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Local convergence of convolution integral

by

JÖRGEN LÖFSTRÖM (Göteborg, Sweden)

Abstract. Writing $f(x) = \lambda^d f(\lambda x)$ we consider the convolution integral $f_\lambda * \varphi$ on open subsets Ω of $\boldsymbol{R}^d$. Under given conditions on f we characterize the class of all functions φ for which $\lambda^s f_\lambda * \varphi$ is bounded in $L_p(\Omega)$ as a function of λ.

0. Introduction. Let f be a given kernel and put $f_\lambda(x) = \lambda^d f(\lambda x)$. Moreover, let Ω be an open subset of the d-dimensional Euclidean space $\boldsymbol{R}^d$. Then we shall consider the space $A_p^s(\Omega)$ of all $\varphi \in L_p$ such that

$$\|f_\lambda * \varphi;\, \Omega\|_p = \mathcal{O}(\lambda^{-s}), \qquad \lambda \to \infty.$$

Here $\|\Psi;\, \Omega\|_p$ denotes the L_p-norm on the set Ω. We let $A^s_{p\,\mathrm{loc}}(\Omega)$ be the set of all $\varphi \in A_p^s(U)$ for all open bounded sets U such that the closure of U is contained in Ω.

The case $\Omega = \boldsymbol{R}^d$ has been treated by many authors. See, for instance, Butzer [1], Butzer–Nessel [3], [4], Löfström [11], Nessel [12], Shapiro [16]. It turns out that $A_p^s(\boldsymbol{R}^d)$ is a Besov space for most values of the parameter s. Thus $A_p^s(\boldsymbol{R}^d)$ is characterized by Lipschitz or Zygmund conditions on the derivatives of φ. These conditions are global conditions, i.e. conditions on the regularity of φ on the entire space $\boldsymbol{R}^d$.

The aim of this paper is to study the case where Ω is an open and bounded subset of $\boldsymbol{R}^d$. We shall characterize $A^s_{p\,\mathrm{loc}}(\Omega)$ by means of local regularity conditions on Ω (local Besov spaces). Comparing to the global theory mentioned above, we shall need an extra condition on f. In the case $p = 1$ this condition reads

$$\int_{|x|>t} |f(x)|\, dx = \mathcal{O}(t^{-s}), \qquad t \to \infty. \tag{1}$$

We prove that this condition must be satisfied if $A^s_{p\,\mathrm{loc}}(\Omega)$ is defined by local regularity conditions. If (1) or its analogue in the case $p \geqslant 1$ is not satisfied, we show that global conditions on the derivatives of φ of low order combined by local conditions on the derivatives of φ of higher order imply that $A^s_{p\,\mathrm{loc}}(\Omega)$.

We have the ambition to develop a general theory. Thus we shall assume that $f\hat{}(\xi) = H(\xi)g\hat{}(\xi)$, where $H(\xi)$ is a general positive homogeneous function, which is infinitely differentiable for $\xi \neq 0$. Applications are given when $f\hat{}(\xi) = 1-F(H(\xi))$, where F is a given function defined on the positive real axis. We consider some concrete cases, for instance,

$$F(u) = \exp(-u) \quad \text{(generalized Gauss–Weierstrass kernel)},$$

$$F(u) = (1-u)_+^a \quad \text{(Riesz kernel)}.$$

A few results, similar to ours in the one-dimensional case, can be found in Butzer–Nessel [3]. We also mention here the paper by Sunouchi [17], where similar questions on the torus have been suggested. See also de Vore [5] and references given there.

The plan of the paper is the following: In Section 1 we introduce the local analogues of the Besov spaces and related spaces. In Section 2 we prove an alternative definition of these spaces, using simple elements from the theory of pseudo-differential equations as developed in Kohn–Nirenberg [10] and Hörmander [8]. In Section 3 we prove direct theorems, i.e., we find subspaces of $A^s_{p\,\mathrm{loc}}(\Omega)$. In Section 4 we prove converse results, i.e., we give necessary conditions for $\varphi \in A^s_{p\,\mathrm{loc}}(\Omega)$. In Section 5 we study condition (1) and its analogues in the case $p > 1$. Finally, we consider some general examples in Section 6.

1. Local Besov spaces and related spaces. In this section we introduce the basic spaces used in this paper. These spaces consist of functions or distributions defined on an open subset Ω of the d-dimensional Euclidean space $\boldsymbol{R}^d$. However, we first give the definitions of the "global" spaces corresponding to the case $\Omega = \boldsymbol{R}^d$.

The space of infinitely differentiable functions on $\boldsymbol{R}^d$ with rapidly decreasing derivatives will be denoted by $\mathscr{S}$. The dual space $\mathscr{S}'$ is the space of tempered distributions.

In the definition of the global Besov space B^s_p we shall need a standard function $\sigma \in \mathscr{S}$. The Fourier transform $\sigma\hat{}$ is positive on the annulus $2^{-1} < |\xi| < 2$ and vanishes outside this annulus. Moreover, we assume that $\sum_j \sigma\hat{}(2^{-j}\xi) = 1$ for $\xi \neq 0$. Then we put $\sigma_j(x) = 2^{jd}\sigma(2^j x)$ for $j > 0$. Then $\sigma_j\hat{}(\xi) = \sigma\hat{}(2^{-j}\xi)$. Moreover, we put $\sigma_0\hat{}(\xi) = 1-\sum_{j>0}\sigma\hat{}(2^{-j}\xi)$.

Let $\|\varphi\|_p$ stand for the L_p-norm on $\boldsymbol{R}^d$. Then B^s_p consists of all $\varphi \in \mathscr{S}'$ such that $\sigma_j * \varphi \in L_p$ for all $j \geq 0$ and such that

$$\|\varphi\|^s_p = \sup_{j\geq 0} 2^{js}\|\sigma_j * \varphi\|_p \tag{1}$$

is finite. Here s is an arbitrary real number. Note that (1) is defined for all real values of s but we shall only consider the case $s \geq 0$. Then $L_p \subset B^s_p$.

Our definition of B^s_p is due to Peetre. There are other more explicite definitions, which, however, are less efficient in the proof we shall give. For more detailed discussions of the different definitions of B^s_p we refer the reader to Peetre [13], [14], [15]. See also Grevholm [7], Taiblesson [18].

Now we shall define the "local" Besov space $B^s_p(\Omega)$. Thus let Ω be an open subset of $\boldsymbol{R}^d$. The space $B^s_p(\Omega)$ consists of all restrictions to Ω of distributions in the global space B^s_p. More precisely, $\varphi \in B^s_p(\Omega)$ if an only if there is a $\Psi \in B^s_p$, such that $\Psi = \varphi$ on Ω. We write

$$\|\varphi;\ \Omega\|^s_p = \inf\{\|\Psi\|^s_p\colon\ \Psi = \varphi \text{ on } \Omega,\ \Psi \in B^s_p\}. \tag{2}$$

The space $B^s_{p\,\mathrm{loc}}(\Omega)$ consists of all φ such that $\varphi \in B^s_p(U)$ for all open sets U such that $U \subset\subset \Omega$, i.e. such that the closure of U is a compact subset of Ω. We shall give an alternative definition of $B^s_p(\Omega)$ in Theorem 1 below.

The norm on $L_p(\Omega)$ will be denoted by $\|\varphi;\ \Omega\|_p$. Thus

$$\|\varphi;\ \Omega\|_p = \Big(\int_\Omega |\varphi(x)|^p\,dx\Big)^{1/p}, \quad 1 \leq p \leq \infty, \tag{3}$$

with the usual convention when $p = \infty$. With this notation we have the following result.

THEOREM 1. *Suppose that $\varphi \in L_p$ and that $s \geq 0$. Then $\varphi \in B^s_{p\,\mathrm{loc}}(\Omega)$ if and only if*

$$\sup_{j\geq 0} 2^{js}\|\sigma_j * \varphi;\ U\|_p \tag{4}$$

is finite for all open subsets U such that $U \subset\subset \Omega$.

The proof of this theorem will be given in the next section. Here we proceed with the definition of some spaces which are closely related to B^s_p and $B^s_p(\Omega)$.

We shall consider a function $H = H_M$ which is infinitely differentiable and positive outside the origin. We also assume that $H_M(\xi)$ is positively homogeneous of order M, i.e. $H(t\xi) = t^M H(\xi)$ for $t > 0$. If $\varphi \in \mathscr{S}$, we define $H(D)\varphi$ by the formula $(H(D)\varphi)\hat{}(\xi) = H(\xi)\varphi\hat{}(\xi)$. We now let $\bar{D}^H_p$ be the space of all $\varphi \in L_p$ such that there exists a sequence $\varphi_n \in \mathscr{S}$ such that $\varphi_n \to \varphi$ in L_p and $\sup_n \|H(D)\varphi_n\|_p < \infty$. Then we put

$$\|\varphi\|^H_p = \inf\sup_n \|H(D)\varphi_n\|_p + \|\varphi\|_p. \tag{5}$$

The local space $\bar{D}^H_p(\Omega)$ consists of all restrictions to Ω of the distributions in $\bar{D}^H_p$. In analogy with (2) we write

$$\|\varphi;\ \Omega\|^H_p = \inf\{\|\psi\|^H_p\colon\ \psi = \varphi \text{ on } \Omega,\ \psi \in \bar{D}^H_p\}. \tag{6}$$

The space $\bar{D}^H_{p\,\mathrm{loc}}(\Omega)$ consists of all φ, such that $\varphi \in \bar{D}^H_p(U)$ for all open sets U, such that $U \subset\subset \Omega$. We then have the following result.

THEOREM 2. *Suppose that* $\varphi \in L_p$ *and that* $M \geqslant 0$. *Suppose that* Ω *is a bounded open subset of* $\mathbf{R}^d$. *If for every* $U \subset\subset \Omega$ *there is a sequence* $\varphi_n \in L_p$ *such that* $\varphi_n \to \varphi$ *in* L_p *and*

$$\sup_n \|H(D)\varphi_n;\ U\|_p < \infty, \tag{7}$$

then it follows that $\varphi \in \bar{D}^H_{p\,\mathrm{loc}}(\Omega)$.

This theorem will also be proved in the next section. Let us conclude this section with a remark on the space $\bar{D}^H_p$. If $1 < p < \infty$, then $\bar{D}^H_p$ is the domain of the closure of the operator $H(D)$ in L_p. Let D^H_p denote this domain. Thus D^H_p is the space of all $\varphi \in L_p$ such that there is a sequence $\varphi_n \to \varphi$ in L_p, $\varphi_n \in \mathscr{S}$ such that $H(D)\varphi_n$ converges in L_p to a function which we shall denote $H(D)\varphi$. Then we have

$$\bar{D}^H_p = D^H_p \quad \text{if} \quad 1 < p < \infty.$$

If $p = 1$ or $p = \infty$, the space $\bar{D}^H_p$ is the relative completion in L_p of D^H_p. (See Butzer–Berens [2].)

2. Proofs of the theorems of the previous section. Although Theorem 1 could be proved by means of Theorem 2, we shall prefer to prove Theorem 1 first, mainly because the proof of Theorem 1 may throw some light on the proof of Theorem 2.

Proof of Theorem 1. Let U be a fixed non-empty open subset of Ω, such that $U \subset\subset \Omega$. Let V be an open subset of U such that $V \subset\subset U$. First we shall prove that if

$$\sup_{j \geqslant 0} 2^{js} \|\sigma_j * \varphi;\ U\|_p < \infty$$

and if $\varphi \in L_p$, then $\varphi \in B^s_p(V)$.

Let b be a fixed function in $\mathscr{S}$, such that $b = 1$ on V and b has compact support W in U. Put $\Psi = b\varphi$. Then

$$\|\varphi;\ V\|^s_p \leqslant \|\Psi\|^s_p = \sup_{j \geqslant 0} 2^{js} \|\sigma_j * \Psi\|_p.$$

Thus the desired result will follow if we can show that

$$\|\sigma_j * \Psi\|_p \leqslant C\big(\max_{j-2 \leqslant i \leqslant j+2} \|\sigma_i * \varphi;\ U\|_p + 2^{-js}\|\varphi\|_p\big). \tag{1}$$

Since $\|\sigma_j * \Psi\|_p \leqslant C\|\varphi\|_p$, we have only to prove (1) for $j > 2$.

In order to prove (1) we note that

$$\sigma_j * \Psi(x) = \int_{\mathbf{R}^d} \sigma_j(y)\, b(x-y)\, \varphi(x-y)\, dy.$$

By Taylor's formula we have that (with $N > s$)

$$b(x-y) = \sum_{|\alpha|<N} (\alpha!)^{-1} (D^\alpha b)(x) \cdot y^\alpha + \varrho(x, y),$$

where $|\varrho(x, y)| \leqslant C|y|^N$. Writing $\sigma^{(\alpha)}_j(y) = (2^j y)^\alpha \sigma_j(y)$ we get that

$$\sigma_j * \Psi = \sum_{|\alpha|<N} (\alpha!)^{-1} 2^{-|\alpha|j} (D^\alpha b) \cdot \sigma^{(\alpha)}_j * \varphi + R\varphi,$$

where

$$R\varphi(x) = \int_{\mathbf{R}^d} \varrho(x, y)\, \sigma_j(y)\, \varphi(x-y)\, dy.$$

Since $\| |y|^N \sigma_j \|_1 \leqslant C 2^{-jN} \leqslant C 2^{-js}$ for $j \geqslant 0$, we get that $\|R\varphi\|_p \leqslant C 2^{-js}\|\varphi\|_p$ and therefore we conclude that

$$\|\sigma_j * \Psi\|_p \leqslant C\big(\max_{|\alpha|<N} \|\sigma^{(\alpha)}_j * \varphi;\ W\|_p + 2^{-js}\|\varphi\|_p\big). \tag{2}$$

Now (1) follows from the following lemma:

LEMMA 1. *Suppose that* $k \in L_1$ *and that*

$$\sup_{t>0} t^m \int_{|x|>t} |k(x)|\, dx < \infty. \tag{3}$$

Write $k_\lambda(x) = \lambda^d k(\lambda x)$ *and assume that* $W \subset\subset U$. *Let* ε *be the distance from the boundary of* U *to* W. *Then*

$$\|k_\lambda * \varphi;\ W\|_p \leqslant A\big(\|\varphi;\ U\|_p + (\varepsilon\lambda)^{-m}\|\varphi\|_p\big),$$

where A *depends on* k *only.*

Proof of Lemma 1. We have that

$$k_\lambda * \varphi(x) = \int_{|y|<\varepsilon} k_\lambda(y)\varphi(x-y)\, dy + \int_{|y|>\varepsilon} k_\lambda(y)\varphi(x-y)\, dy.$$

If $x \in W$, we clearly have $x - y \in U$ in the first integral. Thus the $L_p(W)$-norm of the first integral is bounded by $\|k\|_1 \|\varphi;\ U\|_p$. The $L_p(W)$-norm of the second integral is bounded by

$$\int_{|y|>\varepsilon} |k_\lambda(y)|\, dy\, \|\varphi\|_p = \int_{|x|>\varepsilon\lambda} |k(x)|\, dx\, \|\varphi\|_p.$$

This term is therefore bounded by a constant (depending on k only) times $(\varepsilon\lambda)^{-m}\|\varphi\|_p$. This completes the proof of Lemma 1.

In order to get from estimate (2) to the desired estimate (1) we use the lemma with $k(y) = \sigma^{(\alpha)}(y) = y^\alpha \sigma(y)$. Then $k \in \mathscr{S}$ and (3) holds for every m. Since $\sigma_i^{(\alpha)}*\sigma_j = 0$ unless $j-2\leqslant i\leqslant j+2$, we then get that

$$\|\sigma_j^{(\alpha)}*\varphi;\, W\|_p \leqslant \sum_{j-2\leqslant i\leqslant j+2} \|\sigma_j^{(\alpha)}*\sigma_i*\varphi;\, W\|_p$$
$$\leqslant C\big(\max_{j-2\leqslant i\leqslant j+2} \|\sigma_i*\varphi;\, U\|_p + 2^{-js}\|\varphi\|_p\big).$$

This estimate implies (1). Note that the constant C appearing in the estimates above depends on ε.

Next we shall prove that if $V\subset\subset U$ and if $\varphi\in B_p^s(U)$, then

$$\sup_{j\geqslant 0} 2^{js}\|\sigma_j*\varphi;\, V\|_p < \infty. \tag{4}$$

For that purpose we choose a $\Psi\in B_p^s$ such that $\Psi=\varphi$ on U and $\|\Psi\|_p^s \leqslant 2\|\varphi;\, U\|_p^s$. Let $c\in\mathscr{S}$ be equal to 1 on the set W (the support of b) and assume that c has compact support in U. Then $\|c\Psi\|_p^s\leqslant C\|\Psi\|_p^s$ and $\|c\Psi\|_p \leqslant \|\varphi\|_p$. Replacing if necessary Ψ by $c\Psi$ we can therefore assume that

$$\|\Psi\|_p^s\leqslant C\|\varphi;\, U\|_p^s,\qquad \|\Psi\|_p\leqslant\|\varphi\|_p.$$

Now we have that

$$\|\sigma_j*\varphi;\, V\|_p\leqslant \|b(\sigma_j*\varphi)\|_p\leqslant C\Big(\|\sigma_j*\Psi\|_p+\big\|b\big(\sigma_j*(\varphi-\Psi)\big)\big\|_p\Big),$$

and

$$b(x)\big(\sigma_j*(\varphi-\Psi)\big)(x) = b(x)\int\sigma_j(y)\big(\varphi(x-y)-\Psi(x-y)\big)\,dy.$$

In this integral we have $x-y\notin U$, since $\varphi=\Psi$ on U. Let W be as above and let δ be the distance between the boundary of U and W. Then we must have $|y|>\delta$ in the integral above. Thus it follows that

$$\big\|b\big(\sigma_j*(\varphi-\Psi)\big)\big\|_p\leqslant C\int_{|y|>\delta}|\sigma_j(y)|\,dy\,\|\varphi-\Psi\|_p\leqslant C(\delta 2^j)^{-s}\|\varphi\|_p.$$

It follows that

$$2^{js}\|\sigma_j*\varphi;\, V\|_p\leqslant C(\|\Psi\|_p^s+\|\varphi\|_p)\leqslant C(\|\varphi;\, U\|_p^s+\|\varphi\|_p).$$

This proves (4) and completes the proof of Theorem 1.

In the proof of Theorem 2 we shall use simple elements from the theory of pseudo-differential operators as developed in Kohn–Nirenberg [10], and Hörmander [8].

Proof of Theorem 2. We shall use the notations of the proof of Theorem 1. The main point in the proof is to show

$$\|H(D)\Psi\|_p\leqslant C\big(\|H(D)\varphi;\, U\|_p+\|\varphi\|_p\big), \tag{5}$$

if $\Psi=b\varphi$. In fact, this will imply that if $\varphi_n\to\varphi$ in L_p, then $\|\varphi;\, V\|_p^H\leqslant\|\Psi\|_p^H$

$\leqslant \sup_n\|H(D)(b\varphi_n)\|_p$ and thus (5) implies

$$\|\varphi;\, V\|_p^H\leqslant C\big(\sup_n\|H(D)\varphi_n;\, U\|_p+\|\varphi\|_p\big)$$

which in turn implies Theorem 2.

In order to prove (5) we shall first modify the operator $H(D)$. Let ζ be an infinitely differentiable function, such that $\zeta(\xi)=1$ for $|\xi|>1$ and $\zeta(\xi)=0$ for $|\xi|<1/2$. Put $K(\xi)=\zeta(\xi)H(\xi)$ and define $K(D)\Psi$ in the natural way. We also write $\mu\hat{\ }(\xi)=H(\xi)-K(\xi)=\big(1-\zeta(\xi)\big)H(\xi)$. Then it is easily seen that $\mu\in L_1$. (This follows for instance from Löfström [11], cf. Corollary 3, Section 5.) Therefore, $\big\|\big(H(D)-K(D)\big)\Psi\big\|_p\leqslant C\|\Psi\|_p \leqslant C\|\varphi\|_p$. Thus (5) will follow if we can prove

$$\|K(D)\Psi\|_p\leqslant C\big(\|H(D)\varphi;\, U\|_p+\|\varphi\|_p\big), \tag{6}$$

for all $\varphi\in\mathscr{S}$. We have that

$$\big(K(D)\Psi\big)\hat{\ }(\xi) = K(\xi)\int_{R^d} b\hat{\ }(\eta)\varphi\hat{\ }(\xi-\eta)\,d\eta. \tag{7}$$

Now we expand $K(\xi)=K(\xi-\eta+\eta)$ in a Taylor series:

$$K(\xi) = \sum_{|\alpha|<N}(\alpha!)^{-1}K^{(\alpha)}(\xi-\eta)\eta^\alpha + r(\xi-\eta,\eta), \tag{8}$$

where $K^{(\alpha)}(\xi)=D^\alpha K(\xi)$ and

$$|D_\tau^\beta D_\eta^\gamma r(\tau,\eta)|\leqslant C|\eta|^{N-|\gamma|}\big(1+\min(|\tau|,|\eta|)\big)^{M-N-|\beta|} \tag{9}$$

if $N>M$. Here the constant C depends on β, γ and N.

Introducing (8) in (7) we find that

$$K(D)\Psi = \sum_{|\alpha|<N}(\alpha!)^{-1}(D^\alpha b)\cdot K^{(\alpha)}(D)\varphi + R\varphi, \tag{10}$$

where the operator R is defined by

$$(R\varphi)\hat{\ }(\xi) = \int_{R^d} b\hat{\ }(\eta)r(\xi-\eta,\eta)\varphi\hat{\ }(\xi-\eta)\,d\eta,$$

i.e.,

$$R\varphi(x) = \int_{R^d}\varrho(x,y)\varphi(y)\,dy,$$

where

$$\varrho(x,y) = (2\pi)^{-d}\int\int e^{i(\langle x,\eta\rangle+\langle x-y,\tau\rangle)}b\hat{\ }(\eta)r(\tau,\eta)\,d\tau\,d\eta.$$

Since $b\hat{\ }(\eta)\to 0$ faster than any polynomial as $|\eta|\to\infty$, we get from (9) that

$$|D_\tau^\beta D_\eta^\gamma b\hat{\ }(\eta)r(\tau,\eta)|\leqslant C(1+|\eta|)^{-n}\big(1+\min(|\tau|,|\eta|)\big)^{M-N-|\beta|},$$

for any number n. It follows that if N is large enough, $|x|^a|x-y|^b|\varrho(x,y)|$ is a bounded function for any choice of a and b such that, for instance, $0 \leqslant a \leqslant 2d$, $0 \leqslant b \leqslant 2d$. Thus

$$|\varrho(x,y)| \leqslant C(1+|x|)^{-2d}(1+|x-y|)^{-2d}$$

which implies $\|R\varphi\|_p \leqslant C\|\varphi\|_p$. Thus we get from (10) that

(11) $$\|K(D)\Psi\|_p \leqslant C\big(\max_{|\alpha|<N} \|K^{(\alpha)}(D)\varphi;\, W\|_p + \|\varphi\|_p\big).$$

Now we shall use following simple version of a well-known inequality by Kolmogorov.

LEMMA 2. *For any $\alpha \neq 0$ we have that*

$$\|K^{(\alpha)}(D)\varphi;\, W\|_p \leqslant C_\alpha\big(\|H(D)\varphi;\, U\|_p + \|\varphi\|_p\big),$$

provided that $W \subset\subset U$.

Combining this result with (11) we clearly get (6). This completes the proof of Theorem 2. It remains, however, to prove Lemma 2.

Proof of Lemma 2. First we note that

$$\|K^{(\alpha)}(D)\varphi;\, W\|_p \leqslant \sum_j \|K^{(\alpha)}(D)\sigma_j * \varphi;\, W\|_p.$$

Since $K^{(\alpha)}(\xi)$ vanishes for $|\xi| \leqslant 1/2$, the terms in the sum vanish for $j < -2$. For $-2 \leqslant j \leqslant 2$ we have $\|K^{(\alpha)}(D)\sigma_j\|_1 \leqslant C$ and thus the terms in the sum for which $-2 \leqslant j \leqslant 2$ can be estimated by $C\|\varphi\|_p$. It remains to estimate the terms for which $j > 2$. But $K^{(\alpha)}(\xi) = H^{(\alpha)}(\xi) = D^\alpha H(\xi)$ for $|\xi| > 1$ so that $K^{(\alpha)}(D)\sigma_j = H^{(\alpha)}(D)\sigma_j$ for $j > 2$. Now let us write

$$\gamma\hat{\ }(\xi) = \sigma\hat{\ }(\xi) H^{(\alpha)}(\xi)/H(\xi)$$

and $\gamma_j\hat{\ }(\xi) = \gamma\hat{\ }(2^{-j}\xi)$. Then $H^{(\alpha)}(D)\sigma_j = 2^{-j|\alpha|}H(D)\gamma_j$. In order to get the estimate of Lemma 2 it is therefore sufficient to show that

(12) $$\|H(D)\gamma_j * \varphi;\, W\|_p \leqslant C\big(\|H(D)\varphi;\, U\|_p + \|\varphi\|_p\big),$$

whenever $W \subset\subset U$.

Suppose that ε is defined as in Lemma 1 and put $a(x) = 1 - \zeta(x)$. Then $a(x) = 0$ for $|x| > 1$ and $a(x) = 1$ for $|x| < 1/2$. Put $a^\varepsilon(x) = a(x/\varepsilon)$ and $\zeta^\varepsilon(x) = \zeta(x/\varepsilon)$. We write $H(D)\gamma_j * \varphi = (a^\varepsilon\gamma_j) * H(D)\varphi + \big(H(D)(\zeta^\varepsilon\gamma_j)\big) * \varphi$. Just as in the proof of Lemma 1 we see that

$$\|(a^\varepsilon\gamma_j) * H(D)\varphi;\, W\|_p \leqslant C\|H(D)\varphi;\, U\|_p.$$

Thus (12) will follow if we can show that

(13) $$\|H(D)(\zeta^\varepsilon\gamma_j)\|_1 \leqslant C_\varepsilon \quad \text{for} \quad j > 2.$$

In order to prove (13) we note that

$$\|H(D)(\zeta^\varepsilon\gamma_j)\|_1 \leqslant C \max_{|\nu|\leqslant n} \|D^\nu(\zeta^\varepsilon\gamma_j)\|_1 \quad \text{if} \quad n > M.$$

Let us write $r = 2^j\varepsilon$ and $u(x) = \zeta(x/r)\gamma(x)$. Then $\|D^\nu(\zeta^\varepsilon\gamma_j)\|_1 = 2^{j|\nu|}\|D^\nu u\|_1$. But since $\gamma \in \mathcal{S}$ and $\zeta(x) = 0$ if $|x| < 1/2$, we have that

$$\|D^\nu u\|_1 \leqslant C \int_{|x|\geqslant r/2} |\gamma(x)|\,dx \leqslant Cr^{-n}.$$

Thus it follows that

$$\|H(D)(\zeta^\varepsilon\gamma_j)\|_1 \leqslant C 2^{nj} r^{-n} = C\varepsilon^{-n}.$$

This proves (11). (A more detailed calculation would show that the constant C_ε in (11) is $O(\varepsilon^{-M})$.) ■

From the proof of Theorems 1 and 2 we get the following immediate corollary.

COROLLARY 1. *Suppose that $\varphi \in \bar{D}^H_{p\,\mathrm{loc}}(\Omega)$ (or $\varphi \in B^s_{p\,\mathrm{loc}}(\Omega)$). For every $U \subset\subset \Omega$ there is a $\Psi \in \bar{D}^H_p$ (or $\Psi \in B^s_p$) such that $\Psi = \varphi$ on U and Ψ has compact support in Ω.*

Proof. If $\Psi \in D^H_p$ and $\Psi = \varphi$ on U, then we can replace Ψ by $b\Psi$ and this latter function will have the desired properties.

3. Direct theorems. For a given kernel f we shall let $A^s_p(\Omega)$ denote the space of all $\varphi \in L_p$ such that $\|f_\lambda * \varphi;\, \Omega\|_p = O(\lambda^{-s})$ as $\lambda \to \infty$. The space $A^s_{p\,\mathrm{loc}}(\Omega)$ consists of all φ such that $\varphi \in A^s_p(U)$ for every open set U such that $U \subset\subset \Omega$. The objective of this section is to find large subspaces of $A^s_{p\,\mathrm{loc}}(\Omega)$.

First we recall the situation in the global case. Let us take $p = \infty$ for simplicity. Assume that $f\hat{\ }(\xi) = H(\xi)g\hat{\ }(\xi)$ and that f and g are bounded measures. Then $f_\lambda * \varphi = \lambda^{-M} g_\lambda * H(D)\varphi$. Thus

$$\|f_\lambda * \varphi\|_\infty \leqslant C\lambda^{-M}\|H(D)\varphi\|_\infty.$$

Then it follows that

(1) $$\|f_\lambda * \varphi\|_\infty \leqslant C\lambda^{-s}\|\varphi\|^s_\infty \quad \text{for} \quad 0 < s < M.$$

This is a well-known result (see Löfström [11], Shapiro [16]). In the proof of the main theorem below we shall reproduce the simple proof of (1).

Now we shall see how one can get a local analogue of global result (1). Thus let Ω be an open, bounded subset of $\boldsymbol{R}^d$. We shall assume that there is a positive number m, such that

(2) $$\sup_{t>0} t^m \int_{|x|>t} |f(x)|\,dx < \infty.$$

Then we shall prove that

(3) $$L_\infty \cap B^s_{\infty\,\mathrm{loc}}(\Omega) \subset A^s_{\infty\,\mathrm{loc}}(\Omega) \quad \text{if} \quad 0 < s < M,\ s \leqslant m.$$

This is a particular case of the main theorem, but we prefer to give the proof here. Thus assume that $V \subset\subset \Omega$ and choose U such that $V \subset\subset U \subset\subset \Omega$. If $\varphi \in L_\infty \cap B^s_{\infty\,\mathrm{loc}}(\Omega)$, we can find a function $\Psi \in L_\infty \cap B^s_\infty$ such that $\Psi = \varphi$ on the set U. Clearly, $f_\lambda * \varphi = f_\lambda * \Psi + f_\lambda * (\varphi - \Psi)$. Now let ε be the distance between the boundary of U to the set V. If $x \in V$, we then have

$$f_\lambda * (\varphi - \Psi)(x) = \int_{|y|>\varepsilon} f_\lambda(y)\big(\varphi(x-y) - \Psi(x-y)\big)\,dy,$$

since if $x \in V$ and $x - y \notin U$ we must have $|y| > \varepsilon$. Thus it follows that

$$\|f_\lambda * (\varphi - \Psi); V\|_\infty \leqslant \int_{|y|<\varepsilon} |f_\lambda(y)\,dy|\,\|\varphi - \Psi\|_\infty.$$

Using (2) we get that

$$\int_{|y|>\varepsilon} |f_\lambda(y)|\,dy = \int_{|y|>\varepsilon\lambda} |f(x)|\,dx \leqslant C(\varepsilon\lambda)^{-m}.$$

Thus (1) implies that

$$\|f_\lambda * \varphi; V\|_\infty \leqslant \|f_\lambda * \Psi\|_\infty + C(\varepsilon\lambda)^{-m}\|\varphi - \Psi\|_\infty$$
$$\leqslant C\big(\lambda^{-s}\|\Psi\|^s_\infty + (\varepsilon\lambda)^{-m}\|\varphi - \Psi\|_\infty\big).$$

This implies (3).

Before we proceed with our main theorem, we shall comment on condition (2). If $m < M$ and if m is as large as possible such that (3) holds, we only get the inclusion

$$L_\infty \cap B^s_{\infty\,\mathrm{loc}}(\Omega) \subset A^m_{\infty\,\mathrm{loc}}(\Omega) \quad \text{if} \quad m < s < M.$$

Thus an improvement in the local regularity of φ will not give an improvement of the local convergence rate of $f_\lambda * \varphi$. We shall now show that it is impossible to get rid of this defect.

THEOREM 3. *Suppose that there exist two open (non-empty) bounded sets U and V such that $V \subset\subset U$ and such that*

(4) $$L_\infty \cap B^\sigma_\infty(U) \subset A^s_\infty(V),$$

for some fixed σ and s. Then (2) holds with $m = s$. In particular, an inclusion of the form $L_\infty \cap B^\sigma_{\infty\,\mathrm{loc}}(\Omega) \subset A^s_{\infty\,\mathrm{loc}}(\Omega)$ (fixed Ω, σ and s) will imply (2) with $m = s$.

Proof. By the uniform boundedness theorem we see that (4) implies

(5) $$\|f_\lambda * \varphi; V\|_\infty \leqslant C\lambda^{-s}(\|\varphi; U\|^\sigma_\infty + \|\varphi\|_\infty).$$

It is easily verified that this estimate holds if we translate U and V. Thus we can assume that 0 is an interior point in V. Moreover, we can assume that $U \subset \{x\colon |x| < R\}$ for some $R > 0$. Now let Y denote the space of all $\varphi \in L_\infty$, such that $\varphi(x) = 0$ for $|x| \leqslant R$. Clearly, Y is a Banach space in the maximum norm. The dual norm on Y is

$$\|f\|_{Y'} = \int_{|y|>R} |f(y)|\,dy.$$

For $\varphi \in Y$ we now define the functional F by the formula $F(\varphi) = f_\lambda * \varphi(0)$. Since $\|\varphi; U\|^\sigma_p = 0$ for all $\varphi \in Y$, we get from (5) that

$$|F(\varphi)| \leqslant C\lambda^{-s}\|\varphi\|_Y.$$

Thus

$$\int_{|y|>R} |f_\lambda(-y)|\,dy \leqslant C\lambda^{-s},$$

i.e.

$$\int_{|y|>R\lambda} |f(x)|\,dx \leqslant C\lambda^{-s}.$$

This implies (2). ■

Now we shall give the main theorem of this section. We shall work with the space M_p of all Fourier multipliers on L_p. Thus M_p is the space of all tempered distributions f such that $\|f * \varphi\|_p \leqslant C\|\varphi\|_p$. The infimum of all possible constants C is a norm on M_p. We shall denote this norm by $|f^\wedge|_p$. Note that $M_1 = M_\infty$ = space of bounded measures. We shall use freely and without explicite reference various well-known properties of the M_p-spaces. The reader is referred to Hörmander [8].

We shall also work with certain subspaces M^m_p of M_p, which are defined by conditions analogous to (2). Let ζ be an infinitely differentiable function such that $\zeta(x) = 1$ if $|x| > 1$ and $\zeta(x) = 0$ if $|x| < 1/2$. Then we define the space M^m_p by means of the norm

(6) $$|f^\wedge|^m_p = \sup_{t>0} t^m |(\zeta^t f)^\wedge|_p + |f^\wedge|_p.$$

Note that if $f^\wedge \in M_p$, then $(\zeta^t f)^\wedge \in M_p$. In fact, we have that

$$(\zeta^t f)^\wedge(\xi) = \int \zeta_t^\wedge(\eta) f^\wedge(\xi - \eta)\,d\eta = \zeta_t^\wedge * f^\wedge(\xi).$$

Since M_p is translation invariant, we get $|(\zeta^t f)^\wedge|_p \leqslant \|\zeta^\wedge\|_1 |f^\wedge|_p$.

Here is the main theorem of this section.

THEOREM 4. *Suppose that H_M is homogeneous of order $M > 0$ and that $g^\wedge \in M_p$. Put $f^\wedge(\xi) = H_M(\xi) g^\wedge(\xi)$ and assume that $f^\wedge \in M^m_p$. Then we have the following inclusions*

(7) $$L_p \cap \bar{D}^{H_M}_{p\,\mathrm{loc}}(\Omega) \subset A^M_{p\,\mathrm{loc}}(\Omega) \quad \text{if} \quad m \geqslant M,$$

(8) $$L_p \cap B^s_{p\,\mathrm{loc}}(\Omega) \subset A^s_{p\,\mathrm{loc}}(\Omega) \quad \text{if} \quad m \geqslant s \quad \text{and} \quad 0 < s < M.$$

Proof. Assume first that $\varphi \in L_p \cap D_{p\,\mathrm{loc}}^{H_M}(\Omega)$. Then there is a $\Psi \in \bar{D}_p^{H_M}$ such that $\Psi = \varphi$ on a given set $U \subset\subset \Omega$. Let $V \subset\subset U$ and write $f_\lambda * \varphi = f_\lambda * \Psi + f_\lambda * (\varphi - \Psi)$. If ε is the distance between the boundary of U to the set V, we have $f_\lambda * (\varphi - \Psi) = (\zeta^{2\varepsilon} f_\lambda) * (\varphi - \Psi)$ on V. Since

$$|(\zeta^t f_\lambda)^\wedge|_p = |(\zeta^{t\lambda} f)^\wedge|_p = O\big((t\lambda)^{-m}\big),$$

we therefore conclude that

$$\|f_\lambda * (\varphi - \Psi);\, V\|_p \leqslant C(\varepsilon\lambda)^{-m} \|\varphi - \Psi\|_p. \tag{9}$$

In order to estimate the term $f_\lambda * \Psi$ we choose a sequence $\Psi_n \in \mathcal{S}$ such that $\Psi_n \to \Psi$ in L_p. Then $f_\lambda * \Psi_n = \lambda^{-M} g_\lambda * H(D)\Psi_n$ and thus

$$\|f_\lambda * \Psi;\, V\|_p \leqslant C\lambda^{-M} \sup_n \|H(D)\Psi_n\|_p.$$

Thus we conclude that

$$\|f_\lambda * \varphi;\, V\|_p < C\big(\lambda^{-M} \|\varphi;\, U\|_p^{H_M} + (\varepsilon\lambda)^{-m} \|\varphi - \Psi\|_p\big).$$

This clearly implies (7).

Next we prove (8). Again we continue φ outside U to a function Ψ. In this case $\Psi \in B_p^s$. Clearly, (9) still holds. We have only to show that

$$\|f_\lambda * \Psi\|_p \leqslant C\lambda^{-s} \|\Psi\|_p^s. \tag{10}$$

In order to show this we note that

$$\|f_\lambda * \Psi\|_p \leqslant \sum_{j \geqslant 0} \|f_\lambda * \sigma_j * \Psi\|_p.$$

But $\|f_\lambda * \sigma_j * \Psi\|_p \leqslant C \min\big(\|\sigma_j * \Psi\|_p,\ \lambda^{-M} \|H_M(D)\sigma_j * \Psi\|_p\big)$. Since $\|H_M(D)\sigma_j * \Psi\|_p \leqslant C2^{jM} \|\sigma_j * \Psi\|_p$, we conclude that

$$\|f_\lambda * \Psi\|_p \leqslant C \sum_{j \geqslant 0} \min\big(1,\, (2^j/\lambda)^M\big) 2^{-js} \|\Psi\|_p^s,$$

which implies (10). ■

In the case $m \geqslant M$, Theorem 4 is quite satisfactory since then a local regularity condition on φ implies a corresponding local behaviour of $f_\lambda * \varphi$. But if $m < M$ this is not true. In the case $m < s < M$ we see from Theorem 3 that no local regularity condition on φ will be sufficient to guarantee that $\varphi \in A_{p\,\mathrm{loc}}^s(\Omega)$. However, we can prove that a global regularity condition on the low order derivatives of φ combined with a local regularity condition of higher order derivatives of φ, will imply that $\varphi \in A_p^s(\Omega)$.

THEOREM 5. *Suppose that* $g^\wedge \in M_p$, $f^\wedge = H_M g^\wedge \in M_p^m$, *where* $0 < m < M$. *Let* H_m *and* H_{M-m} *be homogeneous functions of the orders indicated by the subscripts and suppose that* $H_M(\xi) = H_m(\xi) H_{M-m}(\xi)$. *Assume that* $h^\wedge(\xi) = H_m(\xi) g^\wedge(\xi) \in M_p^m$. *Then we have the following inclusions*

$$D_p^{H_{M-m}} \cap D_{p\,\mathrm{loc}}^{H_M}(\Omega) \subset A_{p\,\mathrm{loc}}^M(\Omega), \tag{11}$$

$$B_p^{s-m} \cap B_{p\,\mathrm{loc}}^s(\Omega) \subset A_{p\,\mathrm{loc}}^s(\Omega) \quad \text{if} \quad m < s < M. \tag{12}$$

Proof. First we note that $f_\lambda * \varphi = \lambda^{-(M-m)} h_\lambda * H_{M-m}(D)\varphi$. Thus if $V \subset\subset U \subset\subset \Omega$ we get from the previous proof that

$$\|f_\lambda * \varphi;\, V\|_p \leqslant C\lambda^{-M}\big(\|H_{M-m}(D)\varphi;\, U\|_p^{H_m} + \|H_{M-m}(D)\varphi\|_p\big).$$

This implies (11).

In order to prove (12) we write $\varphi_j = \sigma_j * \varphi$. Then

$$\|f_\lambda * \varphi;\, V\|_p \leqslant \sum_{j \geqslant 0} \|f_\lambda * \varphi_j;\, V\|_p.$$

By Theorem 4 we have that

$$\|f_\lambda * \varphi_j;\, V\|_p \leqslant C\lambda^{-m}\big(\|\varphi_j;\, U\|_p^m + \|\varphi_j\|_p\big).$$

Now it is easily seen that $\|\varphi_j;\, U\|_p^m \leqslant C2^{j(m-s)}$ and $\|\varphi_j\|_p \leqslant C2^{j(m-s)}$ if $\varphi \in B_p^{s-m} \cap B_{p\,\mathrm{loc}}^s(\Omega)$. Thus it follows that

$$\|f_\lambda * \varphi_j;\, V\|_p \leqslant C\lambda^{-m} 2^{j(m-s)}.$$

Using the formula $f_\lambda * \varphi_j = \lambda^{-(M-m)} h_\lambda * H_{M-m}(D)\varphi_j$ and the previous theorem, we get that

$$\|f_\lambda * \varphi_j;\, V\|_p \leqslant C\lambda^{-M}\big(\|H_M(D)\varphi_j;\, U\|_p + \|H_{M-m}(D)\varphi_j\|_p\big).$$

Writing $H_M(D)\varphi_j = 2^{jM} \tilde{\sigma}_j * \varphi_j$, where $\tilde{\sigma}^\wedge(\xi) = H_{M-m}(\xi)\sigma^\wedge(\xi)$ and using Lemma 1 and the fact that $\tilde{\sigma} \in \mathcal{S}$, we see that

$$\|H_M(D)\varphi_j;\, U\|_p \leqslant C2^{j(M-s)},$$

$$\|H_{M-m}(D)\varphi_j\|_p \leqslant C2^{j(M-s)}.$$

Thus

$$\|f_\lambda * \varphi_j;\, V\|_p \leqslant C\lambda^{-M} 2^{j(M-s)}.$$

It follows that

$$\|f_\lambda * \varphi;\, V\|_p \leqslant C \sum_j \min\big((2^j/\lambda)^m,\ (2^j/\lambda)^M\big) 2^{-js}$$

and thus $\|f_\lambda * \varphi;\, V\|_p \leqslant C\lambda^{-s}$. This proves the result. ■

4. Converse theorems. The space M_p is a Banach algebra for pointwise multiplication. In the case $p = 1$ this is the Wiener algebra of Fourier transforms of bounded measures. The characters (i.e. the continuous multiplicative linear functionals) on that algebra are the point-evaluations of the functions. Thus if $\mu^\wedge(\xi) \neq 0$ on a compact set then $1/\mu^\wedge(\xi)$ agrees on that set with the Fourier transform of a bounded measure. This fact is essential in proving converse theorems in the global case. In the case $1 < p < \infty$ the situation is similar. However, the characters in M_p are possibly not only point-evaluations. Therefore it is necessary to replace M_p by the closure in M_p of $\mathscr{S}$. We shall denote this closure by $\mathscr{M}_p$. On $\mathscr{M}_p$ the characters are just the point-evaluations. In fact, every character γ on $\mathscr{M}_p$ is also a character on the space $L_1^\wedge$ of Fourier transforms of L_1-functions. Thus $\gamma(f^\wedge) = f^\wedge(\xi)$ for $f^\wedge \in L_1^\wedge$. Since $\mathscr{S}$ is dense in $\mathscr{M}_p$, we also have that $L_1^\wedge$ is dense in $\mathscr{M}_p$. Thus $\gamma(f^\wedge) = f^\wedge(\xi)$ for all $f^\wedge \in \mathscr{M}_p$. Conversely, every mapping $f^\wedge \to f^\wedge(\xi)$ is obviously a character on $\mathscr{M}_p$ (and on M_p too).

In order to prove local converse theorems we shall work with the space M_p^m instead of M_p. We shall prove that M_p^m is a Banach algebra for pointwise multiplication. Usually we shall have to replace M_p^m by the closure of $\mathscr{S}$ in M_p^m. We shall denote that closure by $\mathscr{M}_p^m$. In order to illustrate the relations between the spaces M_p, $\mathscr{M}_p$, M_p^m and $\mathscr{M}_p^m$ we take $p = 1$. The space M_1 is the space of Fourier transforms of bounded measures, while $\mathscr{M}_1$ is the space $L_1^\wedge$ of Fourier transforms of integrable functions. The space M_1^m consists of Fourier transforms of bounded measures μ for which

$$\sup_{t>0} t^m \int_{|x|>t} |d\mu| < \infty .$$

The space $\mathscr{M}_1^m$ consists of Fourier transforms of integrable functions f for which

$$\limsup_{t\to\infty} t^m \int_{|x|>t} |f(x)|\,dx = 0 .$$

In general, it is easy to prove that $\mathscr{M}_p^m$ consist of all $f^\wedge \in \mathscr{M}_p$ such that

$$(1) \qquad \limsup_{t\to\infty} t^m |(\zeta^t f)^\wedge|_p = 0 .$$

(The reader will have no difficulty to verify this fact.)

LEMMA 3. *The spaces M_p^m and $\mathscr{M}_p^m$ are Banach algebras for pointwise multiplication. The characters on the space $\mathscr{M}_p^m$ are the functionals $f^\wedge \to f^\wedge(\xi)$, $\boldsymbol{\xi} \in \boldsymbol{R}^d$.*

Proof. The last statement is obvious. We have only to prove that M_p^m is a Banach algebra. Thus assume that $f^\wedge, g^\wedge \in M_p^m$. Then we write

$$\zeta_{2t}^\wedge * (f^\wedge g^\wedge) = \zeta_{2t}^\wedge * \big((\zeta_t^\wedge * f^\wedge) g^\wedge\big) + \zeta_{2t}^\wedge * \big((f^\wedge - \zeta_t^\wedge * f^\wedge) g^\wedge\big).$$

The last term is the Fourier transform of $\zeta^{2t}\big(((1-\zeta^t)f) * g\big) = \zeta^{2t}\big(((1-\zeta^t)f) * (\zeta^{t/2} g)\big)$. Thus we get that

$$\zeta_{2t} * (f^\wedge g^\wedge) = \zeta_{2t}^\wedge * \big((\zeta^t f)^\wedge g^\wedge\big) + \zeta_{2t}^\wedge * \big(((1-\zeta^t)f)^\wedge (\zeta^{t/2} g)^\wedge\big)$$

and thus

$$|\zeta_{2t}^\wedge * (f^\wedge g^\wedge)|_p \leqslant \|\zeta^\wedge\|_1 \big(|(\zeta^t f)^\wedge|_p |g^\wedge|_p + |((1-\zeta^t)f)^\wedge|_p |(\zeta^{t/2} g)^\wedge|_p\big).$$

Since $|((1-\zeta^t)f)^\wedge|_p \leqslant (1+\|\zeta^\wedge\|_1)|f^\wedge|_p$ and since M_p^m is a subspace of M_p, we conclude that

$$|f^\wedge g^\wedge|_p^m \leqslant C |f^\wedge|_p^m |g^\wedge|_p^m . \ \blacksquare$$

COROLLARY 2. *Suppose that $f^\wedge \in \mathscr{M}_p^m$ and that $f^\wedge(\xi) \neq 0$ on the open set U. Let φ be a function in $\mathscr{S}$ such that $\varphi^\wedge$ has compact support in $\mathscr{S}$. Then there is a function $h^\wedge \in \mathscr{M}_p^m$ such that $f * h * \varphi = \varphi$.*

Proof. Let K be the support of φ. We consider the space of all $\mu^\wedge$ such that $\mu^\wedge$ agrees on K with an element of $\mathscr{M}_p^m$. Two functions $\mu^\wedge$ and $\nu^\wedge$ are said to be *equivalent* if $\mu^\wedge = \nu^\wedge$ on K. The space of equivalence classes of such functions will be denoted by $\mathscr{M}_p^m(K)$. In a natural way $\mathscr{M}_p^m(K)$ becomes a Banach algebra with unit element. (Note that the function 1 agrees on K with a function $\varphi^\wedge \in \mathscr{S} \subset \mathscr{M}_p^m$.) Let F be a character on $\mathscr{M}_p^m(K)$. Denote by $\bar{\mu}$ the equivalence class of all $\nu^\wedge$ which agrees on K with $\mu^\wedge$. Write $G(\mu^\wedge) = F(\bar{\mu})$. Then G is a character on $\mathscr{M}_p^m$, thus of the form $\mu^\wedge \to \mu^\wedge(\xi)$. If $\varphi^\wedge \in \mathscr{S}$ is identically 1 on K, we get

$$\varphi^\wedge(\xi)\mu^\wedge(\xi) = G(\varphi^\wedge \mu^\wedge) = F(\bar{\mu}) = G(\mu^\wedge) = \mu^\wedge(\xi).$$

It follows that $\xi \in K$.

Now for the given function $f^\wedge$ we have $F(\bar{f}) = f^\wedge(\xi) \neq 0$ for all characters F on $\mathscr{M}_p^m(K)$. Thus $\bar{f}$ has an inverse element $\bar{h}$ in $\mathscr{M}_p^m(K)$. This means that there is a function $h^\wedge \in \mathscr{M}_p^m$ such that $h^\wedge(\xi) f^\wedge(\xi) = 1$ on K. But then it follows that $f^\wedge(\xi) h^\wedge(\xi) \varphi^\wedge(\xi) = \varphi^\wedge(\xi)$ for all ξ. This gives the result.

THEOREM 6. *Suppose that $g^\wedge \in M_p$ and that $f^\wedge(\xi) = H_M(\xi) g^\wedge(\xi) \in M_p$. Suppose that $g^\wedge(\xi) \neq 0$ in a neighbourhood of $\xi = 0$ and that $g^\wedge$ agrees on this neighbourhood with a function $g_0^\wedge \in \mathscr{M}_p^m$, where $m \geqslant M$. Then*

$$A_{p\,\mathrm{loc}}^M(\Omega) \subset D_{p\,\mathrm{loc}}^{H_M}(\Omega).$$

Moreover, if

$$\lim_{\lambda\to\infty} \lambda^M \|f_\lambda * \varphi; U\|_p = 0$$

for some open set U, then $H(D)\varphi = 0$ on U.

Proof. We suppose that $g^\wedge(\xi) \neq 0$ on $|\xi| < 3\delta$. Choose a function $\Psi \in \mathscr{S}$ so that

$$\Psi^\wedge(\xi) = \begin{cases} 1 & \text{if } |\xi| < \delta, \\ 0 & \text{if } |\xi| > 2\delta. \end{cases}$$

By Corollary 1 we can find a function $h\hat{} \in \mathscr{M}_p^m$ such that $g_0 * h * \Psi = g * h * \Psi = \Psi$. Let us write

$$H(\xi) = \lambda^M H(\lambda^{-1}\xi) = \lambda^M \Psi\hat{}(\lambda^{-1}\xi) H(\lambda^{-1}\xi) + \lambda^M(1 - \Psi\hat{}(\lambda^{-1}\xi)) H(\lambda^{-1}\xi).$$

Now $\Psi\hat{}(2\xi) = 0$ if $|\xi| > \delta$. Thus $\Psi\hat{}(2\xi)(1 - \Psi\hat{}(\xi)) = 0$ and thus

$$\Psi\hat{}(2\lambda^{-1}\xi) H(\xi) = \lambda^M \Psi\hat{}(2\lambda^{-1}\xi)\Psi\hat{}(\lambda^{-1}\xi) H(\lambda^{-1}\xi).$$

Since $\Psi_\lambda = h_\lambda * g_\lambda * \Psi_\lambda$ and $f_\lambda\hat{}(\xi) = H(\lambda^{-1}\xi) g_\lambda\hat{}(\xi)$, we therefore get that

$$H(D)\Psi_{\lambda/2} * \varphi = \lambda^M \Psi_{\lambda/2} * g_\lambda * \Psi_\lambda * f_\lambda * \varphi.$$

Since $g * \Psi = g_0 * \Psi$ and since $g_0 \in \mathscr{M}_p^m$, we therefore get that $h = \Psi_{1/2} * * g * \Psi \in \mathscr{M}_p^m$.

Now let $V \subset\subset U \subset\subset \Omega$ and let ε be the distance between the boundary of U and the set V. Then we can write $h = (1 - \zeta^\varepsilon)h + \zeta^\varepsilon h$ and thus we get

$$\|h_\lambda * \chi;\ V\|_p \leqslant C(\|\chi;\ U\|_p + (\varepsilon\lambda)^{-m}\varrho(\varepsilon\lambda)\|\chi\|_p),$$

where $\varrho(t) \to 0$ as $t \to \infty$, provided that $h\hat{} \in \mathscr{M}_p^m$. (See formula (1).) If $m \geqslant M$, we therefore get that

$$\|H(D)\Psi_{\lambda/2} * \varphi;\ V\|_p \leqslant C_\varepsilon(\lambda^M\|f_\lambda * \varphi;\ U\|_p + \varrho(\lambda)\|\varphi\|_p).$$

If $\varphi \in A^M_{p\,\mathrm{loc}}(\Omega)$, it follows that $\sup_\lambda \|H(D)\Psi_{\lambda/2} * \varphi;\ V\|_p < \infty$ and thus Theorem 2 implies $\varphi \in \bar{D}^H_{p\,\mathrm{loc}}(\Omega)$. (Note that $\Psi_{\lambda/2} * \varphi \to \varphi$ in L_p if $\lambda \to \infty$.) We also see that if $\lim_{\lambda\to\infty} \lambda^M \|f_\lambda * \varphi;\ U\|_p = 0$, then it follows that

$$\lim_{\lambda\to\infty} \|H(D)\Psi_{\lambda/2} * \varphi;\ V\|_p = 0.$$

Let $\chi \in \mathscr{S}$ have compact support in V. Then it follows that

$$0 = \lim_{\lambda\to\infty}\langle H(D)\Psi_{\lambda/2} * \varphi;\ \chi\rangle = \lim_{\lambda\to\infty}\langle \Psi_{\lambda/2} * \varphi;\ H(D)\chi\rangle$$
$$= \langle \varphi;\ H(D)\chi\rangle = \langle H(D)\varphi;\ \chi\rangle.$$

Thus $H(D)\varphi = 0$ on V. Hence $H(D)\varphi = 0$ on U. ∎

Note that in the one-dimensional case the condition $H(D)\varphi = 0$ on U means that φ is a polynomial of degree at most $\langle M\rangle$, where $\langle M\rangle$ is the largest integer n such that $n < M$.

THEOREM 7. *Assume that $f\hat{}(\xi) \neq 0$ on the annulus $2^{-1-i} < |\xi| < 2^{1-i}$ for some integer i and that $f\hat{} = f_0\hat{}$ on that annulus, where $f_0\hat{} \in \mathscr{M}_p^m$. For $0 < s \leqslant m$ we then have that*

$$A^s_{p\,\mathrm{loc}}(\Omega) \subset B^s_{p\,\mathrm{loc}}(\Omega).$$

Proof. Let $\sigma\hat{}$ be the standard function used in Section 1 and put $\sigma_{-i} = 2^{-id}\sigma(2^{-i}x)$. Then $f\hat{}\,\hat{\sigma}_{-i} = f_0\hat{}\,\hat{\sigma}_{-i}$ and thus there is a function $h\hat{} \in \mathscr{M}_p^m$ such that

$$f * h * \sigma_{-i} = \sigma_{-i}.$$

(Corollary 2.) Put $j = i + k$, where $k \geqslant 0$. Writing $\lambda = 2^j$ we then get that $(\sigma_{-i})_\lambda = \sigma_k$ and thus

$$h_\lambda * \sigma_k * f_\lambda * \varphi = \sigma_k * \varphi.$$

Consequently, we get that

$$\|\sigma_k * \varphi;\ V\|_p \leqslant C(\|f_\lambda * \varphi;\ U\|_p + (\varepsilon\lambda)^{-m}\|\varphi\|_p),$$

where $V \subset\subset U \subset\subset \Omega$ and ε is defined as in the proof of Theorem 6. Since $2^{sk} = 2^{sj}\cdot 2^{-is} = \lambda^s 2^{-is}$, we conclude that

$$2^{sk}\|\sigma_k * \varphi;\ V\|_p \leqslant C(\sup_\lambda \lambda^s\|f_\lambda * \varphi;\ U\|_p + \|\varphi\|_p).$$

Using Theorem 1 we see that $\varphi \in A^s_{p\,\mathrm{loc}}(\Omega)$ implies that $\varphi \in B^s_{p\,\mathrm{loc}}(\Omega)$. ∎

5. Subspaces of M_p^m. In this section we let $\hat{\sigma}$ denote the standard function of Section 1, but we shall write $\sigma_j(x) = 2^{jd}\sigma(2^j x)$ for *all* integers j, not only for $j > 0$. We shall work with Besov spaces $\dot{B}_p^{s,q}$ defined by the semi-norms

$$\|\varphi\|_p^{s,q} = \Big(\sum_j (2^{js}\|\sigma_j * \varphi\|_p)^q\Big)^{1/q}.$$

We shall now prove the following theorem, which generalizes results by Peetre [14].

THEOREM 8. *We have the following continuous inclusions:*

(1) $\dot{B}_2^{d/2,1} \subset L_1\hat{}$,

(2) $\dot{B}_2^{d/2,1} \cap \dot{B}_2^{m+d/2,\infty} \subset M_p^m$,

(3) $\dot{B}_r^{d/r,1} \subset M_p$ *if* $r^{-1} > |p^{-1} - 2^{-1}|$, $2 < r < \infty$,

(4) $\dot{B}_r^{d/r,1} \cap \dot{B}_r^{m+d/r,\infty} \subset M_p^m$ *if* r *as above.*

Proof. The proof of (1) and (3) was given in Peetre [14]. Thus we shall only prove (2) and (4) here. In order to prove (2) we note that

$$|\hat{\zeta}_t * \varphi|_1 \leqslant \sum_j |\hat{\zeta}_t * \sigma_j * \varphi|_1 \leqslant C \sum_{2^j \geqslant t/2} |\sigma_j * \varphi|_1.$$

Using (1) we get that the sum on the right-hand side is bounded by a constant times

$$\sum_{2^j \geqslant t/2} 2^{jd/2}\|\sigma_j * \varphi\|_2 \leqslant \Big(\sum_{2^j \geqslant t/2} 2^{-jm}\Big)\|\varphi\|_2^{m+d/2,\infty}.$$

Thus it follows that $|\hat{\zeta}_t * \varphi|_1 \leqslant Ct^{-m}$ if $\varphi \in \dot{B}_2^{m+d/2,\infty}$. This implies (2). Inclusion (4) will follow in the same way, since by (3) we have that

$$|\hat{\zeta}_t * \varphi|_p \leqslant C \sum_{2^j \geqslant t/2} |\sigma_j * \varphi|_p \leqslant C \sum_{2^j \geqslant t/2} 2^{jd/r} \|\sigma_j * \varphi\|_r . \blacksquare$$

Using Theorem 8 it is now possible to prove the following two corollaries. We shall not give the proves here, but refer the reader to Löfström [11].

COROLLARY 3. *Suppose that* $\varphi(\xi) = F(H(\xi))$, *where* $F(u)$ *is infinitely differentiable on* $0 < u < \infty$ *and satisfies*

$$|F(u) - F(0)| \leqslant C_0 u^\alpha, \quad 0 < u < 1,$$
$$|D^j F(u)| \leqslant C_j u^{\beta - j}, \quad 0 < u < 1,\ j \geqslant 1,$$

and

$$|F(u)| \leqslant C_0 u^\beta, \quad 1 < u < \infty,$$
$$|D^j F(u)| \leqslant C_j u^{\beta - j}, \quad 1 < u < \infty,\ j \geqslant 1.$$

Then if $\alpha > 0$, $\beta > 0$ *and* $\alpha M \geqslant m$ *we have that*

$$\varphi \in M_1^m .$$

COROLLARY 4. *Suppose that* $\varphi(\xi) = F(H(\xi))$, *where* F *has compact support on* $0 < u < \infty$ *and is infinitely differentiable except at the point* $u = 1$. *Suppose, moreover, that*

$$|F(u)| \leqslant C_0 |u-1|^\alpha, \quad u \neq 1,$$
$$|D^j F(u)| \leqslant C_j |u-1|^{\alpha - j}, \quad u \neq 1,\ j \geqslant 1.$$

Then if $\alpha \geqslant (d-1)|p^{-1} - 2^{-1}| + m$, $m \leqslant M$ *we have that*

$$\varphi \in M_p^m .$$

6. Applications. In this section we shall discuss some particular cases of the general theory. First we give a rather general example, where the kernel f has compact support. In this case the condition $f^\wedge \in M_p^m$ will cause no trouble, since if $f^\wedge \in M_p^m$ and f has compact support, then $f^\wedge \in M_p^m$.

EXAMPLE 1. Let f be a radial bounded measure with compact support. (The function f is radial if $f(x)$ depends on $|x|$ only.) We shall suppose that

$$\int_{R^d} f(x)\,dx = 0, \tag{1}$$

$$\int_{R^d} |x|^2 f(x)\,dx \neq 0. \tag{2}$$

For instance, we can take $f(x) = \delta_0(x) - c(1-|x|^\beta)_+^\alpha$, where $0 < \beta$ and $-1 < \alpha$, and δ_0 is the Dirac measure at the origin.

LEMMA 4. *Suppose that* f *is a radial function such that* (1) *and* (2) *hold. Then we can write* $f^\wedge(\xi) = |\xi|^2 g^\wedge(\xi)$, *where* $g^\wedge \in M_1^2$ *and* $g^\wedge(0) \neq 0$.

Proof. Let u be the fundamental solution for the Laplace operator $|D|^2$, i.e.

$$u(x) = c_d |x|^{2-d} \quad \text{if} \quad d \neq 2,$$
$$u(x) = c_2 \ln|x| \quad \text{if} \quad d = 2.$$

Then $|\xi|^2 u^\wedge(\xi) = 1$. If we put $g = f * u$, we have that $|D|^2 g = f$. From (1) we get that

$$g(x) = \int f(y)\big(u(x-y) - u(x)\big)\,dy,$$

and hence, since f is radial, that

$$g(x) = \tfrac{1}{2}\int f(y)\big(u(x+y) + u(x-y) - 2u(x)\big)\,dy.$$

It follows that g is a bounded measure on every compact set. Moreover, it is easy to see that g is a radial measure. In order to prove that $g^\wedge \in M_1^2$ it is sufficient to prove that

$$|g(x)| \leqslant C|x|^{-2-d}. \tag{3}$$

But since g is radial we have only to consider the case $x = (s, 0, \ldots, 0)$. By a simple computation it is easy to verify that

$$u(x+y) + u(x-y) - 2u(x) = s^{-d}(|y|^2 - dy_1^2) + s^{-d-2}\varphi(y, s),$$

where φ is bounded as $|s|$ is large and y is in the support of f. Now we note that

$$\int f(y)(|y|^2 - dy_1^2)\,dy = 0$$

and hence

$$g(s, 0, \ldots, 0) = s^{-d-2}\int f(y)\varphi(y, s)\,dy.$$

This gives (3).

It remains to show that $g^\wedge(0) \neq 0$. Now the Fourier transform of f is

$$f^\wedge(\xi) = \sum_{\nu=0}^{\infty} \frac{(-i)^\nu}{\nu!} \int f(y)\,\langle y, \xi\rangle^\nu dy.$$

Using a symmetry argument we see that

$$\int f(y) y_j\,dy = \int f(y) y_i y_j\,dy = 0 \quad \text{if} \quad i \neq j.$$

Moreover,

$$\int f(y) y_j^2\,dy = \frac{1}{d}\int f(y)\,|y|^2\,dy = c,$$

where $c \neq 0$ by (2). Thus it follows that

$$\int f(y)\,dy = \int f(y)\langle y, \xi\rangle\,dy = 0, \qquad \int f(y)\langle y, \xi\rangle^2 dy = c|\xi|^2$$

and hence $f^\wedge(\xi) = -c|\xi|^2 + \mathcal{O}(|\xi|^3)$, $|\xi| \to 0$. Consequently, $g^\wedge(\xi) = |\xi|^{-2} f^\wedge(\xi) = -c + \mathcal{O}(|\xi|)$ as $|\xi| \to 0$, i.e. $g^\wedge(0) \neq 0$. ■

Note that the proof also implies that $g^\wedge(\xi)$ is infinitely differentiable in a neighbourhood of $\xi = 0$. Thus $g^\wedge(\xi)$ agrees on this neighbourhood with a function in say $\mathcal{M}_1^3$. Thus we can use Theorem 7. It is also clear that Theorems 4 and 8 apply. Writing $H(\xi) = |\xi|^2$ we therefore have the following results:

$$\bar{D}^H_{p\,\mathrm{loc}}(\Omega) = A^2_{p\,\mathrm{loc}}(\Omega),$$
$$B^s_{p\,\mathrm{loc}}(\Omega) = A^s_{p\,\mathrm{loc}}(\Omega), \qquad 0 < s < 2.$$

Moreover, we have "local saturation", i.e. if

$$\lim_{\lambda\to\infty} \lambda^2 \|f_\lambda * \varphi; U\|_p = 0$$

for some open subset U of Ω, then φ is a harmonic function on U, i.e. $|D|^2\varphi = 0$ on U.

EXAMPLE 2. This example is related to Corollary 3. Let $F(u)$ be an infinitely differentiable function on $0 \leqslant u < \infty$, such that F is analytic at $u = 0$ with $F(0) = 1$, $F'(0) \neq 0$ and

$$|D^j F(u)| \leqslant C_j u^{-\beta-j}, \qquad j = 0, 1, 2, \ldots, 1 < u < \infty.$$

Then F satisfies the assumptions of Corollary 3 with $\alpha = 1$. Now put $f^\wedge(\xi) = F(H(\xi)) - 1$. Then $f^\wedge \in M_1^M$ and $f^\wedge(\xi) = H(\xi) g^\wedge(\xi)$, where $g^\wedge(\xi) = G(H(\xi))$. Here we have

$$G(u) = u^{-1}(F(u) - 1).$$

Now G satisfies the assumptions of Corollary 3. Thus it follows that $g^\wedge \in M_1$. Since $F'(0) \neq 0$, we also have $g^\wedge(\xi) \neq 0$ in a neighbourhood of $\xi = 0$.

Next we prove that $g^\wedge(\xi)$ agrees in a neighbourhood with a function $g_0^\wedge \in \mathcal{M}_1^M$. If $F(u) = 1 + 1 + c_1 u + c_2 u^2 + \ldots$, we have $G(u) = c_1 + c_2 u + \ldots$ These expansions hold in a neighbourhood of $u = 0$. Let $\chi \in \mathcal{S}$ have compact support in this neighbourhood and assume that $\chi(u) = 1$ in a smaller neighbourhood of $u = 0$. Put $\varphi^\wedge(\xi) = \chi(H(\xi))$. Clearly, we than have $\varphi^\wedge \in \mathcal{S}$ and $H(\xi)\varphi^\wedge(\xi) \in \mathcal{M}_1^M$. Therefore we have that $\Psi^\wedge(\xi) = c_1\varphi^\wedge(\xi) + c_2 H(\xi)\varphi^\wedge(\xi) \in \mathcal{M}_1^M$. Put $F_0(u) = \chi(u)\,(G(u) - c_1 - c_2 u)$. Then

$$|D^j F(u)| \leqslant C_j u^{2-j}, \qquad j = 0, 1, 2, \ldots,$$

and F_0 has compact support. Writing $f_0^\wedge(\xi) = F_0(H(\xi))$ we therefore have $f_0^\wedge \in M_1^{2M}$ (Corollary 3). Thus it follows that $f_0^\wedge \in \mathcal{M}_1^M$. Now put $g_0^\wedge(\xi) = \chi(H(\xi)) g^\wedge(\xi)$. Then $g_0^\wedge = f_0^\wedge + \Psi^\wedge$ and thus $g_0^\wedge \in \mathcal{M}_1^M$.

Now we can use Theorems 4, 7 and 8. In this case $\varphi \in A^s_{p\,\mathrm{loc}}(\Omega)$ if and only if

$$\|f_\lambda * \varphi - \varphi; U\|_p = \mathcal{O}(\lambda^{-s})$$

for all $U \subset\subset \Omega$. We get the following conclusions

$$\bar{D}^H_{p\,\mathrm{loc}}(\Omega) = A^M_{p\,\mathrm{loc}}(\Omega),$$
$$B^s_{p\,\mathrm{loc}}(\Omega) = A^s_{p\,\mathrm{loc}}(\Omega), \qquad 0 < s < M,$$

and

$$\lim_{\lambda\to\infty} \lambda^M \|f_\lambda * \varphi; U\|_p = 0$$

implies $H(D)\varphi = 0$ on U.

As a particular case we take $f = \delta_0 - k$, where

$$k(x) = c\exp(-|x|^2).$$

Then $k^\wedge(\xi) = \exp(-c_1|\xi|^2)$. Thus $f^\wedge(\xi) = 1 - F(H(\xi))$, where $F(u) = \exp(-c_1 u)$ and $H(\xi) = |\xi|^2$. The kernel k is usually the Gauss–Weierstrass kernel. We can also consider the generalized Gauss–Weierstrass kernel $k^\wedge(\xi) = \exp(-H(\xi))$. For instance, if we take $H(\xi) = |\xi|$ we get the Cauchy–Poisson kernel $k(x) = c(1 + |x|^2)^{-1}$.

Another particular case is given by the function $f^\wedge(\xi) = 1 - (1 + H(\xi))^{-\beta}$. If $R_\mu = (\mu + H(D))^{-1}$ denotes the resolvent of $H(D)$, we have $f_\lambda * \varphi = 1 - (\mu R_\mu)^\beta \varphi$, where $\mu = \lambda^M$. Thus we get convergence results for the powers of the operator μR_μ. Note that if $H(\xi) = |\xi|^2$ and $d = 1$ we have $f^\wedge = 1 - k^\wedge$, where $k(x) = c\exp(-c_1|x|)$. This case is connected with Abel-summation for the Fourier transform.

EXAMPLE 3. In this example we shall take $F(u) = 1 - (1-u)_+^\alpha$. Corollary 4 is designed for this function. We see that if $f^\wedge(\xi) = F(H(\xi))$ and if $\alpha = (d-1)|P^{-1} - 2^{-1}| + m$ we have $f^\wedge \in M_p^M$. Let us write $g^\wedge(\xi) = G(H(\xi))$, where $G(u) = u^{-1}(F(u) - 1)$. Now et χ be an infinitely differentiable on the real line, such that $\chi(u) = 0$ for $u > 1/2$ and $\chi(u) = 1$ for $u < 1/4$. Then we write $G = G_0 + G_1 + G_2$, where $G_0(u) = \chi(u) G(u)$, $G_2(u) = (1 - \chi(4u))\, G(u)$ and $G(u) = (1 - \chi(u)\chi(4u))\, G(u)$. Then G_0 and G_2 satisfy the assumptions of Corollary 3 and G_1 satisfies the assumptions of Corollary 4. Thus it follows that

$$g^\wedge \in M_p^{\min(m,M)}.$$

If $0 < m < M$, we put $h^\wedge(\xi) = H^{m/M}(\xi) g^\wedge(\xi)$. Then $h^\wedge(\xi) = F_0(H(\xi))$, where $F_0(u) = u^{1-m/M}(F(u) - 1)$. This function can be decomposed in the same way as G. It follows that $h^\wedge \in M_p^m$.

By the same argument as we used in Example 2 we see that $g^\wedge(\xi) = g_0^\wedge(\xi) \neq 0$ in a neighbourhood of $\xi = 0$, where $g_0^\wedge \in \mathcal{M}_p^{\min(m,M)}$. Thus we get that if $m \geqslant M$

$$\bar{D}^{H}_{p\,\mathrm{loc}}(\Omega) = A^{M}_{p\,\mathrm{loc}}(\Omega),$$
$$B^{s}_{p\,\mathrm{loc}}(\Omega) = A^{s}_{p\,\mathrm{loc}}(\Omega), \quad 0 < s < M.$$

If $0 < m < M$, we write $H_{M-m}(\xi) = H^{1-m/M}(\xi)$. Then we get

$$D^{H_{M-m}}_{p} \cap \bar{D}^{H_M}_{p\,\mathrm{loc}}(\Omega) \subset A^{M}_{p\,\mathrm{loc}}(\Omega) \subset \bar{D}^{H_M}_{p\,\mathrm{loc}}(\Omega),$$
$$B^{s-m}_{p} \cap B^{s}_{p\,\mathrm{loc}}(\Omega) \subset A^{s}_{p\,\mathrm{loc}}(\Omega) \subset B^{s}_{p\,\mathrm{loc}}(\Omega), \quad m < s < M,$$
$$B^{s}_{p\,\mathrm{loc}}(\Omega) = A^{s}_{p\,\mathrm{loc}}(\Omega), \quad 0 < s \leqslant m.$$

We also have the local saturation theorem of Example 2.

In the particular case $H(\xi) = |\xi|^2$ we have that $f = \delta_0 + k$, where

$$k(x) = c_a \frac{J_{a+d/2}(|x|)}{|x|^{a+d/2}},$$

where $J_{a+d/2}$ is the Bessel function of order $a+d/2$. Since

$$J_{a+d/2}(|x|) \sim c\frac{\cos(|x|+\theta)}{|x|^{1/2}},$$

we see that

$$k(x) \sim c\frac{\cos(|x|+\theta)}{|x|^{a+(d+1)/2}}.$$

If we assume that $a = (d-1)/2+m$, we see that $f^{\wedge} \in M^{m}_{1}$ and $f^{\wedge} \notin M^{1}_{s}$ if $s > m$. In the case $m < M$ we see from Theorem 3 that the inclusion $B^{s}_{p\,\mathrm{loc}}(\Omega) \subset A^{s}_{p\,\mathrm{loc}}(\Omega)$ does not hold, at least not in the case $p = \infty$.

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