with a similar expression for the discrete operators. Since $R_j$ has kernel $z_j|z|^{-n-1}$ and $z_j|z|^{-1}$ is a spherical harmonic of degree one, we may apply Lemma 5.2 in order to estimate

$$D_j[(a_{lm}R_j - R_j a_{lm})R_{lm}u] - \partial_{h,j}[(a_{lm}R_{jh} - R_{jh}a_{lm})R_{lm}u].$$

The remaining terms may be estimated easily using Theorem 4.1 of [8] and Lemma 4.3.

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# Discontinuous translation invariant linear functionals on $L^1(G)$

by

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**Abstract.** The main result of this article is the proof of the existence of discontinuous translation invariant linear functionals on the group algebra $L^1(G)$, for all compactly generated locally compact abelian groups.

**0.** One of the well-known results in an abelian harmonic analysis on locally compact groups is that any *continuous* translation invariant linear functional on $L^1(G)$, the space of all Haar integrable functions on $G$, is just a complex multiple of the Haar integral $I$,

$$I(f) = \int f\,d\lambda, \quad f \in L^1(G),\ \lambda \text{ is a Haar measure on } G.$$

We may ask whether this result remains valid if we omit the word 'continuous' in the hypothesis.

The purpose of this note is to prove the following.

THEOREM. *Let $G$ be any compactly generated locally compact abelian group. Then $L^1(G)$ admits discontinuous translation invariant linear functionals.*

The proof will be deferred till the end of Section 3.

The motivation has been provided to a great extent in [6], [7], where related problems have been solved for various function spaces associated with locally compact groups.

**1.** Let $G$ be a locally compact abelian group (LCAG); we will denote by $M(G)$ the algebra of all bounded Borel measures on $G$, with convolution as multiplication, i.e., we define

$$\mu * \nu(E) = \int \mu(E-g)\,d\nu(g) \quad \forall E, \tag{1}$$

where $\mu, \nu \in M(G)$, $E$ is a Borel set in $G$ and $E-g = \{h\colon h+g \in E\}$, with the total variation norm on $M(G)$.

It is well known that $M(G)$ can be identified with the strong dual of $C_0(G)$, the space of all continuous complex valued functions vanishing at infinity.

Also, $L^1(G)$ can be canonically embedded in $M(G)$ and will be a norm closed ideal in $M(G)$. $G$ acts on $L^1(G)$ by translations, the action being defined as

$$\tau_g(f) = \varepsilon_g * f \tag{2}$$

where $\varepsilon_g$ is the unit mass at $g$, or equivalently, $\varepsilon_g$ corresponds to the functional

$$\varphi_g(h) = h(g), \quad h \in C_0(G). \tag{2'}$$

Following [6] we will denote by $\Delta = \Delta(L^1(G))$ the subspace of $L^1(G)$ generated by the finite linear combinations of the differences $f - \varepsilon_g * f$, where $f \in L^1(G)$, $g \in G$, i.e.,

$$\Delta = \Big\{h \in L^1(G):\ h = \sum_{i=1}^{m} (f_i - \varepsilon_{g_i} * f_i),\ g_i \in G,\ f_i \in L^1(G)\Big\}. \tag{3}$$

Meisters ([6], Theorem I, pp. 201–202) has shown that if $\Delta$ is not closed in $L^1(G)$ then there are discontinuous translation invariant linear functionals.

We will denote by $T$ the circle group, realised as the multiplicative group of complex numbers of modulus 1, i.e., $T = \{e^{2\pi i t}:\ t \in \boldsymbol{R}\}$. $\hat{G}$ will stand for $\mathrm{Hom}(G, T)$, the group of all continuous homomorphisms of $G$ into $T$, with the group operation written multiplicatively, so that 1 will stand for the homomorphism $g \mapsto 1_T$ $\forall g \in G$. Also we will write $\gamma^{-1}$ as $\bar{\gamma}$ for $\gamma \in \hat{G}$ as $\gamma^{-1}(g) = \overline{\gamma(g)}$ $\forall g \in G$, where $\bar{\ }$ denotes complex conjugation.

Also $\hat{\ }$ will be the Fourier [Gelfand] transform, which is the map

$$\hat{\ }: L^1(G)\ [M(G)] \to C(\hat{G}),$$

defined by

$$f \mapsto \hat{f}, \quad \hat{f}(\gamma) = \int \gamma(g) f(g)\, d\lambda(g), \quad \gamma \in \hat{G}, \tag{4a}$$

$$\big[\mu \mapsto \hat{\mu},\ \hat{\mu}(\gamma) = \int \gamma(g)\, d\mu(g),\ \gamma \in \hat{G}\big]. \tag{4b}$$

It is immediate (by the translation invariance of a Haar measure $\lambda$) that

$$\Delta \subseteq \mathrm{Ker}\, I = \Big\{f \in L^1(G) \mid \hat{f}(1) = \int f(g)\, d\lambda(g) = 0\Big\}. \tag{5}$$

Lemma 1.1. *$\overline{\Delta(L^1(G))} = \mathrm{Ker}\, I_G$ for all locally compact abelian groups.*

Proof. It is plain that both $\bar{\Delta}$ and $\mathrm{Ker}\, I_G$ are closed ideals in $L^1(G)$. Denote by $Z(J) = \bigcap\{Z(f) \mid f \in J\}$, where $Z(f) = \{\gamma \in \hat{G}:\ \hat{f}(\gamma) = 0\}$, $J$ is a subset of $L^1(G)$. Obviously, $Z\ (\mathrm{Ker}\, I_G) = \{1\}$ and also $Z(\bar{\Delta}) \supseteq \{1\}$. Now suppose $\gamma \in \hat{G}$. Then $\exists f \in L^1(G)$ such that $\hat{f}(\gamma) = 1$. Consider $h_g = f - \varepsilon_g * f$, so that $h_g \in \bar{\Delta}$ $\forall g \in G$ and $\hat{h}_g(\gamma) = \hat{f}(\gamma)\big(1 - \gamma(g)\big) = 1 - \gamma(g)$ and so if $\gamma \in Z(\bar{\Delta})$ we have to have $1 - \gamma(g) = 0$ $\forall g \in G$ which implies $\gamma = 1$. So $Z(\bar{\Delta}) = Z(\mathrm{Ker}\, I_G) = \{1\}$ and since every one point set is a set of spectral synthesis ([8], Theorem 7.5.2 (a), p. 170), we obtain $\bar{\Delta} = \mathrm{Ker}\, I_G$. ■

Thus in order to show that $\Delta$ is not closed it will be necessary and sufficient to show that $\mathrm{Ker}\, I \setminus \Delta \neq \emptyset$.

To show this for all compactly generated groups we split the proof into several stages. First we prove in Section 2 some 'reduction' lemmas which will enable one to study only 'elementary' groups instead of the general case. This investigation is carried out in Section 3, where also the main theorem is proved.



**2.** Before going any further, let us recall some more elementary facts. If $\varphi: G \to H$ is a continuous group homomorphism between two LCAGs, then there is an induced Banach algebra homomorphism $\varphi^*$, say,

$$\varphi^*: M(G) \to M(H)$$

given by

$$\varphi^*(\mu)\,[f] = \mu[f \circ \varphi], \quad f \in C_0(G). \tag{6}$$

The fact that $\varphi^*$ is indeed a Banach algebra homomorphism and $\|\varphi^*\mu\| \leqslant \|\mu\|$ are to be found in [2], Proposition 10, p. 478. By standard limit argument one can extend (6) to hold for $\forall f \in C(G)$. Now it is evident that $\varphi^*\big(L^1(G)\big) \subseteq M(H)$. What is not so evident, but true nevertheless ([2], Theorem 12, p. 480), is that $\varphi^*\big(L^1(G)\big) \subseteq L^1(H)$ if $\varphi$ is open, i.e., $\varphi(U)$ is open in $H$ whenever $U \subseteq G$ is open in $G$.

Now we can state and prove the following lemma.

Lemma 2.1. *Let $G$ and $H$ be two LCAGs and suppose that $\Delta\big(L^1(G)\big)$ is not closed. Then $\Delta\big(L^1(G \times H)\big)$ is not closed.*

Proof. Let $\varphi$: $G \times H \to G$ be the usual projection homomorphism $(g, h) \mapsto g$. Then $\varphi$ is continuous, open and so $\varphi^*$ maps $L^1(G \times H)$ into $L^1(G)$. Note also that if we denote by $F$ the function on $G \times H$ by $(g, h) \mapsto f_1(g) f_2(h)$, where $f_1 \in L^1(G)$, $f_2 \in L^1(H)$, then $F \in L^1(G \times H)$ and moreover

$$\hat{F}(\gamma, \chi) = \hat{f}_1(\gamma)\hat{f}_2(\chi), \tag{7}$$

where $\gamma \in \hat{G}$, $\chi \in \hat{H}$, the dual groups of $G$ and $H$, respectively. Also if $\gamma \in \hat{G}$, then $\gamma \circ \varphi$ is in $(G \times H)\hat{\ } \cong \hat{G} \times \hat{H}$ corresponding to $(\gamma, 1)$. So

$$\begin{aligned} \varphi^*(F)\,[\gamma] &= F[\gamma \circ \varphi] = F[(\gamma, 1)] \\ &= \hat{F}(\gamma, 1) = \hat{f}_1(\gamma)\hat{f}_2(1). \end{aligned} \tag{8}$$

Thus if $f \in L^1(G)$ and $f_0$ is any function in $L^1(H)$ such that $\hat{f}_0(1) = 1$ then

$$\varphi^*(f \times f_0)\hat{\ }(\gamma) = \hat{f}(\gamma), \tag{9}$$

and so $\varphi^*(f \times f_0) = f$ (by the uniqueness of Fourier transform). This shows that $\varphi^*$ maps $L^1(G \times H)$ onto $L^1(G)$.

Now if $\Delta(L^1(G\times H))$ were closed, i.e., $\Delta(L^1(G\times H)) = \{F\colon \hat{F}(1,1) = 0\}$ then if $f\in \operatorname{Ker} I_G \setminus \Delta(L^1(G))$ and $f_0\in L^1(H)$, $\hat{f}_0(1) = 1$

$$f\times f_0 \in \Delta(L^1(G\times H))$$

and so there would exist $\{(g_i, h_i)\in G\times H\colon i = 1,\ldots,m\}$ and $\{F_i\in L^1(G\times H)\colon i = 1,\ldots,m\}$ such that

$$f\times f_0 = \sum_i (F_i - \varepsilon_{(g_i,h_i)} * F_i).$$

From this ensues

$$f = \varphi^*(f\times f_0) = \sum_i \big(\varphi^*(F_i) - \varepsilon_{g_i} * \varphi^*(F_i)\big), \tag{10}$$

since $\varphi^*$ is an algebraic homomorphism of $M(G\times H)$ into $M(G)$ and, by Theorem 12 of [2], p. 408, of $L^1(G\times H)$ into $L^1(G)$, since $\varphi$ is open. But (10) says that $f\in \Delta(L^1(G))$ which is a contradiction and so $f\times f_0\in \operatorname{Ker} I_{G\times H} \setminus \Delta(L^1(G\times H))$. ∎

LEMMA 2.2. *Let $G$ and $H$ be two* LCAGs, *$H$ a closed subgroup of $G$ (such that $G/H$ is paracompact). Then if $\Delta(L^1(G/H))$ is not closed, $\Delta(L^1(G))$ is not closed.*

Proof. Let $\pi\colon G\to G/H$ be the canonical homomorphism of $G$ onto $G/H$. $\pi$ is continuous and open and so as before $\pi^*(L^1(G)) \subseteq L^1(G/H)$; it is in fact onto; see, for example, [1], corollary to Theorem 9, Ch. VII, §2. So if $\Delta(L^1(G/H))$ is not closed, choose $f\in \operatorname{Ker} I_{G/H}\setminus \Delta(L^1(G/H))$ and choose any $F\in L^1(G)$ such that $\pi^*(F) = f$. This immediately gives that $F\in \operatorname{Ker} I_G$; and if $F\in \Delta(L^1(G))$ then, by an argument as in the proof of Lemma 2.1, $\pi^*(F)\in \Delta(L^1(G/H))$ which contradicts the choice of $f$. ∎

**3.** In the sequel we shall need two rather well-known results from harmonic analysis:

(I) *The Fourier transform* ^ *maps $L^1(G)$ into $C_0(\hat{G})$ for all* LCAG (Riemman–Lebesgue lemma).

(II) *Let $G$ be a* LCAG *with character group $\hat{G}$. Let $\psi$ be any non-negative function in $C_0(\hat{G})$. Then there exists a function $f\in L^1(G)$ such that*

$$\hat{f}(\gamma) \geqslant \psi(\gamma) \qquad \forall \gamma\in\hat{G}.$$

This can be found together with further references in [4], §32, 47, pp. 385–286.

Let us now suppose that $f\in \Delta(L^1(G))$. So there exist $\{g_i\}$, $\{f_i\}$, $g_i\in G$, $f_i\in L^1(G)$, such that,

$$f = \sum (f_i - \varepsilon_{g_i} * f_i). \tag{11}$$



Taking Fourier transforms (11) becomes

$$\hat{f}(\gamma) = \sum_i \big(1-\gamma(g_i)\big)\hat{f}_i(\gamma), \tag{12}$$

and so

$$|\hat{f}(\gamma)| \leqslant \max_{g_i} |1-\gamma(g_i)| \cdot \sum_i |\hat{f}_i(\gamma)|.$$

Write

$$\varphi(\boldsymbol{g},\gamma) = \max_{g_i} |1-\gamma(g_i)|, \qquad \boldsymbol{g} = (g_1,\ldots,g_m), \qquad g_i\in G \ \forall i.$$

Then $\varphi(\boldsymbol{g},\gamma)\neq 0$ when $\hat{f}(\gamma)\neq 0$ and we can write

$$\varphi(\boldsymbol{g},\gamma)^{-1} |\hat{f}(\gamma)| \leqslant \sum_i |\hat{f}_i(\gamma)| \qquad \forall\gamma \text{ such that } \hat{f}(\gamma)\neq 0 \tag{13}$$

and by (I) the right-hand side is in $C_0(\hat{G})$ and therefore

$$\varphi(\boldsymbol{g},\gamma)^{-1} |\hat{f}(\gamma)| \to 0 \quad \text{as} \quad \gamma\to\infty \quad \text{in } \hat{G}. \tag{14}$$

So if we can find a function $\psi_0\in C_0^+(\hat{G})$ such that

$$\forall m,\ \forall \boldsymbol{g},\ \boldsymbol{g}\in \underbrace{G\times\ldots\times G}_{m \text{ times}}, \qquad \varphi(\boldsymbol{g},\gamma)^{-1}\psi_0(\gamma) \not\to 0 \quad \text{as} \quad \gamma\to\infty,$$

we could conclude that there exists $f\in \operatorname{Ker} I_G\setminus \Delta(L^1(G))$, by (II).

LEMMA 3.1. *Let $G$ be any* LCAG *with dual group $\hat{G}$, $G^{(m)} = G\times\ldots\times G$ ($m$ times). Let $X$ be a finite subset of $\hat{G}$ of cardinality $> N^m$ and such that for any $\gamma_1,\gamma_2\in X$, $\gamma_1\neq\gamma_2$, we have either $\gamma_1\gamma_2^{-1}$ or $\gamma_2\gamma_1^{-1}\in X$. Then for any $\boldsymbol{g}\in G^{(m)}$ there is an element $\gamma\in X$ such that*

$$\varphi(\boldsymbol{g},\gamma) = \max_{g_i} |1-\gamma(g_i)| \leqslant \frac{2\pi}{N}. \tag{15}$$

Proof. Let $\boldsymbol{g}\in G^{(m)}$. Define a map $\Phi\colon \hat{G}\to T^m$ by

$$\gamma \mapsto \big(\gamma(g_i)\big)_{i=1}^m \in T^m.$$

It is plain that $\Phi$ is a continuous group homomorphism between $\hat{G}$ and $T^m$. Consider $\Phi|X$. If $\Phi$ is not injective on $X$, there will be two distinct $\gamma_1,\gamma_2\in X$ such that

$$\Phi(\gamma_1) = \Phi(\gamma_2)$$

and so (since $\Phi$ is a homomorphism)

$$\Phi(\gamma_1\gamma_2^{-1}) = \Phi(\gamma_1^{-1}\gamma_2) = \Phi(1).$$

By the hypothesis on $X$ we obtain that $\gamma_1\gamma_2^{-1}\in X$ (relabelling if necessary) and so

$$\varphi(\boldsymbol{g},\gamma_1\gamma_2^{-1}) = 0 \leqslant \frac{2\pi}{N}.$$

So assume that $\Phi(\gamma_1) \neq \Phi(\gamma_2)$ $\forall \gamma_1 \neq \gamma_2$ in $X$. Realise $T$ as $\{e^{2\pi it}: t \in \boldsymbol{R}\}$. Consider a neighbourhood of 1, $V$ say, to be $V = \{e^{2\pi it}: |t| \leqslant 1/2N\}$. Then $T$ can be covered by $N$ translates of $V$, namely

$$\{V\cdot e^{2\pi ikN^{-1}}\}_{k=0}^{N-1}.$$

So $T^m$ will be covered by $N^m$ translates of $\underbrace{V\times\ldots\times V}_{m \text{ times}} = V^m$.

Since $\Phi$ is now assumed to be injective on $X$ we have card $\Phi(X)$ = card $(X) > N^m$ and so by the 'box principle' there exists $\gamma_1, \gamma_2$ in $X$, $\gamma_1 \neq \gamma_2$, and $\boldsymbol{t}$ in $T^m$ such that $\Phi(\gamma_1) \in V^m + \boldsymbol{t}$ and $\Phi(\gamma_2) \in V^m + \boldsymbol{t}$. Suppose (as we may) that $\gamma_1\gamma_2^{-1} \in X$. Then

$$\Phi(\gamma_1\gamma_2^{-1}) \in V^m(V^m)^{-1} \subseteq (VV^{-1})^m = \left\{e^{2\pi it}: |t| \leqslant \frac{1}{N}\right\}^m$$

and so

$$\gamma_1\gamma_2^{-1}(g_i) \in \left\{e^{2\pi it}: |t| \leqslant \frac{1}{N}\right\}$$

which gives

$$\varphi(\boldsymbol{g}, \gamma_1\gamma_2^{-1}) \leqslant \frac{2\pi}{N}. \ \blacksquare$$

By the fundamental theorem on the structure of LCAGs (compactly generated), any such group is topologically isomorphic to $\boldsymbol{R}^a \times \boldsymbol{Z}^b \times K$, $a$, $b$ finite integers $\geqslant 0$ and $K$ is compact, and for compact groups we can use the following

PROPOSITION 3.2. *Let $X$ be an infinite abelian (discrete) group. Then $X$ contains as a subgroup, $\Lambda$, one of the following*:

(i) $\boldsymbol{Z}$,

(ii) $\boldsymbol{Z}(p^\infty)$,

(iii) *an infinite product of cyclic groups.*

For a proof see [3], Ch. 2, lemma 4.3, pp. 26–27.

Thus if $G$ is any compact group (infinite) then its dual $\hat{G}$ contains one of the groups from Proposition 3.2 as a (trivially) closed subgroup and this is equivalent to saying that $G$ contains a closed subgroup $H$, say, such that $G/H\hat{\ }$ is topologically isomorphic to one of the groups of 3.2 and hence $G/H$ is topologically isomorphic to either $T$, $\Delta p$ (the $p$-adic integers) or a complete direct product $\mathop{\mathsf{P}}\limits_{n=1}^{\infty}(Z/a_nZ)$ of cyclic groups.

So by Lemmas 2.1 and 2.2 to show that $\Delta(L^1(G))$ is not closed for a compactly generated LCAG it will be sufficient to show that $\Delta(L^1(G))$ is not closed for $G = T$, $\boldsymbol{Z}$, $\Delta p$, $\mathop{\mathsf{P}}\limits_{n=1}^{\infty}\boldsymbol{Z}/a_n\boldsymbol{Z}$ (since $\boldsymbol{R}/\boldsymbol{Z} \cong T$).



LEMMA 3.3. *$\Delta(L^1(G))$ is not closed if $G$ is one of the following*:

(i) $G = T$,

(ii) $G = \Delta p$,

(iii) $G = \mathop{\mathsf{P}}\limits_{n=1}^{\infty}\boldsymbol{Z}/a_n\boldsymbol{Z}$,

(iv) $G = \boldsymbol{Z}$.

Proof. *Case* (i). Let $\psi$ be a function on $\boldsymbol{Z}$ defined by

$$\psi(n) = \begin{cases} (\log|n|)^{-1}, & |n| \geqslant 2, \\ 0, & |n| < 2. \end{cases}$$

Take any $f \in L^1(T)$ such that $\hat{f}(n) \geqslant \psi(n)$. This is possible by (II). Then $h = f - \hat{f}(1)\lambda_T$ is also such that $\hat{h}(n) \geqslant \psi(n)$, and moreover, $\hat{h}(1) = 0$. So $h \in \operatorname{Ker} I_T$, and we will show that

$$\varphi(\boldsymbol{t}, n)^{-1}\hat{h}(n) \nrightarrow 0 \tag{16}$$

for any $\boldsymbol{t} \in T^m$, any $m$, which will imply by (11)–(14) that $h \notin \Delta(L^1(T))$. This we do by utilising 4.3 of [6], which says in our notation

$$\forall m\ \forall \boldsymbol{t} \in T^m\ \exists \{n_k\}_{k=1}^{\infty} \subset \boldsymbol{Z}^+, \quad n_{k+1} > n_k\ \forall k$$

such that

$$\varphi(\boldsymbol{t}, n_k) \leqslant 2\pi n_k^{-1/m}. \tag{17}$$

Then

$$\varphi(\boldsymbol{t}, n_k)^{-1}\psi(n_k) = \frac{n_k^{1/m}}{2\pi\log n_k} \to \infty \quad \text{as} \quad n_k \to \infty$$

and since

$$\varphi(\boldsymbol{t}, n_k)^{-1}\hat{h}(n_k) > \varphi(\boldsymbol{t}, n_k)^{-1}\psi(n_k),$$

(16) is valid, which proves that $\Delta(L^1(T))$ is not closed.

*Case* (ii). Write

$$\hat{G} = \boldsymbol{Z}(p^\infty) = 1 \cup \bigcup_{n=1}^{\infty} X_n,$$

where

$$X_n = \{e^{2\pi ilp^{-n}}: l = 1, 2, \ldots, p^n - 1\}.$$

Then cardinality of $X_n = p^n - 1$ and $X_n$ satisfy the hypothesis of Lemma 3.1. Thus

(18) $\forall m\ \forall \boldsymbol{g} \in \Delta_p^m\ \exists \{\gamma_n\}_{n=1}^{\infty} \gamma_n \in X_n$ such that

$\varphi(\boldsymbol{g}, \gamma_n) \leqslant 2\pi\left(E\left((p^n-1)^{\frac{1}{m}}\right)\right)^{-1}$, where $E(x)$ is the entire part of $x$.

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Define $\psi$ on $\hat{G}$ be to $\psi(\gamma) = \big(\log(p^n-2)\big)^{-1}$ for $\gamma \in X_n$ and 0 if $p^n-2 < 2$. As before there exists $h \in L^1(\Delta p)$ such that

$$\hat{h}(1) = 0 \quad \text{and} \quad \hat{h}(\gamma) \geqslant \psi(\gamma) \quad \forall \gamma \in \hat{G},$$

and

$$\hat{h}(\gamma_n)\varphi(\boldsymbol{g}, \gamma_n)^{-1} > \frac{E\big((p^n-1)^{\frac{1}{m}}\big)}{2\pi \log(p^n-2)} \to \infty \quad \text{as} \quad n \to \infty$$

and so $h \notin \Delta\big(L^1(\Delta p)\big)$, i.e., $\Delta$ is not closed.

*Case* (iii). We will consider two cases

a) $a_n = q$ $\forall n$, i.e.,

$$G = \mathop{\mathsf{P}}_{i=1}^{\infty}(\boldsymbol{Z}/q\boldsymbol{Z})_i = D_q, \qquad \hat{G} = \mathop{\mathsf{P}^*}_{i=1}^{\infty}(\boldsymbol{Z}/q\boldsymbol{Z})_i.$$

Consider any $f \in L^1(D_q)$ such that $\hat{f}(1) = 0$ and $\hat{f}(\gamma) > 0$ $\forall \gamma \neq 1$. This is possible by (II) and the fact that $\hat{G}$ is countable. Observe that every element of $D_q$ has order $\leqslant q$ and so any $m$-tuple of $(g_1, \ldots, g_m)$ will generate a finite (closed) subgroup of $D_q$ (at most $m^q$ distinct elements can be obtained as $\prod_{i=1}^{m} g_i^{a_i}$, $a_i \in \boldsymbol{Z}$). Hence $\exists \gamma \in \hat{D}_q$ such that $\gamma(g_i) = 1$ $\forall i$ but $\gamma \neq 1$. So if

$$f = \sum h_i - \varepsilon_{g_i} * h_i$$

then

$$\hat{f}(\gamma) = \sum \big(1-\gamma(g_i)\big)\hat{h}_i(\gamma) = 0 \quad \text{for some } \gamma \neq 1$$

and this contradicts $\hat{f}(\gamma) > 0$ $\forall \gamma \neq 1$.

b) $a_n \geqslant n+2$, i.e.,

$$G = \mathop{\mathsf{P}}_{n=1}^{\infty} \boldsymbol{Z}/a_n\boldsymbol{Z}, \quad \hat{G} = \mathop{\mathsf{P}^*}_{n=1}^{\infty}(\boldsymbol{Z}/a_n\boldsymbol{Z}) = \mathop{\mathsf{P}^*}_{n=1}^{\infty} X_n,$$

and define $\psi$ on $\hat{G}$ to be $\psi(\gamma) = (\log n)^{-1}$, $\gamma \in X_n \setminus \{1\}$ for $n \geqslant 2$ and 0 elsewhere. Then if $\hat{h}(1) = 0$ and $\hat{h}(\gamma) \geqslant \psi(\gamma)$ $\forall \gamma$ we will get

(19) $\quad \forall m > 0$ $\forall \boldsymbol{g} \in G^m$ $\exists \gamma_n \in X_n \setminus \{1\}$ such that

$$\varphi(\boldsymbol{g}, \gamma_n)^{-1} h(\gamma_n) \to \infty \quad \text{as} \quad n \to \infty$$

because, by Lemma 3.1, $\forall m > 0$ $\forall \boldsymbol{g} \in G^m$ $\exists \gamma_n \in X_n \setminus \{1\}$ such that

$$\varphi(\boldsymbol{g}, \gamma_n) \leqslant \frac{2\pi}{E\big((n-1)^{1/m}\big)} \quad \big(\mathrm{card}(X_n \setminus \{1\}) > n\big)$$



and so (19) follows from the definition of $\psi$, $h$ and the fact that

$$\forall m > 0 \ \frac{E\big((n-1)^{1/m}\big)}{\log n} \to \infty \quad \text{as} \quad n \to \infty.$$

Thus $h \notin \Delta\big(L^1(G)\big)$.

In the general case, i.e., $\{a_n\}$ is an arbitrary sequence of integers $a_n \geqslant 2$, we can plainly choose a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $a_{n_k} = \text{const}$ $\forall k$ if $\{a_n\}$ is bounded, or such that $a_{n_k} \geqslant k+2$ if $\{a_n\}$ is unbounded.

The reduction is then obtained by Lemma 2.2 and the observation that $\hat{G}$ contains as a (closed) subgroup $\mathop{\mathsf{P}^*}_{k=1}^{\infty}(\boldsymbol{Z}/a_{n_k}\boldsymbol{Z})$.

*Case* (iv). $G = \boldsymbol{Z}$ is shown to satisfy $\Delta\big(l^1(\boldsymbol{Z})\big)$ is not closed in [6] but for the sake of completeness we will give a different proof here. It is obvious that since dual group of discrete group is compact, we cannot hope to apply the above method. However, in the case of $G = \boldsymbol{Z}$ a direct proof is easily obtained.

Observe, first of all, that if a group $G$ is finitely generated, i.e., $\exists \{g_1, \ldots, g_n\} \subseteq G$ such that every element of $G$ is expressible as a finite product of powers of $g_i$, then

$$f = \sum h_i - \varepsilon_{x_i} * h_i \quad \text{iff} \quad f = \sum_{j=1}^{n} h_j' - \varepsilon_{g_j} * h_j'.$$

In the case $G = \boldsymbol{Z}$, $\boldsymbol{Z}$ is singly generated and so it will be sufficient to construct $f \in l^1(\boldsymbol{Z})$ such that $\hat{f}(1) = 0$ (i.e., if $f = \{a_n\}_{n=-\infty}^{\infty}$, $\sum |a_n| < \infty$, $\hat{f}(1) = \sum a_n$) and $f$ is not expressible as $g - \varepsilon_1 * g$ for $g \in l^1(\boldsymbol{Z})$.

If $f(n) = g(n) - g(n-1)$, $n \in \boldsymbol{Z}$, $f, g \in l^1(\boldsymbol{Z})$, then

$$\sum_{-\infty}^{k} f(n) = g(k).$$

Consider now

$$F(n) = +\frac{1}{n^2} \quad \text{if} \quad n < 0,$$

$$F(0) = 0,$$

$$F(n) = -\frac{1}{n^2} \quad \text{if} \quad n > 0.$$

If $F = G - \varepsilon_1 * G$ we have

$$G(k) = \sum_{n=k}^{\infty} \frac{1}{n^2} \quad \text{for} \quad k < 0 \quad \text{and} \quad G(k) \geqslant \int_k^{\infty} \frac{dx}{x^2} = \frac{1}{k}$$

and so $G$ cannot be in $l^1(\boldsymbol{Z})$. ∎

So by putting together [6], Theorem 1, p. 201 and Lemmas 2.1, 2.2 and 3.1, 3.2, we obtain

THEOREM. *Let $G$ be any compactly generated* LCAG. *Then $L^1(G)$ admits discontinuous translation invariant linear functionals.*

Remarks. (i) In order to extend this result to *all* locally compact abelian groups it would suffice to prove that $\Delta(L^1(G))$ is not closed for all *discrete* groups; alternatively one might try to prove an 'extension' lemma:

*Let $G_0$ be a compact open subgroup of a locally compact abelian group $G$. Suppose that $\Delta(L^1(G_0))$ is not closed. Then also $\Delta(L^1(G))$ is not closed.*

We would conjecture that the above is true, but so far have been unable to prove it.

(ii) It is interesting to note that in the case of compact groups $K$ we have the following corollary:

*There is a linear mapping $T: L^1(K)\to L^1(K)$ such that $T(\varepsilon_g * f) = \varepsilon_g * Tf$ $\forall g\in K$, $f\in L^1(K)$ and $T$ is not continuous.*

Proof. Let $a$ be any discontinuous translation invariant linear functional, and let $f_0$ be the constant function $f_0(g) = 1$ $\forall g$. Then $T$: $L^1(K)\to L^1(K)$, given by $f\mapsto a(f)\cdot f_0$ plainly commutes with translations since $\varepsilon_g * f_0 = f_0$ $\forall g$ and is not continuous.

This is in contrast with the known fact that any linear map from $L^1(\boldsymbol{R})$ into $L^1(\boldsymbol{R})$ which commutes with translations is continuous, viz. [5].

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# Power factorization in Banach algebras with a bounded approximate identity

by

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**Abstract.** Let $A$ be a Banach algebra with a bounded left approximate identity, let $X$ be a left Banach-$A$-module, let $x$ be in the closure of $A\cdot X$, and let $(a_n)$ be a sequence tending to infinity with $a_n > 1$ for all $n$. Then there is an $a$ in $A$ and a sequence $(y_n)$ in $X$ such that $x = a^n y_n$ and $\|y_n\| \leqslant a_n^n\|x\|$ for all $n$. This is used to show that a radical Banach algebra $A$ with bounded approximate identity cannot have $\|x^n\|^{1/n}$ tending to zero uniformly in the unit ball of $A$.

## 1. Introduction.

In this paper we show that an element $x$ in a Banach algebra $A$ with bounded left approximate identity may be factorized as $x = a^n y_n$ for some $a, y_1, y_2, \ldots$ in $A$ with some control of the growth of the sequence of norms $\|y_1\|, \|y_2\|, \ldots$ Like all factorization results concerning bounded approximate identities (see [2], [6]) the method is an adaption of that of P. J. Cohen [3]. P. C. Curtis and H. Stetkaer [5] have shown that for each $x$ in $A$ and each positive integer $n$ there are $a$, $y$ in $A$ such that $x = a^n y$. J. K. Miziotek, T. Müldner, and A. Rek [8] have proved that a radical Banach algebra with a bounded approximate identity cannot satisfy a condition that forces the growth of the products $\|x_1 x_2 \ldots \ldots x_n\|^{1/n}$ uniformly to zero for certain sequences $(x_n)$. We strengthen this result by showing that if $A$ is a radical Banach algebra for which there is a positive sequence $(a_n)$ converging to zero such that, for each $x$ in $A$, $\liminf \|x^n\|^{1/n}/a_n$ is finite, then $A$ does not have a bounded left approximate identity. We obtain this from the factorization $x = a^n y_n$.

If $A$ is a Banach algebra recall that $A$ has a *bounded left approximate identity* [for a Banach-$A$-module $X$] *bounded by $d$* if for each finite subset $\{x_1, \ldots, x_n\}$ of $A$ [of $X$] and $\varepsilon > 0$, there is an $e$ in $A$ such that $\|e\| \leqslant d$ and $\|x_j - ex_j\| < \varepsilon$ for $j = 1, 2, \ldots, n$. This form of the definition is equivalent to the usual form that there is a net $\{e(\lambda): \lambda\in\Gamma\}$ in $A$ bounded by $d$ such that $x = \lim e(\lambda)x$ for all $x$ in $A$. The former definition is more convenient for our applications as it simplifies the notation slightly. For a discussion of bounded approximate identities and known results