E$ be an infinite-dimensional $F$-space; the following are equivalent:

(i) If $L$ is an infinite-dimensional closed subspace of $E$ and $G$ is a finite-dimensional subspace of $L$, then there is an infinite-dimensional subspace $M$ of $L$ with $M \cap G = \{0\}$.

(ii) Let $L$ be an infinite-dimensional closed subspace of $E$ and $G$ a finite-dimensional subspace of $L$. If $\varphi$ is a linear functional on $G$, there is an infinite-dimensional closed subspace $K$ of $L$ containing $G$, and a continuous linear functional $\psi$ on $K$ extending $\varphi$.

(iii) $E$ has RHEEP.

**Proof.** (iii) $\Rightarrow$ (i). We prove (i) by induction on $\dim G$. Certainly (i) is true for $\dim G = 1$. Now suppose it is true for $\dim G = k$. Suppose $\dim G = k+1$, and let $G_k$ be any subspace of $G$ of dimension $k$. Choose a closed infinite-dimensional subspace $M_k$ of $L$ such that $G_k \cap M_k = \{0\}$. Let $L_k = M_k + G_k$ and suppose $\varphi + G_k$. Then there is an infinite-dimensional closed subspace $M$ of $L_k$ with $x \not\in \varphi$. Let $M = \{M_k \cap L_k \mid \dim M_k = k\}$; then $\dim M = \dim L_k$, and clearly $M \cap G = \{0\}$.

(i) $\Rightarrow$ (ii). Choose $M$ as in (i) and let $M = M + G$; we extend $\varphi$ by $\varphi(x) = \varphi(x)$, $x \in G$, and $\varphi(x) = 0$, $x \in M$. Then $\varphi^{-1}(0) = \varphi^{-1}(0) + G$ is closed, and so $\varphi$ is continuous.

(ii) $\Rightarrow$ (i). Suppose $x \in L$; let $G = \overline{\text{lin}}(x)$ and define $\varphi(\lambda x) = \lambda$. Extending $\varphi$ as in (ii) we take $M = \varphi^{-1}(0)$.

**Theorem 3.5.** An $F$-space $E$ has RHEEP if and only if every closed infinite-dimensional subspace contains a basic sequence.

**Proof.** Suppose $E$ has RHEEP and let $L_n$ be a closed infinite-dimensional subspace of $E$; we may suppose $E_n$ separable. We may determine a collection $\mathcal{L}$ of closed infinite-dimensional subspaces of $E_n$ maximal with respect to the property that any two subcollections have infinite-dimensional intersection. Let $G = \cap \mathcal{L}$. If $\dim G = \infty$, then $G \neq \{0\}$ by maximality; however, by RHEEP, $G$ contains a proper closed infinite-dimensional subspace $G_0$ of $G$ and $G_0 \neq \{0\}$ by the maximality of $\mathcal{L}$. Hence $\dim G < \infty$. Then $E_n \cap G = \bigcup (E_n \cap L_n \in \mathcal{L})$, and as $E_n \cap G$ is a Lindelöf space, there is a countable subset $\{L_n \in \mathcal{L} \mid x \in \mathcal{L}\}$ of $\mathcal{L}$ such that $\bigcup (E_n \cap L_n) = E_n \cap G$. Let $\mathcal{M} = \{M_n \in \mathcal{L} \mid x \in M_n \}$ be the family of $M_n$. We may select a subsequence $M_{n0}$ such that $M_{n0}$
span; or equivalently, if every block basis for \((e_n)\) is shrinking. Clearly, every hyper-shrinking basis is shrinking, and the converse holds for pseudo-Fréchet spaces (it seems unlikely that the converse should hold for general \(F\)-spaces, but we have not been able to find a counter-example).

In this section we show that every \(F\)-space with a hyper-shrinking basis is pseudo-Fréchet, and we use this result to construct examples of pseudo-Fréchet spaces that are not locally convex.

We call an \(F\)-space pseudo-reflexive if it has enough continuous linear functionals to separate points, and every bounded subset is relatively compact in the weak topology of its closed linear span. It is easy to see that every pseudo-reflexive \(F\)-space is pseudo-Fréchet. It follows from standard results ([3], §23, Sec. 5, p. 383) that a Fréchet space is pseudo-reflexive if and only if it is reflexive. We show that an \(F\)-space with a basis is pseudo-reflexive if and only if the basis is boundedly complete and hyper-shrinking (a basis \((e_n)\) is boundedly complete if the series \(\sum a_ne_n\) converges whenever \((a_n)\) is a scalar sequence for which the collection of partial sums \(\sum_{1 \leq n \leq N} a_ne_n\) is bounded). This generalizes a result of James for Banach spaces [4], Chapter V, §2, Theorem 2, and allows us to construct examples of non-locally convex pseudo-reflexive \(F\)-spaces.

Before getting to the proofs we note some simple properties of equivalent basic sequences. If \((x_n)\) and \((y_n)\) are equivalent basic sequences in \(F\)-spaces there is a linear homeomorphism \(T: \text{lin}(x_n) \to \text{lin}(y_n)\) such that \(Tx_n = y_n\). The following lemma is then immediate.

**Lemma 4.1.** Suppose \((x_n)\) and \((y_n)\) are equivalent basic sequences in \(F\)-spaces:

1. If \(x_n\) tends to zero in the weak topology of its closed linear span, then so does \(y_n\).
2. If \(x_n\) is of type \(P\), then so is \(y_n\).

We begin our study of pseudo-Fréchet spaces with the promised non-examples.

**Proposition 4.2.** \(P\) is not a pseudo-Fréchet space for \(0 < p < 1\).

**Proof.** Fix \(0 < p < 1\), let \((e_n)\) be the standard unit vector basis for \(P\), and let \[f = \sum_{n=1}^{\infty} (f(n))e_n\] be \(P\). Since the pairing \[
\langle f, g \rangle = \sum_{n=1}^{\infty} f(n)g(n) \quad (f \in P^*, \ g \in P^*)
\] identifies \(l^p\) as the dual of \(P\), the basis \((e_n)\) does not tend weakly to zero in \(P^*\). Note that it is easy to find a block basis \((f_n)\) for \((e_n)\) such that \(\|f_n\| = 1\) for all \(n\), but \(\|f_n\| = 0\). In particular, \((f_n)\) tends to zero weakly in \(P^*\). But, clearly, \((f_n)\) is a basic sequence equivalent to \((e_n)\), so \((f_n)\) does not tend to zero in the weak topology of its closed linear span. Thus \(\psi(P, l^p)\) does not coincide with \(\psi(S, S')\) on the bounded subset \((f_n)\) of \(S\), hence \(P^*\) is not pseudo-Fréchet; and the proof is complete.

Note that every closed subspace of a pseudo-Fréchet space is again pseudo-Fréchet. In [7], Sec. 4, Prop. 1, it was observed that the Hardy spaces \(H^p\) of analytic functions in the unit disc contain a subspace isomorphic to \(P^*\) for \(0 < p < 1\). In particular, \(H^p\) is not pseudo-Fréchet for \(0 < p < 1\).

In order to move toward more positive results we require two simple lemmas, both of which are known for Banach spaces.

**Lemma 4.3.** Suppose \(E\) is an \(F\)-space with basis \((e_n)\), and let \(\gamma\) denote the topology induced on \(E\) by the coordinate functionals of the basis. Then \((e_n)\) is shrinking if and only if \(\gamma\) coincides with \(\omega(E, E')\) on every bounded subset of \(E\).

**Proof.** Every block basis for \((e_n)\) is \(\gamma\)-convergent to zero, so certainly \((e_n)\) is shrinking whenever \(\gamma\) coincides with \(\omega(E, E')\) on bounded sets.

Conversely, suppose \((e_n)\) is shrinking; it is enough to show that if \(x_n\) is bounded and \(x_n \to 0\) \((\gamma)\), then \(x_n \to 0\) \((\omega(E, E'))\). If \[a_n = \sum_{k \geq n} t_{nk}e_k\] and \(|\gamma(a_n)| \geq \varepsilon\) for some \(\varepsilon \in E'\), then by a gliding hump argument (see [6], p. 59) we find increasing sequences \(m_n, n\) such that \[
\|a_m - \sum_{k=1}^{m-1} t_{mk}e_k\| < t_n
\] (where \(\|\cdot\|\) is an \(F\)-norm determining the topology on \(E\)). The sequence \[x_{m+1} = \sum_{k=n}^{m} t_{mk}e_k\] is a block basic sequence and is bounded since the partial-sum operators \[S_{m+1} = \sum_{k=1}^{m} t_{mk}e_k, \quad \sum_{k=1}^{m} t_{mk}e_k = a_n\] are equicontinuous. Hence \[
\lim_{n \to 0} \sum_{k=1}^{m} t_{mk}e_k = 0, \quad \omega(E, E').
\] It follows that \(\lim \gamma(a_{m+1}) = 0\), contrary to the assumption.
Lemma 4.4 (cf. [8], Theorem 12.2, p. 369, for Banach spaces). Suppose $E$ is an $F$-space with a basis $(e_n)$. Then $(e_n)$ is hyper-shrinking if and only if no bounded block basis for $(e_n)$ is of type $P^*$. 

Proof. If $(e_n)$ is hyper-shrinking, then every bounded block basis tends to zero in the weak topology of its closed linear span, hence cannot be of type $P^*$. Conversely, suppose $(e_n)$ is not hyper-shrinking, so there exists a bounded block basis $(f_n)$ which does not tend to zero in the weak topology of $S = \text{lin}(f_n)$. By passing to a subsequence if necessary, we may assume that there exists $y \in S'$ with $\inf_n \|y(f_n)\| > 0$. Thus the vectors $f_n/f(f_n)$ form a bounded block basis for $(e_n)$ of type $P^*$, and the proof is complete.

We now give our main criteria for an $E$-space to be pseudo-Fréchet.

Theorem 4.5. Every $F$-space with a hyper-shrinking basis is pseudo-Fréchet.

Proof. Suppose $E$ is an $F$-space with hyper-shrinking basis $(e_n)$, $S$ is a subspace of $E$, and $B$ is a bounded subset of $S$. We want to show that $w(B, E)$ coincides with $w(B, S')$. Suppose otherwise, i.e., suppose $w(B, S')$ is properly stronger than $w(B, E)$. By Lemma 4.3 the coordinate topology $\gamma$ agrees on $B$ with $w(B, E)$, and is therefore properly weaker than $w(B, S')$. Since $\gamma$ is metrizable, it follows that there is a $\gamma$-convergent sequence in $B$ that is not $w(B, S')$-convergent. After translating this sequence by its $\gamma$-limit (which by definition lies in $B$, hence in $S$) we arrive at a bounded sequence in $S$ which is $\gamma$-convergent to zero but not $w(B, S')$-convergent. By passing to a subsequence if necessary we may further assume that our sequence is $w(S, S')$-regular, hence regular for the original topology of $E$. By Theorem 2.1, this sequence contains a M-basic subsequence $(h_k)$: thus $(h_k)$ is a bounded M-basic sequence in $S$ that is $\gamma$-convergent to zero, but $w(S, S')$-regular.

By a gliding hump argument (cf. [6], p. 52) there is a subsequence $(h_{k_n})$ and a block basis $(e_{k_n})$ for $(e_n)$ such that $\sum |h_{k_n} - h_{k_n+1}| < \infty$, where $\|\cdot\|$ is an $F$-norm inducing the topology of $E$. According to [1], Lemma 4.3 and its proof, $(h_{k_n})$ is therefore a basic sequence equivalent to $(e_n)$. Thus Lemma 4.1 and the remarks preceding it show that $(h_n)$ is bounded but not convergent to zero in the weak topology of its closed linear span, which contradicts the fact that $(e_n)$ is hyper-shrinking. Thus $w(S, S')$ coincides on $B$ with $w(E, E')$, and the proof is complete.

We can at last give examples of non-locally convex pseudo-Fréchet spaces. For $0 < p < q$ and $f = (f(n))/p$ a complex sequence, let

$$
\|f\|_{p,q} = \left( \sum_{n=1}^{\infty} \left( \sum_{k \in \mathbb{N}} |f(k)|^p \right)^{q/p} \right)^{1/q}.
$$

when $q < \infty$, and let

$$
\|f\|_{p,\infty} = \sup_n \left( \sum_{k \in \mathbb{N}} |f(k)|^p \right)^{1/p}.
$$

Define $F(p)$ to be the collection of sequences $f$ such that $\|f\|_{p,q} < \infty$, and let $c_0(p)$ denote those members of $F(p)$ for which

$$
\lim_{n \to \infty} \sum_{k \in \mathbb{N}} |f(k)|^p = 0.
$$

For $p \geq 1$ the functional $\|\cdot\|_{p,q}$ is a norm which makes $F(p)$ into a Banach space. For $0 < p < 1$, $\|\cdot\|_{p,q}$ is a quasi-norm in the sense of [3], p. 159, and the sets

$$
\{ f \in F(p) : \|f\|_{p,q} < \varepsilon \} \quad (\varepsilon > 0)
$$

form a local base for a complete, Hausdorff, locally bounded topology on $F(p)$. So in any case $F(p)$ is a locally bounded $E$-space in the topology induced by $\|\cdot\|_{p,q}$. Now $c_0(p)$ is easily seen to be a closed subspace of $F(p)$, so it is also a locally bounded $E$-space.

Proposition 4.7. $F(p)$ and $c_0(p)$ are not locally convex if $0 < p < 1$.

Proof. We need only find a bounded set whose convex hull is unbounded. Define $f_k$ by

$$
f_k(n) = \begin{cases} 2^{-k} & \text{if } 2^k n < \varepsilon^{k+1} \\
0 & \text{otherwise,} \end{cases}
$$

for $n = 0, 1, 2, \ldots$ Then each $f_k$ is a convex combination of the standard unit vectors $(e_n)$, where

$$
\varepsilon_k(m) = \delta_{m,n},
$$

for $n = 1, 2, \ldots$ Moreover, for $0 < p < q < \infty$:

$$
\|f_k\|_{p,q} = 2^{k(p-q-1)} \quad (k = 0, 1, 2, \ldots)
$$

and $\|f_k\|_{p,q} = 1$ for all $n$. Thus $(e_n)$ is bounded in $F(p)$ and $c_0(p)$, but when $0 < p < 1$, its convex hull is not. This completes the proof.

Note that the standard unit vectors $(e_n)$ defined in the above proof form a basis for $F(p)$ and $c_0(p)$ when $q < \infty$.

Theorem 4.8. $(e_n)$ is a hyper-shrinking basis for $F(p)$ ($1 < q < \infty$) and $c_0(p)$.

Proof. Suppose $(f_k)$ is a block basis for $(e_n)$, say

$$
f_k = \sum_{n \in C_k} a_{n_k} e_{n_k},
$$

where $1 < n_1 < n_2 < \ldots$, and $(a_n)$ is a scalar sequence. Choose integers
$0 \leq p_1 < p_2 < \ldots$ and a subsequence $(k_j)$ such that

$$2^p < s_{k_1} < s_{k_2} < \cdots \leq 2^{p+1}.$$

Then the vectors $g_j = f_{k_j}$ form a block basis for $(e_n)$, the $j$th member of which is "supported" on the integers $2^{p+1} \leq n < 2^{p+2}$, i.e.,

$$g_j = \sum_{a \leq c < 2^{p+2}} b_{k_j} a_n \quad (j = 1, 2, \ldots)$$

for an appropriate scalar sequence $(b_n)$. Now if $g < \infty$ and $(t_j)$ is a scalar sequence, then letting $g_j = p_{j+1} - p_j - 1$ we have

$$\left\| \sum_{j} t_j g_j \right\|_{p,a}^p = \left\| \sum_{j} t_j \left( \sum_{a \leq c < 2^{p+2}} b_{k_j} a_n \right) \right\|_{p,a}^p
= \sum_{j} |t_j|^p \left( \sum_{a \leq c < 2^{p+2}} \left| b_{k_j} a_n \right| \right)^{2p}$$

In particular, if $(f_j)$ is regular and bounded, then $(g_j)$ is equivalent to the standard unit vector basis of $P$. Now since $1 < q < \infty$, this latter basis tends weakly to zero in $F$, hence by Lemma 4.1, $(g_j)$ tends to zero in the weak topology of its closed linear span. Thus every bounded block basis for $(e_n)$ tends to zero in the weak topology of its closed linear span, hence $(e_n)$ is a hyper-shrinking basis for $F(p)$.

For $e_n(p)$ a calculation similar to the one above shows that every bounded regular block basis for $(e_n)$ has a subsequence equivalent to the standard unit vector basis of $e_n$, which is a shrinking basis. By the argument just given, $(e_n)$ is a hyper-shrinking basis for $e_n(p)$, and the proof is complete.

**Corollary 4.8.** $F(p)$ is a non-locally convex pseudo-Fréchet space for $0 < p < q < \infty$. The same is true of $e_n(p)$ for $0 < p < 1$.

**Proof.** The result follows immediately from Theorem 4.5, Proposition 4.6, and Theorem 4.7.

We next turn to pseudo-reflexive $F$-spaces. To set the stage for our main result recall that a Banach space with a basis is reflexive if and only if the basis is boundedly complete and shrinking ([4], Chapter V, §2, Theorem 2).

**Theorem 4.9.** An $F$-space with a basis is pseudo-reflexive if and only if the basis is boundedly complete and hyper-shrinking.

**Proof.** Let $E$ be an $F$-space with basis $(e_n)$, and let $\gamma$ denote the topology induced by the coordinate functionals for this basis. It is not difficult to see that $(e_n)$ is boundedly complete if and only if every bounded subset of $E$ is relatively $\gamma$-compact (see [7], Lemma 1, p. 1002, for a proof).

Now suppose $E$ is pseudo-reflexive. We will show that $(e_n)$ is boundedly complete. Let $B$ be a bounded subset of $E$, and let $S$ be the closed linear span of $B$. Then the $w(S, S')$-closure $C$ of $B$ is $w(S, S')$-compact, hence $\gamma$-compact since $\gamma$ is Hausdorff and $\subseteq w(S, S')$ on $S$. It follows easily that $C$ is also the $\gamma$-closure of $B$, so $B$ is relatively $\gamma$-compact, hence $(e_n)$ is boundedly complete. Note for future reference that $\gamma = w(S, S')$ on $C$, hence on $B$.

To see that $(e_n)$ is hyper-shrinking suppose that $(f_j)$ is a bounded block basis for $(e_n)$ and let $S = \overline{\operatorname{lin}}(f_j)$. By the above remark, $\gamma = w(S, S')$ on $(f_j)$. Clearly, $(f_j)$ is $\gamma$-convergent to zero, hence $w(S, S')$-convergent to zero, so $(e_n)$ is hyper-shrinking.

Conversely, suppose $(e_n)$ is hyper-shrinking and boundedly complete. If $E$ is not pseudo-reflexive, then there is a bounded subset $B$ that is not relatively $w(S, S')$-compact, where $S = \overline{\operatorname{lin}} B$. Since $(e_n)$ is boundedly complete, the $\gamma$-closure of $B$ is $\gamma$-compact. We claim that $C \subseteq S$. Indeed, $C$ is bounded in $E$, since the original topology of $E$ is $\gamma$-polar (this follows easily from the fact that $(e_n)$ is a basis). Now if $C$ were contained in $S$ we would have $\gamma = w(S, S')$ on $C$ because $(e_n)$ is hyper-shrinking (4.3 and 4.5), so $C$ would be $w(S, S')$-compact, hence $B$ would be relatively $w(S, S')$-compact: a contradiction. Thus there exists a vector $b \in C \cap S$, and since $\gamma$ is metrizable, there is a sequence $(b_n)$ in $B$ that is $\gamma$-convergent to $b$.

Now the sequence $(b_n)$ is bounded, $\gamma$-convergent to zero, and regular, so it follows as in the proof of Theorem 4.5 that there is a subsequence equivalent to a block basis $(e_n)$ for $(e_n)$. We may as well assume this subsequence is $(b_n - b_{n-1})$ itself. We claim that $(b_n)$ is of type $E^\diamond$. To see this, define a linear functional $\phi$ on $T = \overline{\operatorname{lin}}(S, b)$ by letting $\phi = 0$ on $S$, and $\phi(b) = 1$. Now $T$ is closed in $E$ ([3], §15, sec. 5, p. 128), hence $T \subseteq \overline{\operatorname{lin}}(b_n - b_{n-1})$. Moreover, $\phi$ is continuous on $T$, hence $\sup \phi = B$ is closed in $T$; and finally, $\phi(b_n - b_{n-1}) = 1$ for all $n$. Thus $(b_n)$ is a basic sequence of type $E^\diamond$ which is bounded in $E$, hence $(e_n)$ is a bounded block basis of type $E^\diamond$. By Lemma 4.4, $(e_n)$ is not hyper-shrinking: a contradiction. Thus $E$ is pseudo-reflexive, and the proof is complete.

**Corollary 4.10.** $F(p)$ is pseudo-reflexive for $0 < p < q < \infty$. $e_n(p)$ is not pseudo-reflexive ($0 < p < \infty$).

**Proof.** We observed in Theorem 4.7 that the standard unit vector basis $(e_n)$ is hyper-shrinking for all the spaces mentioned above. It is easy to see that it is also boundedly complete for $F(p)$, but not for $e_n(p)$. By Theorem 4.9 the proof is complete.
Since $N(p)$ and $o(p)$ are not locally convex when $0 < p < 1$, we have:

**Corollary 4.11.** There exist pseudo-reflexive locally bounded $F$-spaces that are not locally convex. There exist non-locally convex locally bounded pseudo-Frèchet spaces that are not pseudo-reflexive.

A number of results in [1], Sec. 5, deal with vector topologies on an $F$-space compatible with (i.e., having the same closed subspaces as) the original topology. The Hahn–Banach theorem guarantees that the weak topology of a locally convex space is compatible with the original one, but it follows from [1], Corollary 5.3, that this fails in every non-locally convex $F$-space. So it is not obvious that a non-locally convex $F$-space can have a weaker compatible vector topology.

Our next result shows that every locally bounded, pseudo-reflexive $F$-space does have such a topology: the bounded weak topology. The bounded weak topology on an $F$-space $E$ is the strongest topology on $E$ that agrees with the weak topology on bounded sets.

**Theorem 4.12.** The bounded weak topology of a locally bounded, pseudo-reflexive $F$-space is a vector topology compatible with the original one.

**Proof.** Let $\beta$ denote the bounded weak topology on the locally bounded, pseudo-reflexive $F$-space $E$. Since every bounded subset of $E$ is weakly relatively compact, it follows from [3], Proposition 3.3, or [10], Proposition 6.2, p. 48, that $\beta$ is a vector topology. To see that $\beta$ is compatible with the original topology of $E$, suppose $S$ is a closed subspace of $E$: we will show that $S$ is $\beta$-closed, that is, $S \cap E$ is relatively weakly closed in $E$ for every bounded subset $B$ of $E$. Indeed, $B \cap S$ is $w(S, E')$-relatively compact, so its $w(S, E')$-closure $C$ is $w(S, E')$-compact, hence $w(E, E')$-compact. Recall that every pseudo-reflexive space has, by definition, a Hausdorff weak topology; so $C$ is $w(E, E')$-closed. Since $B \cap S = B \cap C$, we see that $B \cap S$ is $w(E, E')$-closed in $B$, which completes the proof.

We remark that the bounded weak topology on a Hausdorff locally bounded space coincides with the original topology only when the space is finite dimensional. For if the two topologies coincide, then the space has a compact neighbourhood of zero, and must therefore be finite dimensional ([3], §15, Sec. 7, p. 155). In particular, the bounded weak topology on the space $F(p)$ for $0 < p < 1 < q < \infty$ is strictly weaker than, yet compatible with, the original topology.

We close with an application of Theorem 4.12 to basis theory. In [1], Theorem 5.5, it is shown that if a sequence in an $F$-space is a basis for a weaker vector topology compatible with the original one, then it is also a basis for the original topology. This, along with Theorem 4.12, yields the following “bounded weak basis theorem”:

**Corollary 4.13.** In a locally bounded, pseudo-reflexive $F$-space every bounded weak basis is a basis.

This result contrasts sharply with the main result of [7] which states that if a locally bounded, non-locally convex $F$-space has a weak basis, then it has a weak basis that is not a basis.

References


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